Notation:

- Let $\rho : G \to GL(V) = GL(n, \mathbb{F})$ be a **faithful, linear** representation of a **finite group** $G$ of order $d$.
- Here, $V$ is an $n$-dimensional vector space over a field $\mathbb{F}$ with basis $\{e_1, \ldots, e_n\}$. We have that $G$ acts on $V$ via $\rho$.
- This induces an action of $G$ on $V^* = \text{Hom}_\mathbb{F}(V, \mathbb{F})$, where $V^*$ has dual basis $\{x_1, \ldots, x_n\}$ with $x_i = e_i^*$.
- We extend this action to get an action of $G$ on $\mathbb{F}[V] := \mathbb{F}[x_1, \ldots, x_n]$, where $\mathbb{F}[V]_{(1)} = V^*$.

We want to understand the algebraic structure of $\mathbb{F}[V]^G$.

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<th>$G$</th>
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<td>degree of generators $\leq \max{n, \binom{n}{2}}$</td>
<td>++ Gobel</td>
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<td>$G$</td>
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<tr>
<td>Finite group $G$</td>
<td>any representation</td>
<td>(for now) generated by orbit Chern classes ${c_i(\ell)}_{\ell \in V^*}$ not necessarily finitely generated as an $F$-algebra</td>
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<tr>
<td>Finite group $G$</td>
<td>any representation</td>
<td>finitely generated as an $F$-algebra generators are not nec. algebraically independent number of generators $\leq \binom{n+d}{d}$ $n \leq$ minimal # of generators (&quot;embedding &quot;) degree of generators $\leq</td>
<td>G</td>
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<tr>
<td>Cyclic group $\mathbb{Z}/(d)$</td>
<td>any complex representation ((\text{given by } \rho(1) = \text{diag}(\omega^{a_1}, \ldots, \omega^{a_n})) where $\omega = \text{primitive } \sqrt[d]{1}$ and $0 \leq a_i \leq d - 1$ for all $i$)</td>
<td>generated by monomials $x_1^{i_1} \ldots x_n^{i_n}$ so that $\sum_{j=1}^n i_j a_j \equiv 0 \mod d$</td>
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<td>Reflection Group $G$</td>
<td>(pseudo)reflection representation</td>
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Examples of (pseudo)reflection groups (i.e. groups that admit a pseudoreflection representations):

- Cyclic groups
- Dihedral groups
- Symmetric groups
- Groups of type $G(m, p, n)$
Algebraic properties of \( \mathbb{F}[V]^G \):

- \( \mathbb{F}[V]^G \) is integrally closed (in its field of fractions) \[\text{Proposition 9.11}\]
- The extension \( \mathbb{F}[V]^G \hookrightarrow \mathbb{F}[V] \) is integral \[\text{Proposition 9.16}\]
- The extension \( \mathbb{F}[V]^G \hookrightarrow \mathbb{F}[V] \) has the lying-over, going up, and going-down properties \[\text{Corollary 9.26}\]
- Given a prime ideal \( q \) of \( \mathbb{F}[V]^G \), the \( G \)-orbit of prime ideals of \( \mathbb{F}[V] \) lying over \( q \) is a finite set \[\text{Theorem 9.27}\]
- The extension \( \mathbb{F}[V]^G \hookrightarrow \mathbb{F}[V] \) is finite (that is, \( \mathbb{F}[V] \) is finitely generated as an \( \mathbb{F}[V]^G \)-module) \[\text{Theorem 10.16}\]
- \( \mathbb{F}[V]^G \) is Noetherian \[\text{Theorem 10.16, Corollary 10.13}\]
- Krull dimension of \( \mathbb{F}[V]^G = \dim_{\mathbb{F}}(V) \) \[\text{Corollary 10.20}\]
- [Molien’s Theorem] The Poincaré series of \( \mathbb{F}[V]^G \) is \( P(\mathbb{F}[V]^G, t) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(I - \rho(g)t)} \) \[\text{Theorem 11.16}\]
- Given a Noether norm \( A \) of \( \mathbb{F}[V]^G \), get “Hironaka decomp’n”: \( \mathbb{F}[V]^G = \bigoplus_{i=1}^s A/f_i \), for some \( f_i \in \mathbb{F}[V]^G \) \[\text{Proposition 12.17}\]
- If \( A = \mathbb{F}[h_1, \ldots, h_n] \) (Noether norm of \( \mathbb{F}[V]^G \)) with \( \deg h_i = d_i \), and \( \deg(f_i) = D_i \), then \( P(\mathbb{F}[V]^G, t) = \sum_{j=1}^s \frac{t^{D_j}}{(1 - t^{d_1}) \cdots (1 - t^{d_n})} \).
- For \( \rho : G \to GL(n, \mathbb{F}) \), the top Chern classes of a Dade basis for \( V^* \) is a syst. of param. for \( \mathbb{F}[V]^G \) \[\text{Proposition 12.23}\]
- The degree of \( \mathbb{F}[V]^G \) is \( 1/|G| \) \[\text{Theorem 12.29}\]
- If \( \mathbb{F}[V]^G \) contains a system of parameters \( f_1, \ldots, f_n \) with \( \prod_{i=1}^n \deg(f_i) = |G| \), then \( \mathbb{F}[V]^G = \mathbb{F}[f_1, \ldots, f_n] \) \[\text{Proposition 12.31}\]
Errata in Neusel:

- **Exercise 3.1**: $\mathbb{F}[V]_{ij}^G$ should be $\mathbb{F}[V]$ [Found by several students]

- **Chapter 4**: The formula for extension of the $\Sigma_n$-action on basis elements of $V$, given by $\sigma(e_i) = e_{\sigma(i)}$, to an $\Sigma_n$-action on all of $V$ is incorrect. The correct formula for the extension is $\sigma(v_1, \ldots, v_n) = (v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(n)})$. To not change the calculations in Chapter 4, just define the $\Sigma_n$-action on $V$ by $\sigma(v_1, \ldots, v_n) = (v_{\sigma(1)}, \ldots, v_{\sigma(n)})$. [Found by Alexander Siegenfeld]

- **Exercise 7.6**: $\text{GL}(3, \mathbb{C})$ should be $\text{GL}(2, \mathbb{C})$ [Found by Jingwen Chen and Jiacheng Feng]

- **Exercise 7.6(v)**: The eigenvalues are not $\{1, \zeta \neq 1\}$, for $\zeta$ a root of unity. Change to $R_5 = iS$. [Found by Dai Yang]

- **Exercise 7.11**: Change matrices $M_2$ and $M_3$ to the following: [Found by Dai Yang]

  $$M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad M_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- **Page 188, line -5**: “functions $\mathbb{F}[V]$” should be “functions $\mathbb{F}(V)$” [Found by Yongyi Chen]

- **Page 192, line 12**: $\mathbb{F}[V][t]$ should be $\mathbb{F}[V][X]$.

- **Pages 196-197**: Role of $q$ and $p$ should be reversed for Theorem 9.27

- **Page 197, line -8**: $\mathbb{F}[V]^G$ should be $\mathbb{F}[V]$ [Found by Yongyi Chen and Alexander Siegenfeld]

- **Exercise 9.5**: Assume that $(S : R)$ is nonzero. [Found by Yongyi Chen and Alexander Siegenfeld]

- **Page 206, lines 1-2**: $x^{n-\ell}$ should be $x^{n-\ell_i}$ [Found by Jiacheng Feng]

- **Page 209, line 1**: Noetherian rings can have infinite Krull dimension, but Noetherian local (rings with unique maximal ideal) rings have finite Krull dimension. [Found by Alexander Siegenfeld]

- **Exercise 10.8**: Should assume that $R$ is Noetherian.
- Chapter 11: Should mention that field extension does not change vector space dimension. This is why we are allowed to assume that \( \mathbb{F} \) is algebraically closed (e.g. \( \mathbb{C} \) for char 0) so that we can employ diagonalization.

- Chapter 11: Should assume that the characteristic of \( \mathbb{F} \) does not divide \(|G|\) when we divide by \(|G|\). Or remind the reader that we’re working in characteristic 0.

- Corollaries 12.4, 12.5: Should require that \( A \) is an integral domain (need this to employ 9.25, 10.19)

- Page 247, lines -3, -1: Swap \( \leq \) and \( \geq \).

- Page 249, line -3,-2: I am unsure of the connection to Corollary 12.5; the focus of this section should be that there is a geometric characterization for having a homogeneous system of parameters.

- Page 250, line -5: cannot be empty “for \( I \) a proper ideal”.

- Page 255, line -9: \( C \) should be \( \mathbb{F} \)

- Page 264, line 8: \( \deg(A) = \frac{1}{d_1 \cdots d_n} \).

- Page 264: The proof of Proposition 12.31 needs more detail. To show that \( A \) and \( \mathbb{F}[V]^G \) have the same fraction field, first observe that we have a field extension \( \text{Frac}(A) \subset \mathbb{F}(V)^G \subset \mathbb{F}(V) \). Here, \( \text{Frac}(\mathbb{F}[V]^G) = \mathbb{F}(V)^G \) by Lemma 9.10. Then, by Theorem A.14, it suffices to show that \( |\mathbb{F}(V) : \text{Frac}(A)| = |\mathbb{F}(V) : \mathbb{F}(V)^G| \). By replacing \( \mathbb{F}(V)^G \) with \( \text{Frac}(A) \) in the proof of Theorem 12.29, we get that \( |\mathbb{F}(V) : \text{Frac}(A)| = 1/\deg(A) \).

Now, \( |\mathbb{F}(V) : \text{Frac}(A)| = 1/\deg(A) = |G| = |\mathbb{F}(V) : \mathbb{F}(V)^G| \). The second equality is shown earlier in the proof of the proposition, and the last equality follows from Theorem A.20.

- Exercise 12.1: form a system of parameters “of \( \mathbb{F}[x_1, \ldots, x_n] \)”. [Found by Jingwen Chen]

- Exercise 12.14: \( V(I \cup J) \) should be \( V(I + J) \), as the union of two ideals is not necessarily an ideal. Recall that \( I + J \) is the smallest ideal containing both \( I \) and \( J \).

- Exercise 12.17: \( M = N \oplus \ker(\phi) \) should be \( M \cong N \oplus \ker(\phi) \) [Found by Jingwen Chen]

- Page 269, line 16: \( f - \Delta_R(f) \psi_R \)

- Page 273, line 7: for any elements \( h_{\alpha} \in \mathbb{F}[V] \)

- Page 274, lines 7-26: To prove that \( A_j \in I \), it suffices to show that \( \frac{\partial f_i}{\partial x_j} \) is in \( I \), for all \( i \).

Have that (*) \( \frac{\partial f_i}{\partial x_j} = F_{i1} h_1 + \cdots + F_{i\ell} h_\ell \), for some \( F_{i\ell} \in \mathbb{F}[V]^G \). Make a change in notation:
take \( p_j := \frac{\partial p}{\partial f_j} \bigr|_{(x_1, \ldots, x_n)} \) for \( j = 1, \ldots, m \). Now the chain rule applied to \( \frac{\partial}{\partial x_j} p(f_1, \ldots, f_m) = 0 \) yields
\[
\sum_{i=1}^{m} \frac{\partial p}{\partial f_i} \bigr|_{(x_1, \ldots, x_n)} \frac{\partial f_i}{\partial x_j} = \sum_{i=1}^{m} p_i \frac{\partial f_i}{\partial x_j} = 0.
\]
Now substitute (*) and proceed as in the book.

- Page 276, line 12: \( \lambda_g \neq 1, 0 \)