ACTIONS OF SOME POINTED HOPF ALGEBRAS 
ON 
PATH ALGEBRAS OF QUIVERS

Chelsea Walton
Massachusetts Institute of Technology 
JMM San Antonio 2015

Joint with Ryan Kinser, arXiv: 1410.7696 (version 3)
Goal:

To understand examples of Hopf Actions on Algebras

We say that a Hopf algebra $H$ acts on an algebra $A$ if

$A$ is an $H$-module algebra:

$A$ is an $H$-module, and the multiplication and unit maps of $A$ are $H$-morphisms.

We also need a notion of faithfulness:

$H$ acts on $A$ inner faithfully

if there is not an induced action of $H/I$ on $A$ for any nonzero Hopf ideal $I$ of $H$. In other words, the Hopf action does not factor through a smaller Hopf quotient.
Two types of results

Fix a field $\mathbb{k}$.
Let $\mathcal{H}$ be a class of Hopf algebras over $\mathbb{k}$.
Let $\mathcal{A}$ be a class of algebras over $\mathbb{k}$.

[No Quantum Symmetry]
If $H \in \mathcal{H}$ acts inner faithfully on any $A \in \mathcal{A}$,
then $H$ must be cocommutative.
(e.g. the Hopf action factors through the action of a cocommutative Hopf algebra)

[Honest Quantum Symmetry]
Classify all pairs $(H, A)$ so that $H \in \mathcal{H}$ acts inner faithfully on $A \in \mathcal{A}$.
Here, at least one $H$ is non-cocommutative.

This problem is more tractable when either:
the size of the class of Hopf algebras $\mathcal{H}$ is limited, or
the size of the class of algebras $\mathcal{A}$ is limited.
Our Setting

[Honest Quantum Symmetry] with \( \mathcal{H} \) limited, \( \mathcal{A} \) vast

\( \mathbb{k} = \) containing a primitive \( n \)-th root of unity \( \zeta \) (char \( \mathbb{k} \) is coprime to \( n \))

\( H = \) Taft algebra \( T(n) \)

-generated by grouplike element \( g \) and \( (1,g) \)-skew primitive element \( x \), subject to relations: \( g^n = 1, x^n = 0, \) and \( xg = \zeta gx \)

\( (H = u_q(sl_2) \) for \( q \) a primitive \( 2n \)-th root of unity, or \( D(T(n)) \), later)

\( \mathcal{A} = \) path algebras \( \mathbb{k}Q \), of a quiver \( Q \)

- \( Q \) is a directed graph consisting of a set of vertices \( Q_0 \), a set of arrows \( Q_1 \), and start/target maps \( s/t : Q_1 \to Q_0 \).
- Basis elements of \( \mathbb{k}Q \) are paths in \( Q \). Multiplication of basis elements is the composition of paths where defined, or 0 otherwise.
Standing Hypotheses

• The quiver $Q$ is finite ($|Q_0|, |Q_1| < \infty$)

• $Q$ is loopless

• $Q$ is Schurian ($\forall i, j \in Q_0, \exists$ at most one $a \in Q_1$ with $s(a) = i$ and $t(a) = j$)

- The action of $T(n)$ preserves the path length filtration on $\kappa Q$
  (e.g. for $x \in T(n)$ and $a \in Q_1$, we allow $x \cdot a \in \kappa Q_0$)
Theorem 1 $[T(n)]$-actions on $kQ$

Given any quiver $Q$ that admits a faithful action of $\mathbb{Z}_n$ (by quiver automorphism),

we have a classification of (e.g. precise formulae for) inner faithful actions of $T(n)$ on $kQ$ that extend the given $\mathbb{Z}_n$-action on $Q$.

Example: We classify Sweedler $(T(2))$-actions on the path algebra of $Q$ below.

Here, the action of $\mathbb{Z}_2$ is given by $\bullet \rightarrow \rightarrow \rightarrow \bullet$.
Steps of Proof of Theorem 1

I. Decompose $Q$ into a certain union of subquivers $\{Q^\ell\}$ so that $Q^\ell \cap Q^{\ell'} \subset Q_0$ for $\ell \neq \ell'$.

II. Have explicit formulae for $T(n)$-action on $kQ^\ell$

III. We obtain $T(n)$-action on $kQ$ from the set of $T(n)$-actions on $kQ^\ell$, by making identifications of the vertices in intersections $Q^\ell \cap Q^{\ell'}$ with $\ell \neq \ell'$.

Further, the *inner faithful* actions of $T(n)$ on $kQ$ are those for which $x$ does not act by zero.
Steps of Proof of Theorem 1

I. Decompose \( Q \) into a certain union of subquivers \( \{Q^\ell\} \) w/ \( Q^\ell \cap Q^{\ell'} \subset Q_0 \) for \( \ell \neq \ell' \).

- We can take each \( Q^\ell \) to be a \( \mathbb{Z}_n \)-stable subquiver of a complete digraph or complete bipartite graph, that is *maximal* with respect to partial ordering given by inclusion.
- Call such \( Q^\ell \) a \( \mathbb{Z}_n \)-component of \( Q \).
- Such a decomposition of \( Q \) is unique up to relabelling.

Example continued:

\[Q \text{ (that admits faithful } \mathbb{Z}_2 \text{-action)} \quad \text{ } \quad \mathbb{Z}_2 \text{-components of } Q\]
Steps of Proof of Theorem 1

II. Have explicit formulae for $T(n)$-action on $kQ^\ell$

- Take $e_i$ to be the trivial path at $i \in Q_0$, and take $\mathbb{Z}_n = \langle g \rangle$.

- Take $\Theta = \{1, \ldots, m\}$ to be a $\mathbb{Z}_n$-orbit of vertices, for $m|n$.

- Relabel vertices so that $g \cdot e_i = e_{i+1}$, with indices taken modulo $m$.

- We get that $x \cdot e_i = \gamma \zeta^i (e_i - \zeta e_{i+1})$ for $\gamma \in k$.

- The $\mathbb{Z}_n$-action on arrows is given by quiver automorphism, up to scalar multiple.

- For $a \in (Q^\ell)_1$, we get that $x \cdot a = \alpha a + \beta (g \cdot a) + \lambda \sigma(a)$, where $\alpha, \beta, \lambda \in k$ depends on the configuration of $a$ and $g \cdot a$, and $\sigma(a)$ is a path with start $= s(a)$ and target $= t(g \cdot a)$ (if it exists, or 0 otherwise).

We illustrate this with the running Example. Brace yourself!
Steps of Proof of Theorem 1

II. Have explicit formulae for $T(n)$-action on $\mathbb{k}Q^\ell$

$$\sigma(a) = e_1 \quad \sigma(b) = e_2$$

$$g \cdot e_1 = e_2 \quad g \cdot e_2 = e_1$$
$$g \cdot a = \mu b \quad g \cdot b = \mu^{-1}a$$
$$x \cdot e_1 = -\gamma(e_1 + e_2) \quad x \cdot e_2 = \gamma(e_1 + e_2)$$
$$x \cdot a = \gamma a - \gamma \mu b + \lambda e_1$$
$$x \cdot b = -\gamma b + \gamma \mu^{-1}a - \lambda \mu^{-1}e_2$$

$$\sigma(c) = e \quad \sigma(d) = f \quad \sigma(e) = c \quad \sigma(f) = d$$

$$g \cdot e_1' = e_2' \quad g \cdot e_2' = e_1'$$
$$g \cdot e_3' = e_4' \quad g \cdot e_4' = e_3'$$
$$g \cdot c = \mu' d \quad g \cdot d = \mu'^{-1}c$$
$$g \cdot e = \mu'' f \quad g \cdot f = \mu'''^{-1}e$$
$$x \cdot e_1' = -\gamma'(e_1' + e_2') \quad x \cdot e_2' = \gamma'(e_1' + e_2')$$
$$x \cdot e_3' = -\gamma''(e_3' + e_4') \quad x \cdot e_4' = \gamma''(e_3' + e_4')$$
$$x \cdot c = -\gamma''c - \gamma' \mu' d + \lambda' e$$
$$x \cdot d = \gamma''d + \gamma' \mu'^{-1}c - \lambda' \mu'^{-1} \mu'' f$$
$$x \cdot e = \gamma'' e - \gamma' \mu'' f + \lambda'' c$$
$$x \cdot f = -\gamma'' f + \gamma' \mu'^{-1}e - \lambda' \mu' \mu''^{-1}d$$

for $\gamma, \gamma', \gamma'', \lambda, \lambda', \lambda'' \in \mathbb{k}$, $\mu, \mu', \mu'' \in \mathbb{k}^\times$, with $(\gamma')^2 = (\gamma'')^2 + \lambda' \lambda''$
(You’ll remember all of those details, of course)
Steps of Proof of Theorem 1

III. We obtain the $T(n)$-action on $\mathbb{k}Q$ from the set of $T(n)$-actions on $\mathbb{k}Q^\ell$, by making identifications of the vertices in intersections $Q^\ell \cap Q^{\ell'}$.

In the running Example, identity the pairs of vertices

$1 \ & 1'$ and $2 \ & 2'$

of the $\mathbb{Z}_2$-components of $Q$, to yield the quiver $Q$. 
Steps of Proof of Theorem 1

II. Have explicit formulae for $T(n)$-action on $kQ^\ell$

$\sigma(a) = e_1 \quad \sigma(b) = e_2$

$\begin{align*}
g \cdot e_1 &= e_2 \quad g \cdot e_2 = e_1 \\
g \cdot a &= \mu b \quad g \cdot b = \mu^{-1}a \\
x \cdot e_1 &= -\gamma(e_1 + e_2) \quad x \cdot e_2 = \gamma(e_1 + e_2) \\
x \cdot a &= \gamma a - \gamma \mu b + \lambda e_1 \\
x \cdot b &= -\gamma b + \gamma \mu^{-1}a - \lambda \mu^{-1}e_2
\end{align*}$

$\sigma(c) = e \quad \sigma(d) = f \quad \sigma(e) = c \quad \sigma(f) = d$

$\begin{align*}
g \cdot e_1' &= e_2' \quad g \cdot e_2' = e_1' \\
g \cdot e_3' &= e_4' \quad g \cdot e_4' = e_3' \\
g \cdot c &= \mu'd \quad g \cdot d = \mu'^{-1}c \\
g \cdot e &= \mu''f \quad g \cdot f = \mu'''^{-1}e \\
x \cdot e_1' &= -\gamma'(e_1' + e_2') \quad x \cdot e_2' = \gamma'(e_1' + e_2') \\
x \cdot e_3' &= -\gamma''(e_3' + e_4') \quad x \cdot e_4' = \gamma''(e_3' + e_4') \\
x \cdot c &= -\gamma''c - \gamma' \mu'd + \lambda' e \\
x \cdot d &= \gamma''d + \gamma' \mu'^{-1}c - \lambda' \mu'^{-1} \mu''f \\
x \cdot e &= \gamma''e - \gamma' \mu''f + \lambda''c \\
x \cdot f &= -\gamma''f + \gamma' \mu'^{-1}e - \lambda' \mu' \mu''^{-1}d
\end{align*}$

for $\gamma, \gamma', \gamma'', \lambda, \lambda', \lambda'' \in k$, $\mu, \mu', \mu'' \in k^\times$, with $(\gamma')^2 = (\gamma'')^2 + \lambda' \lambda''$
Steps of Proof of Theorem 1

III. We obtain the $T(n)$-action on $\mathbb{k}Q$ from the set of $T(n)$-actions on $\mathbb{k}Q^\ell$, by making identifications of the vertices in intersections $Q^\ell \cap Q^{\ell'}$.

In the running **Example**, identity the pairs of vertices

$$1 & 1' \quad \text{and} \quad 2 & 2'$$

of the $\mathbb{Z}_2$-components of $Q$, to yield the quiver $Q$.

As a result, we must impose the following restriction on the scalar parameters of the two actions above:

$$\gamma = \gamma'$$

Further, the *inner faithful* actions of $T(n)$ on $\mathbb{k}Q$ are those for which $x$ does not act by zero.
Actions of $u_q(\mathfrak{sl}_2)$ and $D(T(n))$

We can extend the Taft actions on $\mathbb{k}Q$ in Theorem 1 to actions of the following Hopf algebras:

Let $q$ be a $2n$-th root of unity. The Frobenius-Lusztig kernel $u_q(\mathfrak{sl}_2)$ is generated by grouplike $K$, $(1,K)$-skew-primitive $E$, and $(K^{-1},1)$-skew-primitive $F$, with relations

$$KE = q^2EK, \quad KF = q^{-2}FK, \quad K^n = 1, \quad E^n = F^n = 0, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

By work of H.-X Chen (1999), the Drinfeld double $D(T(n))$ of the $n$-th Taft algebra is generated by $g, x, G, X$, subject to relations:

$$xg = \zeta gx, \quad GX = \zeta XG, \quad gX = \zeta Xg, \quad xG = \zeta Gx, \quad gG = Gg,$$

$$g^n = G^n = 1, \quad x^n = X^n = 0, \quad xX - \zeta Xx = \zeta(gG - 1).$$

Here, $g$ and $G$ grouplike, $x$ is $(1,g)$-skew primitive, and $X$ is $(1,G)$-skew primitive.
Theorem 2 [Extended actions of $u_q(\mathfrak{sl}_2)$, $D(T(n))$ on $\mathbb{k}Q$]

Since $u_q(\mathfrak{sl}_2)$ and $D(T(n))$ are both generated by Hopf subalgebras that are isomorphic to Taft algebras, namely, take $\langle K, E \rangle, \langle K, F \rangle$ for $u_q(\mathfrak{sl}_2)$, and $\langle g, x \rangle, \langle G, X \rangle$ for $D(T(n))$.

we have the following result.

Fix an action of $\mathbb{Z}_n$ on a quiver $Q$.

Additional restraints on parameters are determined so that the Taft actions on $\mathbb{k}Q$ produced in Theorem 1 extend to an action of $u_q(\mathfrak{sl}_2)$ and to an action of $D(T(n))$. 
On the category of Yetter Drinfel'd modules over $T(n)$

As a consequence of Theorem 2, we obtain that $\mathbb{k}Q$, in the case where $Q$ admits $\mathbb{Z}_n$-symmetry, is an algebra in the category of Yetter-Drinfeld modules over $T(n)$.

Motivated by the Radford-Majid biproduct construction, we ask:

Let $Q$ be a quiver that admits $\mathbb{Z}_n$-symmetry.

When does $\mathbb{k}Q$ admit the structure of a bialgebra/ Hopf algebra in the category of Yetter-Drinfeld modules over $T(n)$?