Improved AGSP tools and sub-exponential algorithm for 2D frustration-free uniformly gapped spin systems

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Abstract

We give an improved analysis of approximate ground space projectors in the setting of local Hamiltonians with a degenerate ground space. This implies a direct generalization of the AGSP⇒entanglement bound implication of [Arad, Landau, and Vazirani '12] from unique to degenerate ground states. We use the improved analysis to give a simple algorithm for frustration-free spin systems provided an AGSP with structure as a matrix product operator. We apply our tools to a recent 2D subvolume law of [Anshu, Arad, and Gosset '20], generalizing the result to sub-exponentially degenerate ground spaces and giving a sub-exponential-time classical algorithm to compute the ground states. This time complexity cannot be improved beyond sub-exponential assuming the randomized exponential time hypothesis, even for the special case of classical constraint satisfaction problems on the 2D grid.
1 Introduction

Quantum spin systems are the most basic quantum many-body systems and are widely studied in condensed-matter physics and quantum complexity theory [GHLS14, AAV13]. The state of a quantum spin system lives in the tensor product $\mathcal{H} = \bigotimes_{v \in V} \mathcal{H}_v$ of vector spaces $\mathcal{H}_v \cong \mathbb{C}^d$ (typically with constant $d$) associated to the individual spins. The interactions between the spins are described by a Hermitian operator known as a Hamiltonian. Physically relevant Hamiltonians have additional structure, most importantly locality, which implies that they can be described succinctly despite the exponentially large dimension $d^n$ of the Hilbert space (vector space) $\mathcal{H} \cong (\mathbb{C}^d)^\otimes n$. This leads to the notion of the local Hamiltonian problem: given a Hamiltonian $H = \sum_i H_i$ defined as a sum of terms $H_i$, each of which acts on at most a constant $k$ number of spins, compute the ground state energy (lowest eigenvalue) of $H$. By locality the size of the input is polynomial in the number of spins $n$.

The local Hamiltonian problem is known to be complete for the complexity class QMA in general [KKR06] and even under restrictive assumptions on the interaction geometry, namely when restricting to interactions on the line [HNN13]. On the other hand, combining the interaction geometry of a line with the additional assumption of a spectral gap leads to a tractable problem [LVV15]. This fact relies on the area law [Has07] which holds for ground states of gapped spin chains (local Hamiltonians with the geometry of a line). Area laws are the strongest possible entanglement bounds for ground states of local Hamiltonians and have so far only been rigorously proven in the case of spin chains, where they coincide with a constant entanglement bound. In this one-dimensional case the constant entanglement bound further implies an efficient representation of the state as a matrix product state [Vid03]. Higher-dimensional analogues of matrix product states are knowns as PEPS, and more generally tensor networks. In higher dimensions the property of having an efficient representation as a PEPS state is strictly stronger than satisfying an area law [GE16]. On the other hand a widely believed conjecture says that efficient PEPS representations should exist for gapped ground states—this can be viewed as a strong version of the area law conjecture.

Approximate ground space projectors (AGSPs) are an indispensable tool for proving entanglement bounds on ground states of gapped local Hamiltonians [ALV12, AKLV13, AAG20] and for constructing polynomial-time algorithms [LVV15, CF16, ALVV17] for gapped spin chains. They are operators which shrink excited states of the Hamiltonian but not the ground states. In the setting of algorithms, AGSPs are applied to the vectors in $\delta$-viable sets. A $\delta$-viable set is a set of states of a subsystem with span $\mathcal{V}$ such that the extension $\mathcal{V} \otimes \mathcal{H}_{\text{rest}}$ approximately contains the ground states. Here $\mathcal{H}_{\text{rest}}$ is the Hilbert space of the spins outside the subsystem. For simplicity we will always identify $\delta$-viable sets with their span, referring to them as $\delta$-viable subspaces.

In results using AGSPs it is often assumed that the ground state be unique [ALV12, AKLV13, AAG20]. Indeed for unique ground states the existence of a $(\Delta, R)$-AGSP with $\Delta R \leq 1/2$ immediately implies an entanglement bound $O(\log R)$ by applying a lemma of Arad, Landau, and Vazirani ([ALV12] corollary III.4). Here $R$ is the entanglement rank of the AGSP and $\Delta$ the shrinking factor. We call this implication the (non-degenerate case) readymade bound. It reduces the task of proving an area law to that of constructing an AGSP.

Generalizing area laws and algorithms for 1D gapped Hamiltonians from the setting of a unique ground state to a ground space with degeneracy (i.e., dimension)
$D > 1$ has been a focus of several works, starting with the case of a constant degeneracy [CF16, Hua14] and later generalized further to polynomial degeneracy [ALVV17]. AGSPs have been used before to prove an area law for polynomially degenerate ground spaces [ALVV17] of gapped spin chains, but no direct analogue of the ready-made bound follows from existing tools and analyses. Indeed, inspecting the proofs of degenerate-case entanglement bounds one finds the necessary assumption on a $(\Delta, R)$-AGSP to be $R^C \Delta \leq 1/2$ [ALVV17] where $C$ can be taken no smaller than $C = 2$.

1.1 Our contribution

In this paper we generalize the readymade entanglement bound of [ALV12] to degenerate ground spaces with no strengthening of the assumed parameter tradeoff. To obtain this generalization we prove the optimal bound on how the error of a $\delta$-viable space improves when applying an AGSP. We then use this error reduction bound to give a simple randomized classical algorithm for computing ground states of a spin system given an AGSP provided as a matrix product operator (MPO), i.e., with an explicit one-dimensional entanglement structure. The possibility of using a single AGSP for the entire algorithm with a uniform bound on the MPO bond dimension is mainly relevant to the setting of frustration-free systems. In this setting the detectability lemma (DL) operator takes the place of the more cumbersome truncations necessary in the frustrated case.\footnote{The author thanks Anurag Anshu for helping him appreciate the difference in AGSP constructions between the frustrated and unfrustrated settings, in particular as relating to the DL operator.}

1.1.1 Application to 2D systems

In a recent breakthrough [AAG20] by Anshu, Arad, and Gosset it was shown that a sub-volume law holds for 2D frustration-free spin system satisfying a uniform (or local) gap condition [GS17]. A sub-volume law is a quantitatively weaker analogue of an area law, and the result of [AAG20] represents significant progress towards understanding the conjectured area law in 2D. The readymade bound proved in our paper immediately extends the result of [AAG20] from unique to sub-exponentially degenerate ground states.

We then apply our algorithm with a modified version of the AGSP from [AAG20]. This allows us to compute the ground states of a uniformly gapped frustration-free 2D spin system in sub-exponential time. This is the first sub-exponential algorithm for computing the ground states of gapped lattice Hamiltonians beyond the one-dimensional setting [LVV15]. Assuming the randomized exponential-time typothesis sub-exponential time complexity is the best one can hope for in the present 2D setting, even in the classical special case.

2 Basic definitions

Let $B(\mathcal{H})$ is the space of linear operators on Hilbert space $\mathcal{H}$ and $I \in B(\mathcal{H})$ the identity. Let $\preceq$ denote the Loewner order on operators and write $Z \preceq \mathcal{H}$ when $Z$ is a subspace of $\mathcal{H}$.

Given a Hamiltonian $H \in B(\mathcal{H})$, an AGSP for $H$ is an operator $K$ which shrinks the excited states but not the vectors in the ground space $Z \preceq \mathcal{H}$ of $H$. We do not directly invoke the Hamiltonian itself, as the AGSP property can be captured in
terms of just the ground space $Z$. Given a subspace $Z \preceq H$ let $P_Z$ be the projection onto $Z$. The standard definition of an AGSP is the following:

**Definition 2.1** ([ALV12, AKLV13]). A standard $\Delta$-AGSP with target space $Z \preceq H$ is an operator $K \in \mathcal{B}(H)$ which commutes with $P_Z$ and such that $KP_Z = P_Z$ and $\|KP_Z\| \leq \sqrt{\Delta}$.

In the interest of wide applicability we also use a less restrictive definition of an AGSP when stating our error-reduction bound and the subsequent readymade entanglement bound. This more general AGSP is mainly useful for the frustrated case; in particular it generalizes the spectral AGSP of [ALVV17] as well as standard AGSPs.

**Definition 2.2.** A general $\Delta$-AGSP with target space $Z \preceq H$ is an operator $K \in \mathcal{B}(H)$ which commutes with $P_Z$ and satisfies

1. $P_Z K^1 K P_Z \geq P_Z$, i.e., $K$ is a dilation on $Z$.
2. $\|KP_Z\| \leq \sqrt{\Delta}$.

A spectral AGSP [ALVV17] for a Hamiltonian $H$ corresponds to definition 2.2 with the additional requirement that $K \geq 0$, and that $K$ and $H$ be simultaneously diagonalizable. The condition that $K$ commute with $P_Z$ is equivalent with requiring that $Z$ and $Z^\perp$ be closed under $K$, i.e., following two conditions\(^2\) from [ALV12, AKLV13]:

$$|z\rangle \in Z \implies K|z\rangle \in Z \quad \text{and} \quad |y\rangle \in Z^\perp \implies K|y\rangle \in Z^\perp.$$  \hspace{1cm} (1)

A $(\Delta, R)$-AGSP on a bipartite Hilbert space $\mathcal{H}_L \otimes \mathcal{H}_R$ is a $\Delta$-AGSP $K \in \mathcal{B}(\mathcal{H}_L) \otimes \mathcal{B}(\mathcal{H}_R)$ with entanglement rank at most $R$, i.e., it is the sum of $R$ tensor products. It will often be useful to consider just the span of the left tensor factors arising in such a decomposition using the following definition:

**Definition 2.3.** For a bipartite $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$ we say that an operator subspace $K \subseteq \mathcal{B}(\mathcal{H}_L)$ is a $\Delta$-AGSP with target space $Z \preceq H$ if there exists a $\Delta$-AGSP $K \in K \otimes \mathcal{B}(\mathcal{H}_R)$ with target space $Z$.

A $(\Delta, R)$-AGSP on a bipartite Hilbert space gives rise to an AGSP $K \subseteq \mathcal{B}(\mathcal{H}_L)$ in the sense of definition 2.3 with $\dim K \leq R$.

### 2.1 Overlaps and errors

Let $S(H)$ be the sphere of unit vectors in $H$. Let $Z,V \preceq H$ be subspaces.

**Definition 2.4.** $V$ is $\mu$-overlapping onto $Z$ (written $V \succeq_\mu Z$) if $\|P_V|z\rangle\|^2 \geq \mu$ for all $|z\rangle \in S(Z)$. $V$ is $\delta$-viable for $Z$ if $\|P_{V^\perp}|z\rangle\|^2 \leq \delta$ for all $|z\rangle \in S(Z)$.

$V$ is $\delta$-viable for $Z$ iff it is $\mu$-overlapping onto $Z$ with $\mu = 1 - \delta$. The overlap of $V$ onto $Z$ is $\mu = \min_{|z\rangle \in S(Z)} \|P_V|z\rangle\|^2$ and the error of $V$ onto $Z$ is $\delta = \max_{|z\rangle \in S(Z)} \|P_{V^\perp}|z\rangle\|^2$. Two subspaces are $\delta$-close ($\simeq_\delta$) if each is $\delta$-viable for the other.

We say that $V \subseteq H$ covers $Z \preceq H$ if $P_Z(V) = Z$. Equivalently the range of $P_Z P_V$ is $Z$, or $P_Z P_V P_Z \geq \mu P_Z$ for some $\mu > 0$. That is, $V$ covers $Z$ if its overlap onto $Z$ is $\mu > 0$.

\(^2\)Indeed, (1) imply $KP_Z = P_Z KP_Z = P_Z K - P_Z KP_Z = P_Z K$, where the last equality is because $K$ sends $Z^\perp$ to itself. In the special case where $K$ is Hermitian it suffices to check one of the implications (1).
Definition 2.5 (Bipartite case). Given a target subspace of a bipartite space \(\mathcal{Z} \leq \mathcal{H}_L \otimes \mathcal{H}_R\), a subspace \(\mathcal{V} \leq \mathcal{H}_L\) of the left tensor factor is said to be \(\delta\)-viable for \(\mathcal{Z}\) iff \(\mathcal{V} \otimes \mathcal{H}_R\) is \(\delta\)-viable for \(\mathcal{Z}\).

Typically in the literature the word \(\delta\)-viable refers exclusively to this bipartite case. But our terminologies are in fact equivalent as one can take \(\mathcal{H}_R = \mathbb{C}\).

2.2 Entanglement

The von Neumann entropy \(S(\rho)\) of a density matrix \(\rho\) is the Shannon entropy \(\sum_i \lambda_i \log(1/\lambda_i)\) of its eigenvalues. For a pure state \(|\psi\rangle \in \mathcal{H}_L \otimes \mathcal{H}_R\) in a bipartite space its entanglement entropy is \(S(\rho^V_L)\) where \(\rho^V_L = \text{tr}_R(|\psi\rangle\langle\psi|)\) is the reduced density matrix on \(\mathcal{H}_L\). This quantity is unchanged if switching the roles of \(\mathcal{H}_L\) and \(\mathcal{H}_R\).

3 Results

The entanglement bound and simple algorithm given in this paper both rely on the precise analysis of how overlap is improved when applying an AGSP. This analysis is straightforward when the target is a single vector, but the exact bound was not established previously in the degenerate setting.

3.1 Error reduction bound

Consider two subspaces \(\mathcal{Z}, \mathcal{V} \leq \mathcal{H}\) such that \(\mathcal{V}\) covers \(\mathcal{Z}\). Let \(\mu > 0\) be the overlap of \(\mathcal{V}\) onto \(\mathcal{Z}\) and let \(\delta = 1 - \mu\), then define the error ratio \(\varphi\) of \(\mathcal{V}\) onto \(\mathcal{Z}\) as \(\varphi = \delta/\mu < \infty\).

Denoting the largest principal angle \([\text{GH06, BI67}]\) between \(\mathcal{Z}\) and \(P_V(\mathcal{Z}) \leq \mathcal{V}\) as \(\theta\) one has that \(\mu = \cos^2 \theta\) and \(\delta = \sin^2 \theta\), so we can equivalently write the error ratio of \(\mathcal{V}\) onto \(\mathcal{Z}\) as
\[
\varphi = \delta/\mu = \tan^2 \theta.
\]

Lemma 3.1. Let \(K\) be a general \(\Delta\)-AGSP for \(\mathcal{Z} \leq \mathcal{H}\), and suppose \(\mathcal{V} \leq \mathcal{H}\) covers \(\mathcal{Z}\) with error ratio \(\varphi\). then \(\mathcal{V}' := KV = \{K|v\} : |v\rangle \in \mathcal{V}\) covers \(\mathcal{Z}\) and the error ratio \(\varphi'\) of \(\mathcal{V}'\) onto \(\mathcal{Z}\) satisfies
\[
\varphi' \leq \Delta \cdot \varphi.
\]

This bound is clearly sharp.\(^3\) Because \(\delta' = \frac{\varphi'}{1+\varphi'} \leq \varphi'\), lemma 3.1 implies:

Corollary 3.2. If \(\mathcal{V}\) is \(\delta\)-viable for \(\mathcal{Z}\) with \(\delta = 1 - \mu < 1\), then \(KV\) is \(\delta'\)-viable for \(\mathcal{Z}\) with error \(\delta' \leq \Delta \delta/\mu\).

The best previous error reduction bound for the general degenerate-case AGSPs ([ALVV17] lemma 6) bounded the post-AGSP viability error by
\[
\delta'_{\text{literature}} = \Delta/\mu^2.\tag{2}
\]

The post-AGSP error bound \(\delta'\) in corollary 3.2 improves on (2) by a factor \(\mu \cdot \Delta\), which is particularly significant when starting in either the small-overlap \(\mu \ll 1\) or small-error regime \(\delta \ll 1\).

The subtlety in proving lemma 3.1 comes from the following: While an AGSP is defined in terms of an orthogonal decomposition with respect to the target space \(\mathcal{Z}\),

\(^3\)Consider the \(\Delta\)-AGSP \(K = |0\rangle\langle 0| + \sqrt{\Delta} |1\rangle\langle 1|\) on \(\mathbb{C}^2\) and subspaces \(\mathcal{Z}, \mathcal{V} \leq \mathbb{C}^2\) spanned by \(|z\rangle = |0\rangle\) and \(|v\rangle = \frac{1}{\sqrt{1+\sigma}} (|0\rangle + \sqrt{\sigma} |1\rangle)\).
the overlap is conversely defined in terms of orthogonal decompositions with respect to the covering subspace \( \mathcal{V} \). To prove lemma 3.1 we replace \( \mathcal{V} \) with a subspace \( \mathcal{Y} \subset \mathcal{V} \) and establish a symmetry between \( \mathcal{Y} \) and \( \mathcal{Z} \).

In appendix C we also include an alternative proof of lemma 3.1 which is more similar in structure to the analysis in [ALVV17] lemma 6. In this case we obtain the strengthened bound by improving the 'lifting' lemmas (1 and 2) of [ALVV17] to have quadratically better dependence on the overlap \( \mu \).

3.2 Readymade bound for degenerate ground spaces

We combine our error reduction bound 3.1 with the bootstrap [ALV12] to obtain the readymade entanglement bound in the degenerate setting.

**Proposition 3.3.** Suppose there exists a general \((\Delta, R)\)-AGSP \( K \in \mathcal{B}(\mathcal{H}_L \otimes \mathcal{H}_R) \) such that

\[ R\Delta \leq 1/2. \]

Let \( \mathcal{Z} \) be the target space of \( K \) and \( D = \dim(\mathcal{Z}) \) its degeneracy. Then the maximum entanglement entropy of any state \( |\psi\rangle \in \mathcal{Z} \) satisfies the bound

\[ \max_{|\psi\rangle \in \mathcal{S}(\mathcal{Z})} S(\rho_L^\psi) = 1.01 \cdot \log D + O(\log R), \]

where \( \mathcal{S}(\mathcal{Z}) \) is the set of unit vectors in \( \mathcal{Z} \) and \( S(\rho_L^\psi) \) is the entanglement entropy of \( |\psi\rangle \) between subsystems \( \mathcal{H}_L \) and \( \mathcal{H}_R \).

Proposition 3.3 is proved in section 7.2. In the case of a frustrated Hamiltonian the typical AGSP contraction involves spectral truncations of parts of the Hamiltonian, incurring an error in the target space of the AGSP. We therefore also prove a version (lemma 7.5) of proposition 3.3 which is applicable to the frustrated case by allowing the target space to be approximate.

Possible improvements Given our formulation of the entanglement bound in proposition 3.3 as a uniform bound over all vectors in \( \mathcal{S}(\mathcal{Z}) \) it is clear that the bound must include a term \( \log D \) corresponding to the degeneracy; consider for example the zero Hamiltonian. On the other hand, since the zero Hamiltonian does not enforce entanglement on its ground states, one may wish to avoid the \( \log D \) term at the cost of the bound holding in a weaker sense, say, for a basis. We conjecture that under the conditions of proposition 3.3 \( \mathcal{Z} \) can be written as the span of \( D \) vectors \( |\psi_i\rangle \) satisfying the entanglement bound \( \max_{i=1, \ldots, D} S(\rho_L^\psi) = O(\log R) \).

Even if such an improved entanglement bound holds it seems likely that the uniform bound is the correct notion for algorithms, as it bounds the dimension of a viable space. Note also that this non-uniform definition of an entanglement bound is not preserved under taking subspaces which makes it potentially more difficult to analyze.

3.3 Simple algorithm given implementable AGSP

Consider a multipartite Hilbert space \( \mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_w \) and a standard \( \Delta \)-AGSP \( K \in \mathcal{B}(\mathcal{H}) \) represented as a matrix product operator (MPO) with bond dimension \( R \). We write \( \mathcal{H}_{[i,j]} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_i \). If the bond dimension of the MPO satisfies an appropriate bound, say subexponential, then we call \( K \) an implementable AGSP.

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4The author thanks Anurag Anshu and David Gosset for a discussion about this non-uniform statement.
When applying the algorithm in the 2D case each $H_i$ will correspond to a column of spins.

Let $\bar{d} = \max\{\dim(H_1), \ldots, \dim(H_w)\}$ and let $D$ be an upper bound on the degeneracy $\dim(Z)$. For each $i = 1, \ldots, w$ let $K_{[1,i]} \subseteq B(H_{[1,i]})$ be the operator subspace encoded by the left part of the MPO for $K$ where the cut bond is left open. Then $\dim(K_{[1,i]}) \leq R$, and $K_{[1,i]}$ is a $\Delta$-AGSP in the sense of definition 2.3.

Applying our algorithm to the 2D case makes it especially important that the complexity is polynomial in the entanglement rank; this was less essential in the 1D case where the entanglement is constant, and indeed the first algorithms [LVV15] were exponential in the entanglement rank due to the enumeration over boundary contractions. To achieve the polynomial dependence on entanglement rank we use the random sampling method of [ALVV17].

The algorithm keeps a $\delta$-viable subspace $\mathcal{Y}_{[1,i]} \subseteq H_{[1,i]} = H_1 \otimes \cdots \otimes H_i$ for the leftmost $i$ sites, similarly to the early algorithm [LVV15] for spin chains. In an iteration it extends with all of the next site $H_{i+1}$, samples a subspace as in [ALVV17], and applies the left half of the AGSP. Given a viable space $V$ and an AGSP $K$ as in definition 2.3 this means replacing $V$ with $V' = K V = \{L|v|: L \in K, |v| \in V\}$. As with existing algorithms for spin chains [LVV15, CF16, ALVV17], trimming operations are interspersed; we define the trimming procedure $\mathcal{Y} \mapsto \text{Trim}_\epsilon(\mathcal{Y})$ in a way that allows a simple self-contained analysis (section 6).

\begin{algorithm}
\caption{}
Input: $\Delta$-AGSP $K$ given as MPO. Parameters $V \in \mathbb{N}, \varepsilon, \delta > 0$

Set $\mathcal{Y}_{[1]} = C$

for $i = 1, \ldots, w$ do
\begin{itemize}
\item Sample $\mathcal{Y}_{[1,i-1]} \leq \mathcal{Y}_{[1,i-1]} \otimes H_i$ with $\dim(\mathcal{V})_{[1,i]} = V$.
\item Set $\mathcal{Y}_{[1,i]} = \text{Trim}_\varepsilon(K_{[1,i]} \mathcal{V}_{[1,i]})$.
\end{itemize}

Output: Let $\tilde{H} = I - K^\dagger K$ and $\mathcal{Y} = \mathcal{Y}_{[1,w]}$, and output $\tilde{Z}$, the combined eigenspaces of $\tilde{H}|_{\mathcal{Y}}$ corresponding to eigenvalues $\leq \delta$ for $\tilde{H}$.
\end{algorithm}

In the last line of algorithm 1 we use the notation $A|_{\mathcal{Y}} = \Gamma A \Gamma^\dagger$ where $\Gamma : H \to \mathcal{Y}$ is the (surjective) projection onto $\mathcal{Y} \subseteq H$ and $\Gamma^\dagger : \mathcal{Y} \to H$ the inclusion map. This line selects the states which are approximately preserved by $K$, i.e., the target space of the AGSP.

\begin{proposition}
Given an error parameter $0 < \delta_{\text{goal}} < 1$ suppose $R \Delta \leq \frac{\delta_{\text{goal}}/d}{0.4}$. Let $D$ be the degeneracy of the target space $Z$ of $K$. Then there exists a choice of parameters $V, \varepsilon, \delta$ such that with probability at least $1/2$ the output $\tilde{Z}$ of algorithm 1 satisfies $\tilde{Z} \approx_{\delta_{\text{goal}}} Z$ and such that the time complexity (and bond dimension of the output) is polynomial in $DdRw/\delta_{\text{goal}}$.
\end{proposition}

3.4 Application to 2D spin systems

Consider a system of $n = wh$ qudits, each with local dimension $d$. The spins are arranged in a $w \times h$ lattice with vertex set $\mathbb{Z}^2 \cap ([1, w] \times [1, h])$, and we consider a frustration-free local Hamiltonian $H = \sum_i H_i$ with local interactions $0 \leq H_i \leq 1$, each of which involves only qudits on the grid within a constant diameter.

A recent breakthrough of Anshu, Arad, and Gosset [AAG20] proved a subvolume law for the unique ground state of a frustration-free local Hamiltonian on a 2D lattice
in terms of the uniform (or local) gap $\gamma$ [GS17, AAG20], i.e., the smallest gap of a subsystem.

**Definition 3.5** ([GS17, AAG20]). Given a rectangle $B = ([a, b] \times [c, d]) \cap \mathbb{Z}^2$ in the 2D lattice let $H_B$ be the sum of interactions acting on spins in $B$. Let $\tilde{\gamma}(H_B) = \min(\text{spec } H_B \setminus \{0\})$ be the spectral gap of $H_B$. The uniform (or local) gap of $H$ is $\gamma := \min_B \tilde{\gamma}(H_B)$ where the minimum is over all rectangles. The uniform gap condition posits that $\gamma = \Omega(1)$.

The specific class of subsystems to be included in definition 3.5 depends on the application; rectangles suffice for [AAG20]. The use of the uniform gap assumption in [AAG19] is motivated by finite-size criteria [Kna88, GM16, Lem20, LSW19]. In the context of such criteria a Hamiltonian is gapped precisely because it is locally gapped. The sub-volume law represented significant progress in understanding the entanglement structure of local Hamiltonian systems with gap conditions in 2D, providing evidence in favor of the conjectured area law. The theorem states (slightly paraphrased):

**Theorem 3.6** ([AAG20]). Let $H$ be a frustration-free Hamiltonian on a $w \times h$ lattice of qudits (each with Hilbert space $\mathbb{C}^d$) and uniform gap $\gamma$. If the ground state $|\psi\rangle$ of $H$ is unique, then the entanglement entropy $S(\rho^\psi_{\text{left}})$ of $|\psi\rangle$ across a vertical cut (of height $h$) satisfies

$$S(\rho^\psi_{\text{left}}) = O\left((h/\sqrt{\gamma})^{5/3} \log^{7/3}(dh)\right).$$

Inspecting the proof by [AAG20] it is clear that the analysis of the shrinking and entanglement parameters of their AGSP do not depend on the degeneracy of the ground space. Applying proposition 3.3 to the AGSP of [AAG20] one obtains:

**Corollary 3.7.** Let $H$ be a 2D lattice Hamiltonian satisfying the conditions of theorem 3.6, except now allow the ground space $Z = \text{Ker } H$ to have arbitrary dimension $D = \dim(Z)$. Then,

$$\max_{|\psi\rangle \in S(Z)} S(\rho^\psi_{\text{left}}) = O\left((h/\sqrt{\gamma})^{5/3} \log^{7/3}(dh) + \log D\right),$$

where $S(\rho^\psi_{\text{left}})$ is the entanglement entropy of $|\psi\rangle$ across an arbitrary vertical cut.

In particular,

- The entanglement bound is the same as [AAG20] when the degeneracy has growth at most $D = 2^{O(h^{5/3})}$.

- In the parameter regime for $w, h, \gamma$ where [AAG20] yields a subvolume law (e.g., $w = h$ and $\gamma = \Omega(1)$), a subvolume law still holds for any sub-exponential degeneracy $D = 2^{o(wh)}$.

In appendix A we modify the AGSP of [AAG20] to arrive at an AGSP with an MPO encoding as in section 3.3. Applying algorithm 1 to this AGSP yields:

**Theorem 3.8.** Let $H$ be a frustration-free Hamiltonian with uniform gap $\gamma$ on the $w \times h$ lattice with $n = wh$ qudits. Suppose $w/h$ is at most polynomial in $n$. Let $D$ be a bound on the degeneracy and $\delta$ an accuracy parameter. Then there exists a randomized algorithm with time complexity $D/\delta^{O(1)} \exp\left[O\left(h^{5/3} \gamma^{-5/3} \log^{7/3}(dh)\right)\right]$ which outputs an MPS representing a subspace $\tilde{Z} \preceq \mathcal{H}$ such that $\tilde{Z} \approx_\delta Z$ with probability at least 1/2.
Without loss of generality we may rotate the lattice such that $w \geq h$ and therefore $h \leq \sqrt{n}$. The time complexity in theorem 3.8 is therefore sub-exponential in $n$, being bounded by

$$\left(\frac{D}{\delta}\right)^{O(1)} \exp \left[ O\left(\left(\frac{n}{\gamma}\right)^{\frac{5}{6}} \log^{\frac{3}{2}} \left(\frac{dn}{\gamma}\right)\right)\right].$$  (3)

The error probability in theorem 3.8 is easily reduced by repetition. Indeed, the combined output will be the span of all outputs $\tilde{Z}_i$ on which $H$ has energy 0.

It is important to ask whether the output of theorem 3.8 can be used to compute the expectation values of local observables. In fact we may modify the algorithm with a post-processing step which prepares a list of all such expectation values on the ground states.

**Corollary 3.9 (Post-processing).** Let $S \subset \{\sigma_i\}_{i=1}^{d^2} \otimes^n$ be the set of Pauli observables which act nontrivially on at most $k \leq \frac{n}{5/6}$ spins. The algorithm of theorem 3.8 can be modified to output a 3-dimensional table $T$ such that, for some basis $\{|z_i\rangle\}$ for $Z$, $|T_{ij}^\sigma - \langle z_i | \sigma | z_j \rangle| \leq \delta$ for each $\sigma \in S$ and $i,j = 1, \ldots, D$ with probability at least $1/2$. The time complexity of the modified algorithm is still (3).

**Proof.** The modified algorithm runs the algorithm of theorem 3.8 and then contracts the resulting MPS to compute each entry of $T$. Contracting the MPS is polynomial in the bond dimension and linear in $n$ [Vid03, PKS + 19]. Moreover, The number of entries of $T$ is $D^2 \binom{n}{k}$, so we can absorb the time complexity in (3). \qed

**The possibility of space-efficiency** It is an interesting open problem whether an algorithm can benefit from making more use of the 2-dimensional geometry. The most enticing perspective would be if this could be used to achieve polynomial space complexity. This question is related to the conjectured possibility of representing of ground states as tensor networks, known as PEPS for 2D lattices. The property of having such a representation is a strictly stronger property than that of satisfying an area law, which itself is a conjecture. More precisely, Ge and Eisert [GE16] showed that most states which satisfy area laws do not have efficient representations. Moreover, Huang showed that 2D (but not gapped) local Hamiltonians with area laws are QMA-complete [Hua20]. Nevertheless, it is widely held that the ground states of gapped lattice Hamiltonians have efficient PEPS representations.

**3.5 Lower bounds**

To state the strongest lower bound we should show hardness in as restrictive a special case as possible. We therefore consider the case when $H$ is a satisfiable classical 3SAT-formula and moreover the degeneracy is $D = 1$, i.e., the satisfying assignment is promised to be unique. Then the local gap is $\gamma = 1$, and the satisfying assignment can be found by computing the 1-local observables using corollary 3.9 to constant accuracy $\delta$.

**Lemma 3.10.** Let $A$ be the set of 3SAT instances on a 2D grid and let $uA \subset A$ be the set of such instances with exactly 1 satisfying assignment.
• Suppose there exists a polynomial-time algorithm which given an instance from $uA$ outputs the satisfying assignment with probability $1/2$. Then $NP$ equals $RP$ (randomized polynomial time).

• Suppose there exists a $\exp(n^{o(1)})$-time randomized algorithm which given an instance from $uA$ outputs the satisfying assignment with probability $1/2$. Then there exists a $\exp(n^{o(1)})$-time randomized algorithm for for $SAT$ with $n$ variables.

Proof. $SAT$ is parsimoniously reducible to $3SAT$ [Koz12] which itself is parsimoniously reducible to rectilinear planar $3SAT$ [Lic82, KR92, Dem14] (All reductions mentioned are polynomial-time). A rectilinear planar $3SAT$ instance is easily embedded in the 2D grid with 3-local constraints. So there exists a parsimonious reduction $g$ which takes $SAT$ instances to $A$ and unique $SAT$ instances to $uA$.

By the Valiant-Vazirani theorem [VV85] there exists a randomized reduction $f$ from $SAT$ to unique $SAT$. Since $g$ preserves uniqueness of solutions $g \circ f$ gives a randomized reduction from $SAT$ to the problem of computing the solution to an instance of $uA$. Since the size $n$ of the $uA$ instance is polynomial in the number of variables $n_0$ of the initial $SAT$ formula, $\exp(n^{o(1)}) = \exp(n_0^{o(1)})$.

It follows that the running time $(D/\delta)^{O(1)}2^{O(n/\gamma)}$ of theorem 3.8 and corollary 3.9 cannot be improved to polynomial in $n/\gamma$ unless $NP = RP$. And it cannot be improved to $\exp(n^{o(1)})$ assuming the randomized exponential-time hypothesis.

4 Open problems

The readymade entanglement bound, proposition 3.3, raised the question of whether an entanglement bound without the degeneracy term holds with a weaker, non-uniform notion of the entanglement of a space of vectors. For example we conjectured that $Z$ equals the span of $D$ vectors whose entanglement are bounded without the degeneracy term.

Writing the classical time complexity for computing the ground states of frustration-free 2D Hamiltonians as $\exp(O(n^{c}))$, we have shown upper and lower bounds to establish that $0 < c < 1$. We conjecture that the optimal $c$ is $c = 1/2$, which would likely follow from the conjectured area law. In the opposite direction it is possible that more careful reductions could imply the lower bound $c = 1/2$ under assumptions such as the exponential time hypothesis.

In a different direction it is interesting whether the sub-exponential space complexity of theorem 3.8 can be improved further, possibly even so far as to be polynomial. This is a question about the existence of efficient representations of ground states, which have so far been elusive beyond gapped spin chains, and which may present additional challenges beyond establishing area laws [GE16, Hua20].

5 Proof of error reduction bound

Given Hilbert space $\mathcal{H}$ and subspace $V \leq \mathcal{H}$, let $\Gamma_{V} : \mathcal{H} \rightarrow V$ denote the orthogonal projection onto $V$ when viewed as a surjective map $\mathcal{H} \rightarrow V$. Given another subspace $Z \leq \mathcal{H}$ we define the transition map $\Pi_{V \rightarrow Z}$ from $Z$ to $V$ as the restriction of $\Gamma_{V}$ to domain $Z$. Formally:

**Definition 5.1.** Given a subspace $V \leq \mathcal{H}$, let $\Gamma_{V} : \mathcal{H} \rightarrow V$ be the adjoint of the inclusion map $\Gamma_{V}^{!} : V \hookrightarrow \mathcal{H}$. The transition map from $Z$ to $V$ is $\Pi_{V \rightarrow Z} = \Gamma_{V} \Gamma_{Z}^{!}$. 


The overlap $\mu$ of $V$ onto $Z$ equals $\min \text{spec}(\Pi_{Z \leftarrow Y} \Pi_{Y \leftarrow Z})$, where $\text{spec}$ is the spectrum. The principal angles between $Z$ and $V$ are defined [GH06, BI67] as the arccos of the singular values of $\Pi_{Y \leftarrow Z}$. This definition illustrates a symmetry between two subspaces.

**Observation 5.2.** Let $V, Z \leq \mathcal{H}$ be two subspaces such that each covers the other. Then the overlap of $V$ onto $Z$ equals the overlap of $Z$ onto $V$.

**Proof.** Let $M = \Pi_{Z \leftarrow Y}$. Then $\text{spec}(MM^\dagger) \setminus \{0\} = \text{spec}(M^\dagger M) \setminus \{0\}$ (Jacobson’s lemma). The assumed nonzero overlaps then imply $\text{spec}(M^\dagger M) = \text{spec}(MM^\dagger)$ and in particular the overlaps agree $\mu = \min \text{spec}(MM^\dagger) = \min \text{spec}(M^\dagger M)$.

If $V_1 \preceq_\mu V_2$ and $V_1 \succeq_\mu V_2$ then we say that $V_1$ and $V_2$ are mutually $\mu$-overlapping and write $V_1 \|_\mu V_2$.

**Corollary 5.3** (Symmetry lemma). For $V_1, V_2 \leq \mathcal{H}$ and $\mu > 0$,

$$V_1 \preceq_\mu V_2 \quad \text{and} \quad V_1 \text{ covers } V_2 \iff V_1 \|_\mu V_2.$$

**Lemma 5.4.** For subspaces $Z, Y \leq \mathcal{H}$ and $\mu > 0$,

$$V \succeq_\mu Z \iff P_Y(Z) \|_\mu Z.$$

**Proof.** Let $Y = P_Y(Z)$. (\(\Leftarrow\)) is clear since $Y \leq V$. (\(\Rightarrow\)): Since $Y \leq V$, $P_Y(Y) = P_Y(Y) = Y$ and thus $Z$ covers $Y$. On the other hand $P_Z P_Y P_Z = P_Z P_Y P_Z$, so $V \succeq_\mu Z$ implies that $Y \succeq_\mu Z$. So $Y \|_\mu Z$ by the symmetry lemma.

### 5.1 Proof of error reduction bound

**Lemma** (Lemma 3.1 repeated). Let $K$ be a $\Delta$-AGSP for $Z \leq \mathcal{H}$, and suppose $V \succeq \mathcal{H}$ covers $Z$ with error ratio $0 < \varphi < \infty$. then $Y' = K(Y)$ covers $Z$ with error ratio $\varphi' \leq \Delta \cdot \varphi$.

**Proof.** Let $Y = P_Y(Z)$ be the projection of $Z$ onto the covering space $V$, and let $Y' = K Y$. By lemma 5.4 we have that $Y \|_\mu Z$ where $\mu = \frac{1}{1+\Delta \varphi}$. Since $Y' \succeq KV$ it suffices to show that $Y' \|_{\mu'} Z$ where $\mu' = \frac{1}{1+\Delta \varphi}$. We prove this using the symmetry lemma:

First, $Y'$ covers $Z$ because $P_Z(Y') = P_Z(KP_Y Z) = K(P_Z P_Y P_Z Z) = K P_Z = Z$. Here we have commuted $K$ past $P_Z$ and used the fact that $P_Z P_Y P_Z \preceq P_Z$ for $\mu > 0$ since $V$ covers $Z$.

We now compute the overlap of $Z$ onto $Y'$. Given $|y'\rangle \in Y'$ write $\langle y' | y \rangle = K |y\rangle$ for some $|y\rangle \in Y$. Since $Y \|_\mu Z$ we have $\langle y | P_Z | y \rangle \leq \varphi (|y\rangle P_Z |y\rangle)$. Apply the AGSP property $\| KP_Z \| \leq \sqrt{\Delta}$ and the dilation property on $Z$:

$$\| KP_Z |y\rangle \| \leq \sqrt{\Delta} \| P_Z |y\rangle \| \leq \sqrt{\Delta \varphi} \| P_Z |y\rangle \| \leq \sqrt{\Delta \varphi} \| KP_Z |y\rangle \|.$$  \hspace{1cm} (4)

Recognizing the LHS as $\| P_Z |y'\rangle \|$ and the RHS as $\sqrt{\Delta \varphi} \| P_Z |y'\rangle \|$ establishes that $\langle y' | P_Z | y' \rangle \leq \Delta \varphi (|y'\rangle |y'\rangle)$; thus, the error ratio of $Z$ onto $Y'$ is at most $\varphi' = \Delta \varphi$. Since we showed that $Y'$ covers $Z$, $\varphi'$ is a mutual error ratio by the symmetry lemma.
6 Algorithmic complexity reduction

All existing efficient algorithms for spin chains rely in an essential way on bond trimming of matrix product states. We define a trimming procedure which allows for a simple self-contained analysis in the degenerate case. This definition coincides with that of [ALVV17] in the bipartite (as opposed to multipartite) case:

**Definition 6.1** (Bipartite case). Given \( \mathcal{Y} \subseteq \mathcal{H}_{AB} \) and \( \varepsilon > 0 \) introduce the projection \( P_A = \Pi_{[\varepsilon, \infty)}(\rho_A^Y) \) where \( \rho_A^Y = \text{tr}_B(P_Y) \) is the reduced density matrix and \( \Pi \) denotes an indicator function. Then \( \text{trim}^A_\varepsilon(\mathcal{Y}) \) is the image \([P_A \otimes 1_B](\mathcal{Y})\).

The trimmed subspace is contained in \( \mathcal{V} \otimes \mathcal{H}_B \) where \( \mathcal{V} = P_A(\mathcal{H}_A) \), and Markov’s inequality gives the bound

\[
\dim(\mathcal{V}) = \text{rank } P_A \leq \text{tr}(\rho_A^Y)/\varepsilon = \dim(\mathcal{Y})/\varepsilon,
\]

To define the trimming of \( \mathcal{Y} \subseteq \mathcal{H} \) in a multipartite space \( \mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_w \) we simply iterate the bipartite version:

**Definition 6.2.** Given a subspace \( \mathcal{Y} \subseteq \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_j \) define

\[
\text{trim}(\mathcal{Y}) = \text{trim}^1 \circ \text{trim}^{1,2} \circ \cdots \circ \text{trim}^{[1,j-1]}(\mathcal{Y}).
\]

6.1 Analysis of simple trimming procedure

Since our trimming procedure is just an iteration of the bipartite case its analysis reduces to analyzing the bipartite trimming. We consider a tripartite \( \mathcal{H}_{ABC} \) because the subspace \( \mathcal{Y} \subseteq \mathcal{H}_{AB} \) to be subjected to bipartite trimming is itself viable for a target space \( \mathcal{Z} \) on an extended space \( \mathcal{H}_{(AB)C} \).

**Lemma 6.3.** Let \( \mathcal{Z} \subseteq \mathcal{H}_{ABC} \) and let \( \mathcal{Y} \subseteq \mathcal{H}_{AB} \) be \( \delta \)-viable for \( \mathcal{Z} \). If there exists \( \mathcal{V} \subseteq \mathcal{H}_A \) with \( \dim(\mathcal{V}) = V \) which is \( \alpha \)-viable for \( \mathcal{Z} \), then \( \mathcal{Y}_\varepsilon = \text{trim}^A_\varepsilon(\mathcal{Y}) \) is \( \delta' \)-viable for \( \mathcal{Z} \) where \( \delta' = \delta + \sqrt{\varepsilon V} + \sqrt{\alpha} \).

**Proof.** Introduce projectors \( P_+ = \Pi_{[\varepsilon, \infty)}(\rho_A^Y) \) and \( P_- = \Pi_{[0,\varepsilon)}(\rho_A^Y) \) on \( \mathcal{H}_A \). Denote extensions of operators and subspaces as \( \bar{P} = P \otimes 1_{BC} \) and \( \bar{Y} = Y \otimes \mathcal{H}_C \).

Given any \( |z\rangle \in \mathcal{S}(\mathcal{Z}) \) pick \( |y\rangle \in \mathcal{S}(\bar{Y}) \) satisfying \( \langle z|y\rangle \geq 1 - \delta \). Let \( |y'\rangle = \bar{P}_+|y\rangle \) so that \( |y'\rangle \in \mathcal{Y}_\varepsilon \) and \( |||y'||\| \leq 1 \). Then,

\[
\langle z|y'\rangle = \langle z|\bar{P}_-|y\rangle = \langle z|\bar{P}_-\bar{P}_+|y\rangle + \langle z|\bar{P}_-\bar{P}_-|y\rangle.
\]

Bound the first term on the RHS by

\[
||\bar{P}_-\bar{P}_-|y\rangle|| = \sqrt{\text{tr}(P_-P_-\text{tr}_{BC}(|y\rangle\langle y|))} \leq \sqrt{\text{tr}(P_-P_-\rho_A^Y)} \leq \sqrt{2V}.
\]

since \( ||P_-\rho_A^Y|| \leq \varepsilon \) and \( \text{rank } P_Y = V \). Bound the second term on the RHS of (5) by \( ||\bar{P}_-|z\rangle|| \leq \sqrt{\alpha} \). By (5), \( \langle z|y'\rangle \geq \langle z|y\rangle - \sqrt{\varepsilon V} - \sqrt{\alpha} \).

**Corollary 6.4.** Suppose \( \mathcal{Z} \subseteq \mathcal{H}_{1 \ldots j \ldots w} \) is such that for each \( i \) there exists a \( \alpha \)-viable space \( \mathcal{V}_{[1,i]} \subseteq \mathcal{H}_{[1,i]} \) for \( \mathcal{Z} \) with \( \dim(\mathcal{V}_{[1,i]}) \leq V \). If \( \mathcal{Y} \subseteq \mathcal{H}_{[i,j]} \) is \( \delta \)-viable for \( \mathcal{Z} \) then \( \text{trim}_\varepsilon(\mathcal{Y}) \) is \( \delta' \)-viable for \( \mathcal{Z} \) where \( \delta' = \delta + \sqrt{\varepsilon V} + \sqrt{\alpha} \).
6.2 Dimension reduction by sampling [ALVV17]

Having analyzed the AGSP which achieves the improvement of the overlap we now recall a standard tool for entanglement reduction.

**Lemma 6.5** ([ALVV17] lemma 5). Let $\mathcal{Z} \subseteq \mathcal{H}_L \otimes \mathcal{H}_R$ be a subspace with dimension $D$ and let $\mathcal{W} \subseteq \mathcal{H}_L$ be left-\(\mu\)-overlapping onto $\mathcal{Z}$ with $\dim(\mathcal{W}) = W$. Then a Haar-uniformly random subspace $\mathcal{V} \subseteq \mathcal{W}$ of dimension $\nu \leq W$ is left-\(\nu\)-overlapping onto $\mathcal{Z}$ with probability at least $1 - \eta$ where

\[ \nu = \frac{V}{8W} \cdot \mu \quad \text{and} \quad \eta = (1 + 2\nu^{-1/2})^D W e^{-V/16}. \]

Since $1 + 2x \leq 3x$ for $x > 1$ (and in particular for $x = \nu^{-1/2} \geq \sqrt{8}$) we have the bound on the error probability:

\[ \eta < (9/\nu)^{D/2} W e^{-V/16}. \] (6)

Applying the probabilistic method we obtain:

**Corollary 6.6.** Let $\mathcal{W} \subseteq \mathcal{H}_L$ of dimension $W$ be left-\(\mu\)-overlapping onto $\mathcal{Z} \subseteq \mathcal{H}_L \otimes \mathcal{H}_R$ with $\dim(\mathcal{Z}) = D$. For any $0 < \nu \leq \mu$ there exists a subspace $\mathcal{V} \subseteq \mathcal{W}$ which is left-\(\nu\)-overlapping onto $\mathcal{Z}$ and has dimension

\[ V = \left[8\left(W \cdot \frac{\nu}{\mu} \lor (D \log(9/\nu) + 2 \log W)\right)\right] \land W. \] (7)

**Proof.** If $V = W$ then $\mathcal{V} = \mathcal{W}$ suffices. Otherwise let $\tilde{\nu} = \frac{V}{8W} \mu$ be the overlap from lemma 6.5 corresponding to the choice (7) of $V$ and let $\tilde{\eta} = (9/\nu)^{D/2} W e^{-V/16}$. Then $\log \tilde{\eta} = \frac{9}{2} \log(9/\nu) + \log W - V/16 \leq 0$ by the choice of $V$. By (6) the error probability in lemma 6.5 is strictly below $\tilde{\eta} \leq 1$ so by the probabilistic method there exists a left $\tilde{\nu}$-overlapping space. But $\tilde{\nu} \geq \nu$ which proves the claim.

7 Proof of readymade entanglement bound

The bootstrapping argument [ALV12, AKLV13, ALVV17] proves the existence of a subspace $\mathcal{V} \subseteq \mathcal{H}_L$ with small dimension and non-negligible overlap with the target space $\mathcal{Z} \subseteq \mathcal{H}_L \otimes \mathcal{H}_R$. The argument combines a method for reducing the entanglement of a subspace with one for increasing overlap with the target space (i.e., an AGSP) in such a way that $\dim(\mathcal{V})$ does not increase when concatenating the operations.

To offset the dimension growth from the AGSP, the entanglement reduction needs to decrease the entanglement by an factor $R$, which means decreasing the overlap by a factor $\Theta(R)$ using the dimension reduction procedure of [ALVV17] (appendix 6.2). One therefore has to apply the $(\Delta, R)$-AGSP in the low-overlap regime $\mu = c/R$. If we used the error bound $\delta' = \Delta/\mu^2$ of [ALVV17] then we would need $\Delta < \mu^2 = (c/R)^2$ to have any bound on the post-AGSP error, hence requiring a bound of the form $R^2 \Delta < c^2$ on the parameter tradeoff for the AGSP. In contrast, lemma 3.1 weakens this requirement to $\Delta = \mu = cR$. More precisely we will use:

**Corollary 7.1.** Let $K$ be a $(\Delta, R)$-AGSP with target space $\mathcal{Z} \subseteq \mathcal{H}$, and suppose $\mathcal{V} \subseteq \mathcal{H}$ $\mu$-overlaps onto $\mathcal{Z}$ with $\mu \geq \Delta$. Then $\mathcal{V}' = K(\mathcal{V})$ has overlap $\mu' = 1/2$ onto $\mathcal{Z}$.

**Proof.** $\mathcal{V}$ has error ratio $\varphi = \frac{1}{\mu} \leq \frac{1}{\mu}$, so $\mathcal{V}'$ has error ratio $\varphi' \leq \Delta/\mu \leq 1$ by lemma 3.1. This corresponds to overlap $\mu' = \frac{1}{\varphi' + 1} \geq 1/2$. □
The following lemma is proven following the overall argument of [ALVV17] proposition 2 and combining it with the sharp error reduction bound in the form of corollary 7.1 to change the condition from a bound on $R \Delta$ to one on $R \Delta$. In the following $x \lesssim y$ means $x = O(y \lor 1)$ where $\lor$ denotes the maximum.

**Lemma 7.2.** Let $Z \preceq \mathcal{H}_L \otimes \mathcal{H}_R$ be a subspace with degeneracy $\text{dim}(Z) = D$. If there exists a $(\Delta, R)$-AGSP $K \in \mathcal{B}(\mathcal{H}_L) \otimes \mathcal{B}(\mathcal{H}_R)$ with target space $Z$ and parameters such that

$$\Delta \cdot R \lesssim 1/32,$$

then there exists a left $\frac{1}{32R}$-overlapping space $\mathcal{V} \preceq \mathcal{H}_L$ onto $Z$ such that $\text{dim}(\mathcal{V}) \lesssim D \log R$. It follows that there exists $\mathcal{V}''$ of dimension $\text{dim}(\mathcal{V}'') \lesssim DR^2 \log R$ which is left $\Delta$-viable for $Z$.

**Proof.** Let $\mathcal{V}$ be a left $\nu = \frac{1}{32R}$-overlapping space onto $Z$ whose dimension $\nu$ is minimal with respect to this property. Let $K \preceq \mathcal{B}(\mathcal{H}_L)$ be the $\Delta$-AGSP of dimension $R$ associated to the $(\Delta, R)$-AGSP $K$, and let $\mathcal{V}' = K \mathcal{V}$ so that $\nu' = \text{dim}(\mathcal{V}') \leq RV$. $\Delta \leq \nu$ by assumption (9), so corollary 7.1 yields that $\mathcal{V}'$ is $1/2$-overlapping onto $Z$.

By corollary 6.6 there exists $\mathcal{V}'' \preceq \mathcal{V}'$ which is left $\nu = \frac{1}{32R}$-overlapping onto $Z$ and has dimension at most $V/2 + O(D \log R \lor \log V)$ since $8\nu' = \frac{1}{(32R)^2} \leq V/2$. By minimality of $\mathcal{V}$ we have that $V \leq V/2 + O(D \log R \lor \log V)$, and rearranging yields the result about $\mathcal{V}$.

The last remark follows by taking $\mathcal{V}'' = K^2 \mathcal{V} = K \mathcal{V}'$. Then $\mathcal{V}''$ covers $Z$ with error ratio $\varphi'' \leq \Delta$ by lemma 3.1 since $\mathcal{V}'$ has $\varphi' = 1$, and this upper-bounds the viability error. $\square$

**Corollary 7.3.** Let $Z \preceq \mathcal{H}_L \otimes \mathcal{H}_R$ be a subspace with degeneracy $\text{dim}(Z) = D$. If there exists a $(\Delta, R)$-AGSP $K \in \mathcal{B}(\mathcal{H}_L) \otimes \mathcal{B}(\mathcal{H}_R)$ with target space $Z$ and parameters such that

$$\Delta \cdot R \lesssim 1/2,$$

then for any $\alpha > 0$ there exists a $\alpha$-viable $\mathcal{V} \preceq \mathcal{H}_{[1,w]}$ with $\text{dim}(\mathcal{V}) \lesssim \alpha^{-1} DR^{O(1)}$.

**Proof.** Applying lemma 7.2 to $K^5$ there exists a $\mu = \frac{1}{32DR}$-overlapping subspace $\mathcal{V}_0$ with dimension $O(D \log R)$. Let $\mathcal{V} = K^p \mathcal{V}_0$ where $p = \left\lfloor \log_\Delta (\alpha \mu) \right\rfloor$. By lemma 3.1 the viability error of $K^p \mathcal{V}_0$ is at most $\Delta^p / \mu \lesssim \alpha$. We bound the dimension using $p < 1 + \log_R (\frac{1}{\alpha \mu}) = 6 + \log_R (32/\alpha)$ which implies $\text{dim}(\mathcal{V}) \lesssim \frac{32}{\alpha} R^6 D \log R$. $\square$

### 7.1 Subspace overlap $\rightarrow$ entanglement of vectors

The following lemma relates the entanglement of individual ground states to $\delta$-viability.

**Lemma 7.4.** Let $Z \preceq \mathcal{H}_L \otimes \mathcal{H}_R$ and suppose there exists a $\delta$-viable space $\mathcal{V} \subset \mathcal{H}_L$ of dimension $V$ for $Z$. Pick any state $|\psi\rangle \in \mathcal{S}(Z)$ and write the Schmidt decomposition $\sum_i \sqrt{\lambda_i} |x_i\rangle |y_i\rangle \in \mathcal{S}(Z)$ with non-increasing coefficients. Then we have the tail bound

$$\sum_{i=V+1}^{\text{dim}(\mathcal{H}_L)} \lambda_i \leq \sqrt{\delta}.$$

**Proof.** Let $|\phi\rangle \in \mathcal{S}(Z)$ such that $\langle \psi | \phi \rangle^2 \geq 1 - \delta$, and let $\rho_\psi$ and $\rho_\phi$ be the reduced density matrices on $\mathcal{H}_L$ so that $\lambda_i = \lambda_i^\psi$ are the eigenvalues of $\rho_\psi$. Then, since the trace distance contracts under the partial trace:

$$\frac{1}{2} \|\rho_\psi - \rho_\phi\|_1 \leq \frac{1}{2} \| \langle \psi | - | \phi \rangle \|_1 = \sqrt{1 - \langle \psi | \phi \rangle^2} \leq \sqrt{\delta},$$

thus
Let $d\rho = \rho_\psi - \rho_\delta$ and call its non-increasing eigenvalues (not all positive) $\lambda^{d\rho}_i$ and let $\lambda^{\psi}_i$ be the non-increasing eigenvalues of $\phi$. For $V + 1 \leq i \leq \dim(\mathcal{H}_L)$, Weyl’s inequalities imply $\lambda^{\psi}_i \leq \lambda^{d\rho}_{i+1} + \lambda^{d\rho}_{i-V}$. Thus $\sum_{i>V} \lambda_i \leq \sum_{i} (\lambda^{d\rho}_i)_+ = \frac{1}{2}\|d\rho\|_1 \leq \sqrt{\delta}$ where $(x)_+ = x \vee 0$ is the positive part and the middle equality is because $\text{tr}(d\rho) = 0$. \hfill $\square$

### 7.2 Proof of proposition 3.3

In the case of a frustrated Hamiltonian the AGSP contraction involves a spectral truncation of the Hamiltonian on either side of a cut, incurring an error in the target space of the AGSP. We first prove a version of proposition 3.3 which is applicable to the frustrated case by allowing the target space to be approximate. We then specialize to the case of an exact target space to obtain proposition 3.3.

**Lemma 7.5.** Let $Z$ with degeneracy $\dim(Z) = D$ be a subspace of bipartite space $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$. Let $\mathcal{Z}_1, \mathcal{Z}_2, \ldots \subseteq \mathcal{H}$ be a sequence of subspaces such that $\mathcal{Z}_n \simeq_{\delta_n} Z$ where $\delta_1, \delta_2, \ldots$ is a sequence such that $\sum_{n=0}^{\infty} n\sqrt{\delta_n} = O(1)$.

Let $R\Delta \leq 1/2$ and suppose there exists a sequence $K_1, K_2, \ldots$ such that $K_n$ is an $(\Delta^n, R^n)$-AGSP for target space $\mathcal{Z}_n$. Then,

$$
\max_{|\psi\rangle \in S(Z)} S(\rho^{\psi}_{\mathcal{H}_L}) \leq (1.01 + c_\delta) \log D + O(\log R) \quad \text{where} \quad c_\delta = \sum_{n=1}^{\infty} \sqrt{\delta_n}.
$$

**Proof.** For any $m = 5, 6 \ldots$ we show that $S(\rho^{\psi}_{\mathcal{H}_L})$ is bounded by

$$(1 + \epsilon_m + c_\delta) \log D + O(m \log R), \quad \text{where} \quad \epsilon_m = \frac{\Delta^m/2}{1 - \Delta^{1/2}}.
$$

(10) then follows by taking $m = 17$ since that and $\Delta \leq 1/2$ yield $\epsilon_m \leq 0.01$.

Applying lemma 7.2 to $K_n$ yields a left $\Delta^n$-viable space for $\mathcal{Z}$ for each $n \geq m$ since $R^n \Delta^n \leq 1/32$. The lemma implies that $\dim(V_n) \lesssim DR^{2n} \log(R^n)$ and hence $\dim(V_n) \leq CDR^{3n}$ for a constant $C > 0$. $V_n$ is $(\Delta^{n/2} + \delta^{n/2})^2$-viable for $Z$ by the proof of [ALV17] lemma 3. By lemma 7.4 the Schmidt coefficients of any state $|\psi\rangle \in S(Z)$ satisfy $\sum_{i > CDR^{3n}} \lambda_i \leq \delta^{2/3} + \sqrt{\delta_n}$ for each $n \geq 5$.

Let $I_0 = \{1, 2, \ldots, CD \cdot R^{3n}\}$, $I_1, \ldots, I_{m-1} = \emptyset$, and $I_n = N \cap (CD \cdot R^{3n}, CD \cdot R^{3(n+1)})$ for $n \geq m$. By the standard decomposition [ALV12] of the Shannon entropy described in lemma B.1 (appendix B.1),

$$
S(\Lambda_i) \leq \log(CDR^{3n}) + \sum_{n=m}^{\infty} (\Delta^{n/2} + \sqrt{\delta_n}) \log(CDR^{3n+3}) + \sum_{n=m}^{\infty} h(\Delta^{n/2} + \sqrt{\delta_n})
$$

$$
= (1 + \epsilon_m) \log D + O(m \log R) + \sum_{n=m}^{\infty} h(\Delta^{n/2} + \sqrt{\delta_n}).
$$

We finalize by bounding the rightmost sum. Since $h$ is increasing on $[0, 1/e]$, we can bound $h(\Delta^{n/2} + \sqrt{\delta_n})$ by $h(2^{-\frac{n}{2}} + \delta^{\frac{n}{2}})$. This in turn is bounded by

$$
h(2^{-\frac{n}{2}} + \delta^{\frac{n}{2}}) \leq (2^{-\frac{n}{2}} + \sqrt{\delta_n}) \log(2^{\frac{n}{2}}) \leq h(2^{-\frac{n}{2}}) + n\sqrt{\delta_n}.
$$

So $\sum_n h(\Delta^{n/2} + \sqrt{\delta_n}) = O(1)$. This establishes (11). \hfill $\square$

The coefficient 1.01 in lemma 7.5 and proposition 3.3 can be replaced by $1 + \epsilon$ for any fixed $\epsilon > 0$ by taking $m \propto \log(1/\epsilon)$. The implicit constant of $O(\log R)$ then depends logarithmically on $1/\epsilon$. 

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Proof of proposition 3.3. Given AGSP $K$ with $R\Delta \leq 1/2$ apply lemma 7.5 to the sequence of AGSPs $K_n = K^n$, each with the exact target space $\tilde{Z}_n = Z$ such that we can take $\delta_n = 0$. 

8 Analysis of algorithm 1

Corollary 8.1. Suppose $R\Delta \leq 1/2$. Given $\delta$ there exists a choice $\varepsilon = \frac{1}{D}(\frac{\delta}{16Rw})^O(1)$ such that $V$ increases the viability error by at most $\delta$.

Proof. By corollary 6.4 it suffices to verify the existence of $\alpha$-viable subspaces of dimension $V$ such that $w(\sqrt{V} + \sqrt{\alpha}) \leq \delta$. Let $\alpha = \left(\frac{\delta}{16Rw}\right)^2$. By corollary 7.3 we can take $V \lesssim (w/\delta)^2 DR^O(1)$. Then pick $\varepsilon = \frac{1}{D}(\frac{\delta}{16Rw})^2$. 

Lemma 8.2. Given $0 < \delta \leq 1/2$ suppose $R\Delta \leq \frac{\delta}{32R}$. Then there exists a choice $V = \Theta(D \log(D\tilde{d}))$ and $\varepsilon = \frac{1}{D}(\frac{\delta}{16Rw})^O(1)$ such that with probability at least $1/2$ each $\mathcal{Y}_{[1,i]}$ is $\delta$-viable for $Z$ in algorithm 1.

Proof. At the beginning of the $i$th iteration, $\dim(\mathcal{Y}_{[1,i-1]}) \leq \dim(K_{[1,i-1]}\mathcal{Y}_{[1,i-1]}) \leq RV$. $\mathcal{Y}_{[1,i-1]} \otimes H_i$ then has dimension at most $\tilde{Y} = dRV$.

Let $\nu = \frac{1}{16d\tilde{d}}$. The error probability of lemma 6.5 is bounded by $\eta = (9/\nu)^{D/2} \tilde{Y} e^{-V/16}$ (6). Pick $V$ such that

$$V - 16 \log V \geq 8D \log(144R\tilde{d}) + 16 \log(2dRw).$$

Then $\eta \leq (9/\nu)^{D/2} \tilde{Y} e^{-\frac{V}{2} \log(9 \cdot 16R\tilde{d}) - \log(2dRVw)} = \frac{1}{2\nu}$. By a union bound lemma 6.5 succeeds at each iteration with probability at least $1/2$. We perform an induction within this event.

Induction step. By the induction hypothesis $\mathcal{Y}_{[1,i-1]}$ is $1/2$-viable for $Z$. As lemma 6.5 succeeds, $\mathcal{Y}$ has left overlap $\nu = \frac{1}{16d\tilde{d}}$ onto $Z$. By lemma 3.1 $K_{[1,i]}\mathcal{Y}$ is $\delta/2$-viable for $Z$ since $\Delta/\nu = 16dR\Delta \leq \delta/2$. By corollary 8.1 the trimming increases the error only by $\delta/2$, so $\mathcal{Y}_{[1,i]}$ is $\delta$-viable for $Z$. 

Having shown that $\mathcal{Y}_{[1,w]}$ is $\delta$-viable for $Z$ it remains to analyze the restriction on the last line of algorithm 1.

Lemma 8.3. If $\mathcal{Y} = \mathcal{Y}_{[1,w]}$ is $\delta$-viable for $Z$ then the output of algorithm 1 is $2\delta$-close to $Z$.

Proof. We show more precisely that $\tilde{Z}$ is $\delta/\tilde{\gamma}$-close to $Z$ where $\tilde{\gamma} = 1 - \Delta \geq 1/2$. By the symmetry lemma it suffices to show that

1. $Z$ is $\delta/\tilde{\gamma}$-viable for $\tilde{Z}$ and $\dim(\tilde{Z}) \geq \dim(Z)$.

1. By definition $\tilde{Z} \leq \mathcal{Y}$ is such that $H_{[\tilde{Z},\mathcal{Y}]_{\tilde{Z}}} \leq \delta$. Since $K$ is a $\Delta$-AGSP we can write $\tilde{H} = I - K^T K = 0_{\mathcal{Y}} \oplus \tilde{H}_{\mathcal{Y}}$ where $\tilde{\gamma} \leq \tilde{H}_{\mathcal{Y}}$. So $\tilde{\gamma} P_{Z} \tilde{P}_{Z} \perp P_{Z} \leq P_{Z} \tilde{H} P_{Z} \leq \delta P_{Z}$, which implies $Z$ is $\delta/\tilde{\gamma}$-viable for $\tilde{Z}$.

2. Since $\mathcal{Y}$ is $\delta$-viable for $Z$, lemma 5.4 implies that $Z' := P_{Z'} \mathcal{Y} \approx_{\lambda} Z$. Therefore $P_{Z'} \tilde{H} P_{Z} \perp P_{Z'} \tilde{P} \perp P_{Z'} \perp P_{Z} \tilde{P} \perp P_{Z} \leq \delta$. So $Z'$ is a subspace of $\mathcal{Y}$ where $\tilde{H}$ has energy at most $\delta$ which implies $\dim(\tilde{Z}) \leq \dim(Z)$. Item 2 follows since $\dim(Z) = \dim(Z')$. 

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Proof of proposition 3.4. By lemmas 8.2 and 8.3 we can take $V = \Theta(D \log(Rd) + \log w)$, $\varepsilon = \frac{1}{4} \left(\frac{1}{\Delta h}\right)^{O(1)}$, and $\delta = \delta_{\text{coal}}/2$.

Since $\dim(K_{[1,i]}^\perp \mathcal{V}_{[1,i]}) \leq RV/\varepsilon$ the bond dimension of the trimmed space $\mathcal{V}_{[1,i]}$ is bounded by $RV/\varepsilon$ in each iteration. This bounds the bond dimension of $\mathcal{V}_{[1,i]}$ at the beginning of each iteration by $\tilde{d} RV/\varepsilon$, and the same bound holds for the bond dimension of $\mathcal{V}_{[1,i]}$. So the largest bond dimension encountered throughout the algorithm, that of $K_{[1,i]} \mathcal{V}_{[1,i]}$ before trimming, is bounded by $\tilde{d} R^2 V/\varepsilon = (DRw/\delta)^{O(1)} \tilde{d} \log \tilde{d}$.

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\section*{Appendix A Constructing an implementable AGSP}

The sub-exponentially implementable AGSP will be a straightforward modification of the AGSP $K(m,t,k)$ defined by Anshu, Arad, and Gosset to prove the subvolume law [AAG20]. We begin by recalling this AGSP, which we refer to as the subvolume law-AGSP.

\subsection*{A.1 The subvolume law-AGSP of [AAG20]}

Let $t$ and $m$ be integer parameters. Define the\textit{ narrow bands} $B_i = (3it - 2t, 3it + 2t] \times \{1, \tilde{h}\} \cap \mathbb{N}^2$ for $i = 0,1,\ldots,\left\lfloor \frac{w}{3t}\right\rfloor + O(1)$. These are vertical bands of width $4t$ (except $B_0$) such that two neighboring bands have an overlap of width $t$.

Let $Q_i$ be the ground space projection for $H_{B_i}$. The AGSP of [AAG20] is based on the $t$-coarse grained detectability lemma operator [AALV09], $DL(t) = Q_{\text{odd}} Q_{\text{even}}$ where $Q_{\text{odd}} = \prod_i Q_i$ and $Q_{\text{even}} = \prod_i Q_i$. The AGSP construction replaces some factors $Q_i$ in $DL(t)$ with a polynomial in subsystem Hamiltonians $H_{B_i}$ to control the entanglement rank.

\textbf{Inner polynomial approximation.} Based on the Chebyshev polynomials, [AAG20] constructs step polynomials $\text{Step}(\cdot)$ of degree $\Theta(\sqrt{\Delta h}/\gamma)$ such that $\text{Step}(0) = 1$ and $\text{Step}(\frac{1}{C\gamma}, 1) \subset [-\frac{1}{20}, \frac{1}{20}]$ where $C/4 = O(1)$ is a bound on the number of interactions involving a single qudit (so $\| H_{B_i} \| \leq Cth$). Then $\tilde{Q}_i = \text{Step}(\frac{1}{C\gamma} H_{B_i})$ is an approximation to $Q_i$. More precisely, considering an eigenbasis for $H_{B_i}$, it is clear that $\tilde{Q}_i Q_i = Q_i$ and $\|Q_i Q_i^{\perp}\| = \| \tilde{Q}_i - Q_i \| \leq 1/20$ where $Q_i^{\perp} = I - Q_i$.

\textbf{Outer polynomial approximation.} [AAG20] cleverly combine the (approximate) projections $Q_i$ on narrow bands using the\textit{ robust AND polynomial} $p_{\text{AND}}$ [She12], an $m$-variate polynomial of degree $O(m)$ with the property that $p_{\text{AND}}(\vec{1}) = 1$ where $\vec{1}$ is the tuple of $m$ ones, and $|p_{\text{AND}}(\vec{x})| \leq e^{-m}$ for all $\vec{x} \in ([-\frac{1}{20}, \frac{1}{20}] \cup \{1\})^m$ such that $\vec{x} \neq \vec{1}$.

Given a set $\Xi$ of indices $x_1 < \ldots < x_m$ such that $B_{x_1}, \ldots, B_{x_m}$ are disjoint, let $\tilde{P}(\Xi) = p_{\text{AND}}(\tilde{Q}_{x_1}, \ldots, \tilde{Q}_{x_m})$.

\textbf{Definition A.1 ([AAG20]).} Given some integer $c$ representing the\textit{ vertical cut a horizontal position} $3ct$, let $\Xi = (c - m, c + m] \cap \mathbb{N}_{\text{odd}}$ be a set of $m$ odd indices around...
c and let \( Q_{\text{rest}} = \prod_{i \in \Xi} Q_i \) (with even-index \( Q_i \) on the right). Then the subvolume law-AGSP of [AAG20] is \( K(m, t, k) := (\tilde{P}(\Xi)Q_{\text{rest}})^k \).

\[
\begin{array}{ccccccc}
Q_{\text{rest}} & \cdots & \cdots & \cdots & \cdots & \cdots & \tilde{P}(\Xi)
\end{array}
\]

Figure 2: The operator \( K(m = 5, t, k = 1) \) of [AAG20]. Short line segments represent coarse-grained projectors \( Q_i \) on narrow (width 4\( t \)) bands \( B_i \). Wavy line segments indicate inner approximations \( \hat{Q}_i \).

A.2 The implementable AGSP \( \tilde{K} \)

We now modify the AGSP \( K(m, t, k) \) such that we can simultaneously control the entanglement across every vertical cut. We denote the resulting AGSP as \( \tilde{K}(m, t, k) \) or simply \( \tilde{K} \), suppressing the dependence on the parameters. Define the wide bands \( B_j = (6(j-1)m, t, 6jm + [1, h] \cap \mathbb{Z}^2 \) for \( j = 1, 2, \ldots, w \sim \frac{w}{6mt} \). These are disjoint vertical bands of width 6\( mt \).

Definition A.2. Let \( \Xi_j = (2(j-1)m, 2jm, \mathbb{N}_{\text{odd}}) \) be the set of odd indices \( i \) such that the narrow band \( B_i \) is contained in \( B_j \). Define the implementable AGSP as

\[
\tilde{K}(m, t, k) = (\tilde{P}Q_{\text{even}})^k \text{ where } \tilde{P} = \bigotimes_{i=1}^{w} \tilde{P}(\Xi_j).
\]

\[
\begin{array}{cccccc}
\tilde{P}(\Xi_0) & \cdots & \cdots & \cdots & \cdots & \tilde{P}(\Xi_4)
\end{array}
\]

\[
\begin{array}{cccccc}
\tilde{Q}_{\text{even}} & B_0 & \cdots & \cdots & \cdots & B_4
\end{array}
\]

Figure 3: The modified operator \( \tilde{K}(m, t, 1) \).

A.3 Properties of \( \tilde{K} \) adapted from [AAG20]

The entanglement bound [AAG20] theorem 5.1 of the subvolume law-AGSP holds across every cut of the implementable AGSP.

Lemma A.3 (By proof of [AAG20] theorem 5.1). Let \( m, t, k \) be at most polynomial in \( h/\gamma \). Then the Schmidt rank of \( \tilde{K} \) across any vertical cut is at most

\[
R = \frac{(hd/\gamma)^{O(mth+k\gamma^{-1/2}\sqrt{h/t})}}{2^k}.
\]

[AAG20] theorem 4.1 bounded the shrinking factor of the subvolume law-AGSP by \( (e^{-m} + 2e^{-\Omega(t\sqrt{\gamma})})^{2k} \). By a similar argument one has:

Lemma A.4. \( \tilde{K} \) is an AGSP with shrinking factor \( \Delta = (w'e^{-m} + 2e^{-\Omega(t\sqrt{\gamma})})^{2k} \), where \( w' \leq w \) is the number of wide bands.

Corollary A.5. Suppose \( h = n^{O(1)} \). For any \( \delta > 0 \) there exists a choice of parameters \( m, t, k \) such that \( \tilde{K}(m, t, k) \) is a \( \Delta \)-AGSP with entanglement rank most \( R \) across each vertical cut, \( R\Delta \leq \delta \), and \( R = \delta^{-1} \exp \left[ O \left( h^2 \gamma^{-2/3} \log^{2/3} \left( \frac{dh}{\gamma} \right) \right) \right] \).

The proofs of lemmas A.3 and A.4 and corollary A.5, adapted from [AAG20], are given in the next subsection A.4.
A.4 Proofs of properties of \(\tilde{K}\) adapted from [AAG20]

Proof of lemma A.3 (entanglement rank). It suffices to show the bound on the entanglement rank across a cut through the middle of a wide band \(B_j\). Indeed, any cut differs from some such cut by at most \(O(mth)\) sites which can contribute only \(d^{O(mth)}\) to the entanglement rank.

Adapt the proof of [AAG20] lemma 5.6 to the modified AGSP \(\tilde{K}\) by replacing \(Q_{\text{rest}}\) with \(Q_{\text{rest}} := (\otimes_{j' \neq j} \tilde{P}(\Xi_j))Q_{\text{even}}\). Combining lemma 5.6 with corollary 5.5 (with \(N = \frac{frk}{2mt}\) in lemma 5.6) yields the entanglement rank bound

\[
R = d^{O(N+mth)}(\frac{h}{\gamma})^{O(mth+N+k)},
\]

where the factor \((\frac{h}{\gamma})^{O(mth)}\) is from the prefactor of corollary 5.5. \(f = \Theta(\sqrt{th/\gamma})\) is the degree of Step and \(r = \Theta(m)\) the degree of \(p_{\text{AND}}\), so \(N = \frac{frk}{2mt} \propto \frac{frk}{t} \propto k\sqrt{\frac{h}{\gamma}}\).

Proof of lemma A.4 (shrinking factor). By [AAG20] lemma 3.1⁵, \(\|DL(t)P_{Z^+}\| = 2e^{-\Omega(t\sqrt{\gamma})}\). Moreover \(\|K(m,t,1) - DL(t)\| = \|(P - Q_{\text{odd}})Q_{\text{even}}\| \leq \|P - Q_{\text{odd}}\|\). Let \(\tilde{P}_j = \tilde{P}(\Xi_j)\) and \(P_j = \prod_{i \in \Xi_j} Q_i\) for \(j = 1, \ldots, w'\) and write

\[
\tilde{P} - Q_{\text{odd}} = \sum_{j=1}^{w'} \left( \bigotimes_{j' < j} \tilde{P}_{j'} \right) \otimes (\tilde{P}_j - P_j) \otimes \left( \bigotimes_{j' > j} P_{j'} \right).
\]

By the proof of theorem 4.1 of [AAG20] it holds that \(\|\tilde{P}_j - P_j\| \leq e^{-m}\) for each \(j\), so \(\|\tilde{P} - Q_{\text{odd}}\| \leq w'e^{-m}\) by the triangle inequality. Then,

\[
\|\tilde{K}(m,t,1)P_{Z^+}\| \leq \|DL(t)P_{Z^+}\| + \|\tilde{P} - Q_{\text{odd}}\| \leq w'e^{-m} + 2e^{-\Omega(t\sqrt{\gamma})}.
\]

\(Q_{\text{even}}\) and each \(\tilde{P}_j\) act as the identity of \(Z\) takes \(Z^\perp\) to itself, hence so does \(\tilde{K}(m,t,1)\). So \(\tilde{K}(m,t,1)\) is an \((w'e^{-m} + 2e^{-\Omega(t\sqrt{\gamma})})^2\)-AGSP and the result follows by raising to the 4th power.

The choice of parameters for the implementable AGSP is as in [AAG20] for the subvolume law-AGSP. To motivate the relations between the parameters, note that balancing the terms in the shrinking factor bound of theorem A.4 suggests choosing \(m \propto t\sqrt{\gamma}\) so that \(\Delta = e^{-\Omega(mk)} = e^{-\Omega(kt\sqrt{\gamma})}\) if \(m \geq 2\log n\).

Proof of corollary A.5 (tradeoff). Fix the relation \(t \propto \gamma^{-1/2}m\) and let \(m \geq 2\log n\). Bounding the shrinking factor \(\Delta\) using lemma A.4 and the Schmidt rank of \(K\) using lemma A.3 we get \(\Delta = e^{-mk/C}\) and

\[
R = (\frac{hd}{\gamma})^{O(mth+k\gamma^{-1/2}\sqrt{h/m})} \leq (\frac{hd}{\gamma})^{C(m^2\gamma^{-\frac{1}{2}}h + k\gamma^{-\frac{1}{2}}\sqrt{h/m})}.
\]

for some large constant \(C\). We will ensure that the parameters satisfy

\[
C^{-1}mk \geq \log \frac{1}{\Delta} + C(\frac{1}{2} + m^2\gamma^{-\frac{1}{2}}h + k\gamma^{-\frac{1}{2}}\sqrt{h/m}) \log(\frac{hd}{\gamma}).
\]

For this it suffices that

\[
mk \geq 2C\log \frac{1}{\Delta} \vee 4C^2(m^2\gamma^{-\frac{1}{2}}h \vee k\gamma^{-\frac{1}{2}}\sqrt{h/m}) \log(\frac{hd}{\gamma}).  \tag{12}
\]

⁵Lemma 3.1 of [AAG20] is stated in terms of unique ground states, but this condition is not required for its proof. Indeed, it is based on [AAV16] which is explicitly stated for general ground space degeneracies.
Let $\tilde{C} = 4C^2 \log(hd/\gamma)$ and pick
\[ m = \left\lceil \tilde{C} \frac{\gamma}{h} \frac{1}{\gamma} \sqrt{\frac{2C}{C h}} \log \frac{1}{\delta} \vee 2 \log n \right\rceil, \quad k = \lceil \tilde{C} m \gamma^{-\frac{1}{2}} h \rceil. \]
This choice ensures that $mk$ is larger than each of the two rightmost terms in (12). Moreover, expanding the expression for $k$,
\[ mk \geq \tilde{C} m^2 \gamma^{-\frac{1}{2}} h \geq \tilde{C} \left( \frac{\gamma}{h} \sqrt{\frac{2C}{C h}} \log \frac{1}{\delta} \right)^2 \gamma^{-\frac{1}{2}} h = 2C \log \frac{1}{\delta}. \]
So (12) is satisfied, hence $R \Delta \leq \delta$. The bound on $R$ follows from
\[
\begin{align*}
\log R &\leq C \left( m^2 \gamma^{-\frac{1}{2}} h + k \gamma^{-\frac{1}{2}} \sqrt{\frac{h}{m}} \right) \log(\frac{hd}{\gamma}) \\
&= \log \frac{1}{\delta} + O(\gamma^{-\frac{1}{2}} h^3 \log^2(\frac{hd}{\gamma}) + \gamma^{-\frac{1}{2}} h (\log n)^2 \log(hd)).
\end{align*}
\]
Since $h \geq (\log n)^3$ we may absorb the last term in the middle term.

\section*{A.5 MPO for the implementable AGSP}
We represent $\tilde{K}(m, t, k)$ by an MPO with $w$ tensors, each corresponding to a vertical column of qudits. Lemma A.3 gives the existence of such an MPO with bond dimension $R$. However, we need not only for such an MPO to exist, but also for the MPO representation to be computable in subexponential time. Fortunately, this turns out to be easy:

\begin{lemma}
An MPO for $\tilde{K}(m, t, k)$ with a local tensor for each column of qudits can be constructed in time $(hd/\gamma)^{O(mth + k \gamma^{-1/2} \sqrt{h/t})}$.
\end{lemma}

\begin{proof}
We begin by constructing a coarser MPO $\mathcal{T}$ for $\tilde{K}(m, t, k)$ with bond dimension $R$ where each physical index represents operators on a wide band $B_j$.

To construct $\mathcal{T}$ begin by constructing explicit matrices for the operators $\tilde{P}(\Xi_j)$ in time $wd^{O(mth)}$. Since $\tilde{P}$ is product across $H_{B_1} \otimes \cdots \otimes H_{B_w}$, we get an MPO for $\tilde{K}(m, t, 1)$ with bond dimension $d^{O(th)}$ (from $Q_{\text{even}}$).

By lemma A.3 the operators $\tilde{K}(m, t, k')$ with $k' \leq k$ satisfy a uniform bound $R$ on their entanglement rank across any vertical cut. Then, for $k' = 2, 4, \ldots, k$ (assuming $k \in 2^k$ for simplicity), alternate between the following two steps:

1. Squaring $\tilde{K}(m, t, k') \leftarrow \tilde{K}(m, t, k' - 1)^2$.

2. Trim the bonds of the MPO for $k'k'$ to its entanglement rank.

This concludes the construction of $\mathcal{T}$. Finally replace each local tensor on $B_j$ with an MPO with bond dimension $R^2 \dim(H_{B_j})$.
\end{proof}

We conclude the proof of theorem 3.8 by applying algorithm 1 to the implementable AGSP $\tilde{K}$.

\begin{proof}[Proof of theorem 3.8]
Let $\mathcal{H}_i = \mathcal{H}_{(i) \times [1, h]}$ for $i = 1, \ldots, w$ and let $\tilde{d} = \dim(\mathcal{H}_r) = d^h$. Corollary A.5 gives parameters such that $\tilde{K}$ is a $\Delta$-AGSP with an MPO of bond dimension $R$ such that $R \Delta \leq \frac{1}{64d}$ and $R = \frac{64d^h}{\delta} \exp\left(\frac{O(\gamma^{-\frac{1}{2}} \log^2(\frac{d^h}{\gamma}))}{\gamma}\right)$, and we can absorb the factor $64d^h$. Apply proposition 3.4 to $\tilde{K}$. The time complexity is $(\tilde{D}Rwdd^{-1})^{O(1)}$ where we can again absorb $\tilde{d}$ in $R$, and we can absorb $w$ since $\tilde{h} = n^{O(1)}$.
\end{proof}
Appendix B

B.1 Standard entropy bound [ALV12] from partial sums

A bound on the Shannon entropy of a probability distribution can be obtained through a dyadic decomposition by following the argument of [ALV12] lemma III.3. Given a sequence $\Lambda = (\lambda_1, \lambda_2, \ldots) \in [0, 1]^\mathbb{N}$ write the Shannon entropy $S(\Lambda) = \sum_i h(\lambda_i)$ where $h(x) = x \log(x^{-1})$.

**Claim B.1** ([ALV12, AKLV13]). Let $\Lambda = (\lambda_1, \lambda_2, \ldots) \in [0, 1]^\mathbb{N}$ be a sequence with $\sum_i \lambda_i \leq 1$ and write $\Sigma_I = \sum_{i \in I} \lambda_i \leq 1$ for $I \subset \mathbb{N}$. Let $I_0, I_1, \ldots$ be a partition on $\mathbb{N}$ such that $\Sigma_{I_n} \leq \gamma_n$ for some sequence of $\gamma_n \in [0, 1]$. If $|I_n| \geq 3$ for each $n$, then

$$S(\Lambda) \leq \log |I_0| + \sum_{n=1}^{\infty} \gamma_n \log(|I_n|) + \sum_{i=1}^{\infty} h(\gamma_n).$$

**Proof.** Since $h$ is concave Jensen’s inequality states that for any set of indices $I$, $\frac{1}{|I|} \sum_{i \in I} h(\lambda_i) \leq h(\frac{1}{|I|} \sum_{i \in I} \lambda_i)$. Rearranging yields:

$$\sum_{i \in I} h(\lambda_i) \leq |I| \cdot h(\Sigma_I / |I|). \quad (13)$$

$h$ is increasing on $[0, 1/e]$, so if $|I| \geq 3$ and $\gamma \leq 1$ is an upper bound on $\Sigma_I$, then

$$\sum_{i \in I} h(\lambda_i) \leq |I| \cdot h(\gamma / |I|) = \gamma \log(|I| \gamma^{-1}).$$

Apply this bound for each $n = 1, 2, \ldots$. We also have in particular that $\sum_{i \in I} h(\lambda_i) \leq \log |I|$. Apply this for $I_0$. \qed

Appendix C Alternative proof of sharp error reduction

Let $Z, V \preceq \mathcal{H}$ be subspaces such that $P_Z P_V P_Z \geq \mu P_Z$ (i.e., $V \succeq_{\mu} Z$). Lemmas 1 and 2 of [ALV17] state that for every $|z \rangle \in Z$ there exists $|v \rangle \in V$ with norm at most $\|v\| \leq \mu^{-1} \|z\|$ such that $P_Z |v \rangle = |z \rangle$. The alternative proof of the error reduction lemma 3.1 relies on noticing that this statement can be improved quadratically, i.e., we can replace $\mu^{-1}$ with $\mu^{-1/2}$.

C.1 Quadratically improved lifting lemma

**Definition C.1.** Let $Z, V \preceq \mathcal{H}$ be subspaces such that $V$ covers $Z$. Define the lifting operator from $Z$ to $V$ as $\text{lift}_{V \rightarrow Z} = \Pi_{V \rightarrow Z} (\Gamma_Z P_V \Gamma_Z^\dagger)^{-1}$.

**Lemma C.2.** Given subspaces $Z, V \preceq \mathcal{H}$ such that $V \succeq_{\mu} Z$ with $\mu > 0$, the lifting operator $Z \rightarrow V$ satisfies:
1. \( P_Z \circ \text{lift}_{V\leftarrow Z} |z\rangle = |z\rangle \) for any \(|z\rangle \in Z \) (lifting property),

2. \( \| \text{lift}_{V\leftarrow Z} \| \leq \mu^{-1/2} \).

Proof. The restricted projection \( M = \Pi_{Z\leftarrow V} \) is surjective since \( V \) covers \( Z \), so \( M^\dagger (MM^\dagger)^{-1} = \text{lift}_{V\leftarrow Z} \) is a well-defined right-inverse\(^6\) of \( M \). This is the lifting property. For the norm bound we write the polar decomposition \( \Pi_{Z\leftarrow V} = SV^\dagger \) where \( S \) is a positive operator on \( Z \) and \( V^\dagger \) is the adjoint of an isometry \( V: Z \rightarrow V \) (again using that \( M \) is surjective). Since \( V \geq \mu \) \( Z \) we have \( \mu I_Z \leq MM^\dagger = SV^\dagger VS = S^2 \) which implies that \( S \geq \sqrt{\mu} I_Z \). Then \( \| \text{lift}_{V\leftarrow Z} \| = \| VS^{-1} \| \leq \mu^{-1/2} \). \( \square \)

We also write \( \text{lift}_{V\leftarrow Z} \) in the same way when extending its codomain and viewing it as a map \( Z \rightarrow H \). By Pythagoras’ theorem, \( \| z \|^2 + \| \text{lift}_{V\leftarrow Z} |z\rangle \|^2 = \| \text{lift}_{V\leftarrow Z} |z\rangle \|^2 \leq \mu^{-1} \| z \|^2 \). Since \( \varphi = \mu^{-1} - 1 \), rearranging yields:

**Corollary C.3.** Let \( Z, V \leq H \) be such that \( V \) covers \( Z \) with error ratio \( \varphi \). Then for any \(|z\rangle \in Z\),

\[
\text{lift}_{V\leftarrow Z} |z\rangle = |z\rangle + P_{Z^\perp} \text{lift}_{V\leftarrow Z} |z\rangle \quad \text{where} \quad \| P_{Z^\perp} \text{lift}_{V\leftarrow Z} \| \leq \sqrt{\varphi}.
\]

The alternative proof of lemma 3.1 can now be finalized essentially as in the proof of [ALVV17] lemma 6:

**Finishing the alternative proof of lemma 3.1.** Write the AGSP as \( K_Z \oplus K_{Z^\perp} \). Given an arbitrary unit vector \(|z\rangle \in Z\) pick \(|v\rangle = K \circ \text{lift}_{V\leftarrow Z} \circ K_Z^\dagger |z\rangle \in KV\). It suffices to show that \( \langle z|v\rangle^2 \geq \mu' \| v \|^2 \) where \( \mu' = \frac{1}{1 + \Delta \varphi} \): Applying corollary C.3 to \( K_Z^{-1} |z\rangle \) we have the orthogonal decomposition

\[
|v\rangle = |z\rangle + |h\rangle \quad \text{where} \quad |h\rangle = K_{Z^\perp} (P_{Z^\perp} \text{lift}_{V\leftarrow Z} K_Z^{-1} |z\rangle).
\]

where \( \| |h\rangle \| \leq \| K_{Z^\perp} \| \cdot \| P_{Z^\perp} \text{lift}_{V\leftarrow Z} \| \leq \sqrt{\Delta \varphi} \). Then \( \| v\|^2 \leq 1 + \Delta \varphi \) by Pythagoras’, so \( \langle z|v\rangle^2 / \| v \|^2 = 1 / \| v\|^2 \geq \mu' \). \( \square \)

**References**


\(^6\)This right-inverse is a special case of the Moore-Penrose pseudo-inverse, but its role is not analogous to the pseudoinverse in the proof of [ALVV17] lemma 6, which was a pseudoinverse of the *AGSP*. 21


