Research Statement
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My research focuses on the computational complexity of interacting quantum many-body systems and their energy spectra. Such systems are characterized by a local Hamiltonian (LH) which describes the individual few-body interactions. Two different viewpoints meet in the study of local Hamiltonians:

(a) LH are an abstract model of condensed-matter physics, which motivates their study under specific interaction geometries such as lattices. Furthermore, the spectral gap is a quantity commonly studied for condensed-matter physics and turns out to be of central importance in the complexity of such structured systems.

(b) LH are a generalization of constraint satisfaction problems (CSPs) studied in optimization theory and computer science. Taking this view motivates the complexity-theoretic study of LH in a setting with less structure. For example arbitrary interaction geometries without spatial locality may be allowed, and often no spectral gap is assumed.

Most research on local Hamiltonians blend viewpoints (a) and (b) in different proportions. In direction (a) I have given a sub-exponential classical algorithm for solving frustration-free 2D Hamiltonians with a uniform spectral gap. I have furthermore improved the tools for analyzing approximate ground space projectors for degenerate ground spaces. In the direction (b) I gave a simple proof that the spectrum of a local Hamiltonian concentrates for more general geometries.

I plan to continue researching (a) to improve our understanding of entanglement bounds and efficient ground state representations for lattice Hamiltonians and their relation to spectral gaps. I also plan to branch into (b) by exploring the quantum PCP conjecture and its relations to coding theory.

1 Background

A quantum system with \( n \) sites is described by a vector in \( \mathcal{H} = \bigotimes_{v \in V} \mathcal{H}_v \) where \( V \) is an index set with \( n \) elements and \( \mathcal{H}_v \) is a complex vector space with the degrees of freedom at site \( v \). We take \( \mathcal{H}_v \approx \mathbb{C}^d \) for a constant \( d \). A \( k \)-local Hamiltonian is a Hermitian operator with a particular structure \( H = \)
\[ \sum_{e \in E} H_e \otimes I_{V \setminus e} \] where each \( H_e \) is an operator acting on few vertices \( v \in e, |e| \leq k \). The geometry of \( H \) can be viewed as a hypergraph \( G = (V, E) \). \( H \) is a local Hamiltonian when \( k \) is constant. The Hamiltonian has a concise description in the form of the list \( \{ H_e \} \), but diagonalizing it would a priori require time exponential in \( n \) due to the large dimension of the vector space.

Hastings [Has07] showed that for a gapped spin chain,\(^1\) the ground state (eigenstate with lowest eigenvalue, or energy) has particular structure; the entanglement of the ground state across a cut is bounded by a constant. It is an open problem whether an analogous bound holds for other interaction geometries and in particular for higher-dimensional lattice interactions. It is conjectured that the entanglement of gapped ground states across any cut is at most of the order of the number of edges in the cut–such a bound is known as an area law. A quantitatively weaker but non-trivial entanglement bound is called a sub-volume law. The 1-dimensional case enjoys an additional property beyond the provable area law, namely an implication that an area law leads to an efficient classical encoding of the state as a matrix product state (MPS). These properties for the 1-dimensional case were used to give an efficient classical algorithm [LVV15] computing the ground state of gapped spin chains. Later improvements include generalizations to degenerate (non-unique) ground states with constant [CF16] and later polynomial degeneracies [ALVV17]. Prior to my work the only algorithms with non-trivial time complexity\(^2\) for computing the ground states of gapped local Hamiltonians were for spin chains.

2 Local Hamiltonians with structure

We now discuss results of the type (a) mentioned in the introduction. Building on recent work on algorithms for spin chains and a 2-dimensional sub-volume law [AAG20] I gave a subexponential-time classical algorithm for computing ground states of 2D local Hamiltonians under a variant of the gap assumption known as the uniform or local gap [GS17, AAG20].

**Theorem 1** ([Abr20d]) Let \( H \) be a frustration-free Hamiltonian with uniform gap \( \gamma \) on the \( w \times h \) lattice with \( n = wh \) qudits. Let \( D \) be a bound on the degeneracy and \( \delta \). There exists a randomized classical algorithm with time complexity

\[ (D/\delta)^O(1) \tilde{O}((n/\gamma)^{5/6}), \]

which on input \( H \) outputs a matrix product state of bond dimension at most \( (1) \) representing a subspace \( Z \lesssim \mathcal{H} \) which is \( \delta \)-close to \( Z = \ker H \) with probability at least \( 1/2 \).

We should not expect to be able to improve the time complexity beyond sub-exponential \( \exp(n^c) \) for some \( 0 < c < 1 \) (likely \( c = 1/2 \)). Indeed, even the case

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\(^1\)A spin chain is a local Hamiltonian whose interaction graph is a line

\(^2\)Brute-force diagonalization takes time exponential in \( n \). Non-trivial time complexity is any improvement over this.
of classical constraint-satisfaction problems has such a complexity lower bound \( \exp(n^c) \) under standard assumptions (randomized exponential-time hypothesis).

The sub-volume law [AAG20] was proved for unique ground states where it reduces to the construction of an operator known as an *approximate ground state projector* (AGSP) by a lemma from [ALV12]. In the process of proving theorem 1 I generalized this lemma to degenerate (i.e., non-unique) ground states. In the statement below \( \Delta \) is the *shrinking factor* and \( R \) the entanglement rank of the AGSP.

**Lemma 1** ([Abr20b]) Suppose there exists an \((\Delta, R)\)-AGSP \( K \) acting on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) such that

\[
R \Delta \leq 1/2.
\]

Let \( Z \subsetneq \mathcal{H}_1 \otimes \mathcal{H}_2 \) be the target space of \( K \) and \( D = \dim Z \) its degeneracy. Then the maximum entanglement of any state \( |\psi\rangle \in Z \) satisfies the bound

\[
\max_{|\psi\rangle \in Z} S(\rho_1^\psi) = O(\log R + \log D),
\]

where \( S(\rho_1^\psi) \) is the entanglement entropy of \( |\psi\rangle \) between subsystems \( \mathcal{H}_1 \otimes \mathcal{H}_2 \).

To prove lemma 1 I improved [Abr20b] on the existing analysis of AGSPs for degenerate target spaces. More precisely I improved the previously stated post-\( \Delta \)-AGSP error from \( \delta' = \Delta / \mu^2 \) [ALVV17] to the optimal bound \( \delta' / \mu' = \Delta (\delta / \mu) \). Here \( \delta \) and \( \mu = 1 - \delta \) are the pre-AGSP error and overlap, respectively, and \( \delta' \) and \( \mu' = 1 - \delta' \) the post-AGSP error and overlap. This is an improvement by a factor \( \delta \mu \) which is particularly significant when starting from a very good or very poor approximation. Combining lemma 1 with the AGSP constructed in [AAG20] immediately yields a generalization of the sub-volume law of [AAG20] to sub-exponentially degenerate ground states.

I also adapted the AGSP-constructions of [ALV12, AKLV13] and algorithms [LVV15, ALVV17] to show that the ground state for a tree-like Hamiltonian can be computed classically in polynomial time when the tree satisfies a sub-quadratic local volume growth bound\(^3\) [Abr20a]. While the growth assumption is violated for important classes of trees including regular trees, it is satisfied for other important classes such as the *Vicsek fractals* used to model hyperbranching polymers [BvFJK04].

### 3 Local Hamiltonians with less structure

Just as classical constraint satisfaction problems are complete for the complexity class \( \text{NP} \) by the Cook-Levin theorem, Kitaev observed that local Hamiltonians are complete for *quantum Merlin-Arthur* (QMA). The (classical) PCP theorem states that it remains \( \text{NP-hard} \) to even *approximate* the optimal value of a CSP. Here the permissible approximation error is as large as a small constant multiple

\(^3\)[ALV12] had previously noted that sub-volume laws follow under the same growth condition without specifically considering trees.
of the number of constraints. The quantum PCP conjecture [AAV13] posits that the ground state energy of a local Hamiltonian is QMA-hard to approximate even up to error scaling with the number of interactions.

In [Abr20c] I observed that the related approximation problem for the quantiles of the spectrum is easy, except for an exponentially small fraction of the quantiles. Indeed, the quantiles concentrate exponentially around the normalized trace:

**Proposition 1 ([Abr20a] simple version)** Let $H$ be a local Hamiltonian on a $k$-uniform and $g$-regular hypergraph and let $\sigma = d^{-n} \sum_i \delta_{\lambda_i(H)}$ be the spectral probability distribution for $H$. Let $\mu = d^{-n} \text{tr} H$. Then,

$$\sigma([\mu - \varepsilon m, \mu + \varepsilon m]) \leq 2kg \exp\left( -\gamma^2 \left\lfloor n/k^2 \right\rfloor \right).$$

For a local Hamiltonian we have that $k$ is uniformly bounded by a constant. But the degree $g$, i.e., the number of interactions involving a site $v$, need not be bounded. It is therefore essential that [Abr20c] achieves a bound with no $g$-dependence in the exponent, yielding exponential concentration even for unbounded-degree geometries (including the complete graph).

Spectral concentration bounds for local Hamiltonians had been shown previously [Ans16, Kuw16, KS19]. In fact these results treat a more general “directional” question of the energy distribution in a specific state. Coming from a viewpoint of condensed-matter physics, the bounds of [Ans16, Kuw16] require geometric locality to achieve exponential concentration. In contrast, [KS19] gives an exponential concentration bound even for long-range interacting system (similarly to proposition 1) though it is quantitatively weaker than proposition 1. In [Abr20c] I gave a simple proof for the spectral concentration bound using Weyl’s inequalities in contrast with the careful analysis of cluster expansions in [KS19].

### 4 Future outlook

For the sub-exponential algorithm [Abr20d] for 2D Hamiltonians we have that the special case of classical CSPs provides a lower bound $\exp(n^{O(1)})$ on the time complexity. However, in the classical case the algorithm is space-efficient. I am interested in whether the sub-exponential space complexity of theorem 1 can be improved further. This problem is closely related to area laws and their relation to efficient representations such as tensor networks. Such implications from sub-volume laws to PEPS (lattice tensor network) representations have been conjectured by the authors of [AAG20]. I am also interested in quantitative improvements to the sub-volume law [AAG20] and in further clarifying the relation between the gap and the uniform gap [GS17, AAG20]. These questions are guided by the goal of shedding light on the area law conjecture for lattices.

I also plan to explore further the unstructured regime (b) and questions related to the quantum PCP conjecture. In particular I plan to study the weaker
NLTS (no low-energy trivial states) conjecture and its relation to quantum error-correcting codes.

References


