SMALL QUANTUM GROUPS ASSOCIATED TO
BELAVIN-DRINFELD TRIPLES

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ABSTRACT. For a simple Lie algebra \( g \) of type \( A, D, E \) we show that any
Belavin-Drinfeld triple on the Dynkin diagram of \( g \) produces a collection
of Drinfeld twists for Lusztig’s small quantum group \( u_q(g) \). These
twists give rise to new finite-dimensional factorizable Hopf algebras, i.e.
new small quantum groups. For any Hopf algebra constructed in this
manner, we identify the group of grouplike elements, identify the Drin-
feld element, and describe the irreducible representations of the dual in
terms of the representation theory of the parabolic subalgebra(s) in \( g \)
associated to the given Belavin-Drinfeld triple. We also produce Drin-
feld twists of \( u_q(g) \) which express a known algebraic group action on its
category of representations, and pose a subsequent question regarding
the classification of all twists.

INTRODUCTION

Let \( g \) be a simple Lie algebra over \( \mathbb{C} \) of type \( A, D, E \), and let \( \Gamma \) be its
Dynkin diagram. A Belavin-Drinfeld triple on \( \Gamma \) is a choice of two subgraphs
\( \Gamma_1 \) and \( \Gamma_2 \) and an isomorphism \( T : \Gamma_1 \to \Gamma_2 \) satisfying a certain nilpotence
condition. In [5, Ch. 6] Belavin and Drinfeld showed that such a triple gives
rise to solutions to the classical Yang-Baxter equation in \( g \otimes g \), and in [14]
Etingof, Schedler, and Schiffmann showed that any Belavin-Drinfeld triple
gives rise to (Drinfeld) twists of the Drinfeld-Jimbo quantum group \( U_h(g) \).
Such a twist \( J \) of \( U_h(g) \) produces a new quantum group \( U_h(g)^J \) and new
R-matrix, i.e. solution to the Yang-Baxter equation (see Section 2). These
new solutions to the Yang-Baxter equation quantize the classical solutions
of Belavin and Drinfeld, in the sense described in [12, 14]. Furthermore, one
can show that any twist of the Drinfeld-Jimbo quantum group, over \( \mathbb{C}[h] \),
arises as one of the quantizations of [14], up to gauge equivalence.

Here we follow the methods of [14, 4] to produce twists of Lusztig’s small
quantum group \( u_q(g) \) from Belavin-Drinfeld triples. We also produce explicit
twisted automorphisms of \( u_q(g) \) which arise out of an algebraic group action
on its category of representations. The action we consider here first appeared
in the work of Arkhipov and Gaitsgory [3], but can also be derived from
De Concini and Kac’s earlier quantum coadjoint action [8], as is explained
in Section 9 below. Using the Belavin-Drinfeld twists, and those twists

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associated to the algebraic group action, we propose a question regarding the classification of all twists of the small quantum group.

**Belavin-Drinfeld triples and twists of** \( u_q(\mathfrak{g}) \). Recall that the small quantum group is a finite dimensional quasitriangular Hopf algebra produced from the Cartan data for \( \mathfrak{g} \) and a primitive \( l \)th root of unity \( q \). In addition to a triple \((\Gamma_1, \Gamma_2, T)\) for \( \mathfrak{g} \) we need one more piece of data \( S \). The element \( S \) is a choice of solution to a certain equation involving \( T \), which we describe below (see Section 3). Given any Belavin-Drinfeld triple \((\Gamma_1, \Gamma_2, T)\) we will have \( \max\{1, l|\Gamma - \Gamma_1||\Gamma - \Gamma_1| - 1\}/2 \) such solutions \( S \). We show

**Theorem I** (3.1). Any Belavin-Drinfeld triple \((\Gamma_1, \Gamma_2, T)\) for \( \mathfrak{g} \) and solution \( S \) produces a twist \( J = J_{T,S} \) for the small quantum group \( u_q(\mathfrak{g}) \), and an associated Hopf algebra \( u_q(\mathfrak{g})^J \).

This twist \( J_{T,S} \) is given explicitly by the formula

\[
J_{T,S} = (T_+ \otimes 1)(R) \cdots (T_+ \otimes 1)(R)S^{-1}\Omega^{-1/2}(T^\sigma \otimes 1)(\Omega)^{-1} \cdots (T \otimes 1)(\Omega)^{-1}
\]

where \( R \) is the \( R \)-matrix for \( u_q(\mathfrak{g}) \), \( \Omega \) is an element representing the Killing form, and \( T_+ \) is an extension of \( T \) to an endomorphism of the positive quantum Borel in \( u_q(\mathfrak{g}) \). The above theorem is a non-dynamical analog of [13, Sect. 5.2], and a discrete version of [14, Cor. 6.1].

Recall that for any twist \( J \) of a Hopf algebra \( H \) we will have a canonical equivalence between the associated tensor categories of finite dimensional representations \( \text{rep}(H) \xrightarrow{\sim} \text{rep}(H^J) \). In addition to studying the relationship between Belavin-Drinfeld triples and solutions to the Yang-Baxter equation (i.e. \( R \)-matrices) for finite dimensional Hopf algebras, we want to study variances of Hopf structures under tensor-equivalence. With this purpose in mind we give an in depth study of the Hopf algebras \( u_q(\mathfrak{g})^J \) arising from our twists.

For the remainder of the introduction fix \( J = J_{T,S} \) the twist associated to some Belavin-Drinfeld data \((\Gamma_1, \Gamma_2, T)\) and \( S \). Using the new \( R \)-matrix for \( u_q(\mathfrak{g})^J \), in conjunction with the frameworks of [24], we identify the grouplike elements of \( u_q(\mathfrak{g})^J \), show that the Drinfeld element for \( u_q(\mathfrak{g})^J \) is equal to that of the untwisted algebra \( u_q(\mathfrak{g}) \), verify invariance of the traces of the powers of the antipode under the twists \( J = J_{T,S} \), and classify irreducible representations of the dual. (See Corollaries 7.8, 8.2, 8.3, and Theorem 7.4 below.) Our analyses of the Drinfeld element and antipode give positive answers to some general questions of [22] and [26] in the particular case of Belavin-Drinfeld twists of the small quantum group. We describe our result on irreducibles more explicitly below.

In the statement of the following theorem we let \( \mathfrak{p}^{ss} \) be the semisimple Lie algebra associated to the Dynkin diagram \( \Gamma_1 \) appearing in the Belavin-Drinfeld triple \((\Gamma_1, \Gamma_2, T)\).

\[\text{We will need } l \text{ to be coprime to a small number of integers throughout this work.}\]
Theorem II (7.4). There is an abelian group $\mathcal{L}$ of order $l(|\Gamma - \Gamma_1|)$ and bijection

$$\text{Irrep}(C[\mathcal{L}] \otimes u_q(p^{ss})) \xrightarrow{\sim} \text{Irrep}((u_q(g)^J)^*)$$

induced by a surjective algebra map $(u_q(g)^J)^* \to C[\mathcal{L}] \otimes u_q(p^{ss})$.

By comparison, for the untwisted algebra $u_q(g)$ we have that $\text{Irrep}(u_q(g)^J) = (\mathbb{Z}/l\mathbb{Z})^{|\Gamma|}$, and the representation theory of the dual is rather banal from the perspective of irreducibles and the fusion rule. After twisting $u_q(sl_{n+1})$, for example, we can have a copy of the rather rich category $\text{rep}(u_q(sl_n))$ in the category of representations for the dual $(u_q(sl_{n+1})^J)^*$. This will specifically be the case for (what we call) maximal triples on $A_n$. One should compare this result to [13, Thm. 5.4.1].

The Arkhipov-Gaitsgory action and twisted automorphisms. Take $\Theta$ the connected, simply connected, semisimple algebraic group with Lie algebra $g$. In [3] Arkhipov and Gaitsgory show that the category $\text{rep}(u_q(g))$ is tensor equivalent to a de-equivariantization of the category of corepresentations of the quantum function algebra $O_q(\Theta)$. The de-equivariantization is a certain (non-full) monoidal subcategory in $\text{Coh}(\Theta)$ which inherits a natural action of $\Theta$ by left translation (see [2, 15]). From the aforementioned equivalence we then get an action of $\Theta$ on $\text{rep}(u_q(g))$.

According to general principles, any autoequivalence of $\text{rep}(u_q(g))$ should be expressible as a twisted automorphism $(\phi, J)$, i.e. a pair of a twist $J$ and a Hopf isomorphism $\phi : u_q(g) \to u_q(g)^J$. Hence, the action of $\Theta$ should generate twists of $u_q(g)$.

In Section 9 we show that any simple root $\alpha$ of $g$, or its negation $-\alpha$, has an associated 1-parameter family of twisted automorphisms $(\exp\lambda\pm\alpha, J^{\lambda\pm\alpha})$, which then give a 1-parameter subgroup $\omega_{\pm\alpha}$ in the group of autoequivalences of $\text{rep}(u_q(g))$. We identify these 1-parameter subgroups $\omega_{\pm\alpha}$ with the action of Arkhipov and Gaitsgory.

Proposition (9.4). For $\gamma_{\pm\alpha} : C \to \Theta$ the 1-parameter subgroup given by exponentiating the root space $g_{\pm\alpha}$, we have a diagram

$$\begin{array}{ccc}
\gamma_{\pm\alpha} & \xrightarrow{\Theta} & \text{Aut}(\text{rep}(u_q(g))), \\
\text{C} & \xrightarrow{\omega_{\pm\alpha}} & \omega_{\pm\alpha} \, \, \text{AG actn}
\end{array}$$

where $\omega_{\pm\alpha}$ is the 1-parameter subgroup specified by the twisted automorphisms $(\exp\lambda_{\pm\alpha}, J^{\lambda}_{\pm\alpha})$.

This result allows us to produce an explicit action of $\Theta$ on the collection $\mathbb{T}(u_q(g))$ of gauge equivalence classes of twists. We let $\mathbb{BD}(u_q(g)) \subset \mathbb{T}(u_q(g))$ denote the subcollection of Belavin-Drinfeld twists $\{J_{T,S}\}_{T,S}$. We pose the following question, which is also raised in [7].
Question (9.5). Do the Belavin-Drinfeld twists and the 1-parameter subgroups \( \{ \exp^{\lambda} J_{\pm \alpha} \} \) generate all twists of \( u_q(\mathfrak{g}) \)? Equivalently, is the inclusion \( \text{BD}(u_q(\mathfrak{g})) \cdot \Theta \to \text{Tw}(u_q(\mathfrak{g})) \) an equality?

As was stated above, for the Drinfeld-Jimbo algebra \( U_\hbar(\mathfrak{g}) \) one can show that the Belavin-Drinfeld twists are the only twists, up to gauge equivalence. So the appearance of \( \Theta \) here already marks a deviation from the generic setting.

**Organization.** Sections 1 and 2 are dedicated to background. In Section 3 we introduce and prove Theorem I. In Sections 4 and 5 we analyze relations between Radford’s left and right subalgebras \( R^l_J \) and \( R^r_J \) in \( u_q(\mathfrak{g}) \) and the quantum parabolics associated to \( \Gamma_1 \) and \( \Gamma_2 \). We prove an explicit description of the \( R^J(\ast) \) in Section 6, which leads to the proof of Theorem II in Section 7. In Section 8 we discuss the Drinfeld element and antipode of such a twist \( u_q(\mathfrak{g}) \). Section 9 is dedicated to the action of the algebraic group \( \Theta \) on \( \text{rep}(u_q(\mathfrak{g})) \).

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1. The small quantum group, Belavin-Drinfeld triples, and associated subgroups in the Cartan

We introduce the small quantum group \( u_q(\mathfrak{g}) \), then give some information on the Cartan subgroup \( G = G(u_q(\mathfrak{g})) \) and Belavin-Drinfeld triples.

1.1. The small quantum group. Take \( \mathfrak{g} \) a simple and simply laced Lie algebra, i.e. a Lie algebra of type \( A, D, E \). Let \( \Phi \) be a root system for \( \mathfrak{g} \) (in the dual of some Cartan), \( \Gamma \) be a choice of simple roots, and \( l \) be an odd integer coprime to the determinant of the Cartan matrix for \( \mathfrak{g} \). Let \( (?, ?) \) be the scaling of the Killing form so that each \( (\alpha, \beta) \) is the Cartan integer for simple roots \( \alpha, \beta \). Take \( q \) a primitive \( l \)th root of unity.

The small quantum group \( u_q(\mathfrak{g}) \) is the Hopf algebra

\[
\mathbb{C}\langle K_\alpha, E_\alpha, F_\alpha : \alpha \in \Gamma \rangle / (\text{Rels}),
\]

where Rels is the set of relations

\[
[K_\alpha, K_\beta] = 0, \quad K_\alpha E_\beta = q^{(\alpha, \beta)} E_\beta K_\alpha, \quad K_\alpha F_\beta = q^{-(\alpha, \beta)} F_\beta K_\alpha,
\]

\[
[E_\alpha, F_\beta] = \delta_{\alpha, \beta} \frac{K_\alpha - K_\alpha^{-1}}{q - q^{-1}},
\]

\[
[E_\alpha, E_\beta] = [F_\alpha, F_\beta] = 0 \quad \text{when} \quad (\alpha, \beta) = 0,
\]

\[
E_\alpha^2 E_\beta - (q + q^{-1}) E_\alpha E_\beta E_\alpha + E_\beta E_\alpha^2, \quad \text{when} \quad (\alpha, \beta) = -1,
\]

\[
F_\alpha^2 F_\beta - (q + q^{-1}) F_\alpha F_\beta F_\alpha + F_\beta F_\alpha^2 \quad \text{when} \quad (\alpha, \beta) = -1.
\]

\[
K^l_\alpha = 1, \quad E^l_\mu = F^l_\mu = 0 \quad \forall \mu \in \Phi^+.
\]
We will explain the (currently opaque) relations (1) more clearly below. The coproduct is given by
\[ \Delta(K_a) = K_a \otimes K_a, \quad \Delta(E_\alpha) = E_\alpha \otimes 1 + K_a \otimes E_\alpha, \quad \Delta(F_\alpha) = F_\alpha \otimes K_a^{-1} + 1 \otimes F_\alpha \]
and the antipode is given by
\[ S(K_a) = K_a^{-1}, \quad S(E_\alpha) = -K_a^{-1} E_\alpha, \quad S(F_\alpha) = -F_\alpha K_a. \]

We let \( G \) denote the group of grouplikes in \( u_q(\mathfrak{g}) \), \( u^+ \) and \( u^- \) denote the subalgebras generated by the \( E_\alpha \) and \( F_\alpha \) respectively, and \( u_+ \) and \( u_- \) denote the positive and negative quantum Borels in \( u_q(\mathfrak{g}) \). Note that \( G \) is generated by the \( K_a \) and that under the adjoint action of \( G \) on \( u^\pm \) we will have \( u_\pm = u^\pm \rtimes \mathbb{C}[G] \). Note also that \( u_q(\mathfrak{g}) \) and the \( u_\pm \) are graded by the root lattice \( \mathbb{Z} \cdot \Gamma \), where the generators \( E_\alpha, F_\alpha, \) and \( K_a \) have degrees \( \alpha, -\alpha, \) and 0 respectively.

We would like to employ Lusztig’s standard basis for \( u_q(\mathfrak{g}) \), which we review here. Recall that for a reduced expression \( w = \sigma_{\alpha_1} \cdots \sigma_{\alpha_t} \) of the longest word \( w \) in the Weyl group, in terms of the simple reflections, we have \( \text{length}(w) = |\Phi^+| \) and
\[ \Phi^+ = \{ \sigma_{\alpha_1} \cdots \sigma_{\alpha_{i-1}}(\alpha_i) : 1 \leq i \leq \text{length}(w) \}. \tag{2} \]
(See e.g. [28].) For each simple root \( \alpha \) there is an automorphism \( B_\alpha \) of \( u_q \) so that the \( B_\alpha \) together give an action of the braid group \( B(\Gamma) \) on \( u_q \) [20].

Now for each \( \mu \in \Phi^+ \) we take
\[ E_\mu = B_{\alpha_1} \cdots B_{\alpha_{i-1}}(E_{\alpha_i}), \quad F_\mu = B_{\alpha_1} \cdots B_{\alpha_{i-1}}(F_{\alpha_i}), \]
where \( \mu = \sigma_{\alpha_1} \cdots \sigma_{\alpha_{i-1}}(\alpha_i) \). The \( E_\mu \) and \( F_\mu \) defined here are the elements appearing in the above relations (1).

**Theorem 1.1** ([20]). For each \( \mu \in \Phi^+ \), the element \( E_\mu \) (resp. \( F_\mu \)) is homogeneous of degree \( \mu \) (resp. \( -\mu \)) with respect to the root lattice grading on \( u_q(\mathfrak{g}) \). Furthermore, the collection of elements
\[ \{ \prod_{\mu \in \Phi^+} E_{\mu}^{n_\mu} : 0 \leq n_\mu \leq l - 1 \}, \quad \{ \prod_{\nu \in \Phi^+} F_{\nu}^{m_\nu} : 0 \leq m_\nu \leq l - 1 \}, \]
give \( \mathbb{C} \)-bases for \( u^+ \) and \( u^- \) respectively, and
\[ \{( \prod_{\nu \in \Phi^+} F_{\nu}^{m_\nu}) ( \prod_{\mu \in \Phi^+} E_{\mu}^{n_\mu}) : 0 \leq n_\mu, m_\nu \leq l - 1 \} \]
gives a \( \mathbb{C}[G] \)-basis for \( u_q(\mathfrak{g}) \).

Homogeneity of the \( E_\mu \) is equivalent to the statement that \( E_\mu \) is a linear combination of permutations of the monomial \( E_{\alpha_1} \cdots E_{\alpha_k} \), where \( \mu = \alpha_1 + \cdots + \alpha_k \) with the \( \alpha_i \in \Gamma \). The analogous statement holds for the \( F_\nu \) as well. We note that the homogeneity is not covered in [20], but can easily be seen from the fact that each braid group operator \( B_\alpha \) is such that \( \text{deg}(B_\alpha(a)) = \)

\[ \text{deg}(B_\alpha(a)) = \]
\[ \sigma_a(\deg(a)), \] for homogeneous \( a \in u_q \). From the identifications \( u_{\pm} = u^{\pm} \times \mathbb{C}[G] \) the \( \mathbb{C} \)-bases for \( u^{\pm} \) produce \( \mathbb{C}[G] \)-bases for the quantum Borels.

We recall finally that \( u_q(\mathfrak{g}) \) is quasi-triangular. The \( R \)-matrix is
\[
R = \prod_{\alpha \in \Phi^+} \left( \sum_{n=0}^{l-1} q^{-n(n+1)/2} \frac{(1-q^2)^n}{[n]_q!} E_\alpha^{-n} \otimes F_\alpha^{n} \right) \Omega,
\]
where \([n]_q!\) is the standard \( q \)-factorial, \( \Omega \in \mathbb{C}[G] \otimes \mathbb{C}[G] \) is such that \((\mu \otimes \nu)(\Omega) = q^{\langle \mu, \nu \rangle}\) for each \( \mu, \nu \in G^\vee \), and the product is ordered with respect to the ordering on \( \Phi^+ \) given by (2) (see [27]).

1.2. Belavin-Drinfeld triples and subgroups of \( G \). We recall some information from [13]. Let \( h \) be the Cartan subalgebra in \( g \), for \( g, \Phi, \Gamma, \) and \( l \) as above. Let \( h^*_Z = \mathbb{Z} \cdot \Gamma \) be the root lattice in \( h^* \). Take now
\[
\mathcal{G} = h^*_Z / lh^*_Z \quad \text{and} \quad G = G^\vee.
\]
Recall that, since \( g \) is simply laced, there is a unique scaling of the Killing form on \( h^* \) which produces an integer valued form on \( h^*_Z \) with \((\alpha, \alpha) = 2\) for each \( \alpha \in \Gamma \). By our assumption on \( l \) this scaling of the Killing form induces a non-degenerate symmetric bilinear form on the quotient \((?, ?) : \mathcal{G} \times \mathcal{G} \to \mathbb{Z} / l\mathbb{Z} \). This gives an identification \( \mathcal{G} \to G \), \( \alpha \mapsto (\alpha, ?) \), and via this identification we get an induced form on \( G \). We take \( K_\gamma = (\gamma, ?) \) for each \( \gamma \in \mathcal{G} \), so that \((K_\gamma, K_\mu) = (\gamma, \mu)\).

As our notation suggests, we identify \( G \) with the collection of grouplike elements in \( u_q(\mathfrak{g}) \). Since \( G \) will be identified with a set of units in an algebra, we adopt a multiplicative notation \( K_\alpha K_\beta = K_{\alpha + \beta} \).

The following structure was introduced by Belavin and Drinfeld in [5].

**Definition 1.2.** A Belavin-Drinfeld triple (BD triple) on \( \Gamma \) is a choice of two subsets \( \Gamma_1, \Gamma_2 \subset \Gamma \) and inner product preserving bijection \( T : \Gamma_1 \to \Gamma_2 \) which satisfies the following nilpotence condition: for each \( \alpha \in \Gamma_1 \) there exists \( n \geq 1 \) with \( T^n(\alpha) \in \Gamma - \Gamma_1 \).

We often take \( T\alpha = T(\alpha) \). Having fixed some BD triple \((\Gamma_1, \Gamma_2, T)\) we can define a number of subgroups in \( \mathcal{G} \) and \( G \). Via the sequence \( \Gamma \to h^*_Z \to \mathcal{G} \) we identify \( \Gamma \) with a basis for \( \mathcal{G} \). We take
\[
\mathcal{L} = \left( \mathbb{Z} / l\mathbb{Z} \cdot \{ \alpha - T\alpha : \alpha \in \Gamma_1 \} \right)^\perp,
\]
where the perp is calculated with respect to the form, and
\[
L = \left( \mathbb{Z} / l\mathbb{Z} \cdot \{ K_\alpha K_{T^{-1}_\alpha}^{-1} : \alpha \in \Gamma_1 \} \right)^\perp.
\]
Under the identification \( \mathcal{G} \to G \) the subgroups \( \mathcal{L} \) and \( L \) are identified. We take also \( \mathcal{G}_i = \mathbb{Z} / l\mathbb{Z} \cdot \Gamma_i \) and \( \mathcal{G}_i = \mathbb{Z} / l\mathbb{Z} \cdot \{ K_\alpha : \alpha \in \Gamma_i \} \).

We assume that \( l \) is such that restrictions of the form to \( \mathbb{Z} / l\mathbb{Z} \cdot \{ \alpha - T\alpha : \alpha \in \Gamma_1 \} \) and \( \mathcal{G}_i \) are non-degenerate. To find such an \( l \) one simply considers the determinants of the (integer) matrices \( ((\alpha - T\alpha, \beta - T\beta))_{\alpha, \beta \in \Gamma_1} \) and \( ((\alpha, \beta))_{\alpha, \beta \in \Gamma_1} \) and chooses \( l \) coprime to these determinants. This will give
\[ L^\perp = \mathbb{Z}/l\mathbb{Z} \cdot \{ \alpha - T\alpha : \alpha \in \Gamma_1 \} \] and split \( \mathcal{G} \) and \( G \) as \( \mathcal{G} = \mathcal{L} \times \mathcal{L}^\perp = G_i \times G_i^\perp \), \( G = L \times L^\perp = G_i \times G_i^\perp \). We also assume \( l \) is such that the restriction of the form to \( G_i \times \mathcal{L}^\perp \) is non-degenerate, which we can do by [14, Lem 3.1], and which can be checked by considering the determinant of the corresponding matrix. The following lemma was covered in [13, Sect. 5.2] (see also [14, Cor 3.2]).

**Lemma 1.3.** Under the above assumptions on \( l \), there are splittings \( \mathcal{G} = \mathcal{G}_1 \times \mathcal{L} \) and \( \mathcal{G}_2 \times \mathcal{L} \), and a unique extension of \( T : \Gamma_1 \to \Gamma_2 \) to a group automorphism \( T : \mathcal{G} \to \mathcal{G} \) with \( T|\mathcal{L} = \text{id}_\mathcal{L} \). This automorphism preserves the form on \( \mathcal{G} \).

We will denote this extension of \( T \) to an automorphism on \( \mathcal{G} \) simply by \( T \). By a further abuse of notation we let \( T \) also denote the induced automorphism on the dual. That is, \( K_{\alpha} \mapsto K_{T(\alpha)} \).

Preservation of the form means specifically \((T\mu, T\nu) = (\mu, \nu)\) for each \( \mu, \nu \in \mathcal{G} \) and \((T \otimes T)(\Omega) = \Omega\).

Throughout this work we make copious use of the dualities \( \mathcal{G} \sim \sim \mathcal{G}_1 \times \mathcal{L} \), \( L \sim L \), \( G_i \sim G_i \), \( L^\perp \sim L_i^\perp \), \( G_i^\perp \sim G_i^\perp \).

By this we mean both that the duality functor \( (\dashv) \) sends the group on the left to the group on the right, and vice-versa, and that for any \( K_{\mu} \) in the group on the left the function \( (K_{\mu}, \cdot) \) will be an element in the corresponding group on the right, and vice-versa.

## 2. Twists and R-matrices

A (Drinfeld) twist of a Hopf algebra \( H \) is a unit \( J \in H \otimes H \) which satisfies the dual cocycle condition

\[
(\Delta \otimes 1)(J)(J \otimes 1) = (1 \otimes \Delta)(J)(1 \otimes J)
\]

and \((\epsilon \otimes 1)(J) = (1 \otimes \epsilon)(J) = 1\). From such a \( J \) we can define a new Hopf algebra \( H^J \) which is equal to \( H \) as an algebra and has the new comultiplication

\[ \Delta^J(h) = J^{-1}\Delta(h)J. \]

The antipode on \( H^J \) is given by

\[ S_J(h) = Q_J^{-1}S(h)Q_J, \]

where \( Q_J = m((S \otimes 1)(J)) \) and \( Q_J^{-1} = m((1 \otimes S)(J^{-1})) \) and \( m \) is multiplication. (See e.g. [15, 25].)

Recall that a quasitriangular Hopf algebra is a Hopf algebra \( H \) with a unit \( R \in H \otimes H \) satisfying \( R\Delta(h)R^{-1} = \Delta^{op}(h) \) for all \( h \in H \), as well as the relations \((\Delta \otimes 1)(R) = R_{13}R_{23} \) and \((1 \otimes \Delta)(R) = R_{13}R_{12} \). We have the additional relations

\[
(\epsilon \otimes 1)(R) = (1 \otimes \epsilon)(R) = 1, \quad (S \otimes 1)(R) = (1 \otimes S^{-1})(R) = R^{-1}
\]
and $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ [10]. When $H$ is quasitriangular with $R$-matrix $R$, the twist $H^J$ will naturally be quasitriangular with new $R$-matrix $R^J = J_{21}^{-1}RJ$.

2.1. Bicharacters and twists on group rings. Let $\Lambda$ be a finite abelian group. We call an element $B \in \mathbb{C}[\Lambda] \otimes \mathbb{C}[\Lambda] = \mathbb{C}[(\Lambda^\vee \times \Lambda^\vee)^\vee]$ a (symmetric, antisymmetric, etc.) bicharacter if its restriction $B : \Lambda^\vee \times \Lambda^\vee \to \mathbb{C}^\times$ is a (symmetric, antisymmetric, etc.) bicharacter. An easy direct check verifies

Lemma 2.1. Any bicharacter $B \in \mathbb{C}[\Lambda] \otimes \mathbb{C}[\Lambda]$ is a twist for $\mathbb{C}[\Lambda]$.

Indeed, up to so-called gauge equivalence, every twist of an abelian group ring is given by an antisymmetric bicharacter (see e.g. [16]).

3. Twists from Belavin-Drinfeld triples

For the remainder of this study we fix $g$ a simply laced simple Lie algebra with root system $\Phi$ and a choice of simple roots $\Gamma$. We take $l$ as in Section 1.2 and $u_q = u_q(g)$.

Let $(\Gamma_1, \Gamma_2, T)$ be a BD triple. Following [14, 13], we extend the group maps $T^{\pm 1} : G \to G$ constructed in Lemma 1.3 to Hopf endomorphisms of the quantum Borels $T_\pm : u_\pm \to u_\pm$ defined by

$$T_+(E_\alpha) = \begin{cases} E_{T\alpha} & \text{when } \alpha \in \Gamma_1 \\ 0 & \text{when } \alpha \in \Gamma - \Gamma_1 \end{cases}, \quad T_-(F_\beta) = \begin{cases} E_{T^{-1}\beta} & \text{when } \beta \in \Gamma_2 \\ 0 & \text{when } \beta \in \Gamma - \Gamma_2 \end{cases}.$$ 

There will be a unique minimal positive integer $n$ such that $T_\pm^n I_\pm = 0$, where $I_\pm$ is the ideal in $u_\pm$ generated by all the $E_\alpha$, or $F_\alpha$. We call this integer the nilpotence degree of $T_\pm$.

We will be interested in antisymmetric bicharacters $S$ in $\mathbb{C}[G] \otimes \mathbb{C}[G]$ solving the following equation:

$$S^2(\alpha - T\alpha, ?) = \Omega(\alpha + T\alpha, ?) \quad \forall \alpha \in \Gamma_1. \quad \text{(EQ–S)}$$

We verify below that such solutions always exist, and that there are exactly $|L \wedge \mathbb{Z} L|$ of them, which is expected from [5, 14].

This section is dedicated to a proof of the following theorem.

Theorem 3.1. Consider any Belavin-Drinfeld triple $(\Gamma_1, \Gamma_2, T)$ and solution $S$ to (EQ–S). The element

$$J_{T,S} = (T_+ \otimes 1)(R) \cdots (T_+^n \otimes 1)(R)S^{-1}\Omega^{-1/2}L^{-1/2}L^\vee \cdots (T_+ \otimes 1)(R)\Omega^{-1/2}L^{-1/2}L^\vee \cdots (T_+ \otimes 1)(R)$$

is a twist for the small quantum group $u_q(g)$, where $n$ is the nilpotence degree of $T_+$.

This result is a non-dynamical version of [13, Sect. 5.2], and a discrete version of [14, Thm. 6.1]. To clarify our previous point, we have

Lemma 3.2. Antisymmetric bicharacter solutions $S$ to equation (EQ–S) always exist, and there are exactly $|L \wedge \mathbb{Z} L|$ such solutions.
Proof. We decompose $G$ as $L \oplus L^\perp$ to get $G \wedge_{\mathbb{Z}} G = (L^\perp \wedge_{\mathbb{Z}} G) \oplus (L \wedge_{\mathbb{Z}} L)$. Since

$$L^\perp = (L^\perp)^\vee = (\mathbb{Z}/\mathbb{Z} \cdot \{ \alpha - T\alpha : \alpha \in \Gamma_1 \})^\vee$$

we see that the equation (EQ–S) specifies uniquely an element $S_0$ in $L^\perp \wedge_{\mathbb{Z}} G$, which we extend to a bicharacter on $G$ which vanishes on $L \times L$. Whence we have found a solution to (EQ–S). We can add arbitrary elements of $L \wedge_{\mathbb{Z}} L$ to arrive at the set of all solutions $S_0 + L \wedge_{\mathbb{Z}} L$.

One should note that when rank($L$) = |Γ| − 1, the solution $S$ will be unique. Using our nilpotence assumption on $T$ one sees that, up to an automorphism of the Dynkin diagram, this occurs only in type $A$ for the triple

$$\Gamma = A_n, \Gamma = \{ \text{first } n - 1 \text{ roots} \}, \Gamma_2 = \{ \text{last } n - 1 \text{ roots} \}, T(\alpha_i) = \alpha_{i+1}.$$

We will call this the maximal triple on $A_n$.

We will need the following basic property of the $R$-matrix.

**Lemma 3.3.** The $R$-matrix for $u_q(\mathfrak{g})$ satisfies $(T_+ \otimes 1)(R) = (1 \otimes T_-)(R)$.

**Proof.** Any element $W \in u_+ \otimes u_-$ is uniquely specified by the corresponding function $W : u_+^* \otimes u_-^* \rightarrow \mathbb{C}$ and subsequent map $t_W : u_+^* \rightarrow u_-$, $f \mapsto (f \otimes 1)(W)$. By [24, Prop. 2] and the fact that $T_\pm$ is a Hopf map, we see that when $W = (T_+ \otimes 1)(R)$ or $(1 \otimes T_-)(R)$ the $t_W : u_+^* \rightarrow u_-$ are algebra morphisms. Since $u_+$ is cordically graded, $u_+^*$ is generated in degrees 0 and $-1$ as an algebra, with $(u_+^*)_0 = \mathbb{C}[G]^*$ and $(u_+^*)_{-1} = (\sum_\alpha \mathbb{C}[G]E_\alpha)^*$, and we see that the $t_W$ are determined by the restrictions $t_W|(u_+^*)_0$ and $t_W|(u_+^*)_{-1}$. These restrictions are in turn determined by the homogeneous pieces

$$(T_+ \otimes 1)(R)_0 = (T \otimes 1)(\Omega), \quad (1 \otimes T_-)(R)_0 = (1 \otimes T^{-1})(\Omega)$$

and

$$(T_+ \otimes 1)(R)_1 = (q^{-1} - q)(\sum_{\alpha \in \Gamma_1} E_{T\alpha} \otimes F_{\alpha})(T \otimes 1)(\Omega),$$

$$(1 \otimes T_-)(R)_1 = (q^{-1} - q)(\sum_{\beta \in \Gamma_2} E_{\beta} \otimes F_{T^{-1}_{-\beta}})(1 \otimes T^{-1})(\Omega),$$

where we grade $u_+ \otimes u_-$ by the degree on $u_+$. By $T$-invariance of the form $\Omega$, it follows that $(T_+ \otimes 1)(R)_0 = (1 \otimes T_-)(R)_0$ and $(T_+ \otimes 1)(R)_1 = (1 \otimes T_-)(R)_1$. Whence we have the proposed equality. \qed

3.1. **General outline.** In order to prove Theorem 3.1 we will basically repeat the arguments of [14, 13], and so only sketch some of the unoriginal details.

Fix a data $(\Gamma_1, \Gamma_2, T)$. Following the suggestions of [14, Remark 6.1], and the general approach of [4], we will show that $J$ is a twist by showing that both

$$(\Delta \otimes 1)(J)(J \otimes 1) \quad \text{and} \quad (1 \otimes \Delta)(J)(1 \otimes J)$$
solve a certain “mixed ABRR” equation. Solutions to this equation with a specified “initial condition” are shown to be unique, so that we will have

$$(\Delta \otimes 1)(J)(J \otimes 1) = (1 \otimes \Delta)(J)(1 \otimes J).$$

**Remark 3.4.** Our presentation is slightly more complicated than that of [14]. This is a result of our choice to avoid the use of dynamical twists.

### 3.2. Discrete ABRR in 2-components.

For a given solution $S$ let $Z$ denote the restriction of $S$ to $\mathcal{L} \times \mathcal{L}^\perp$, $\Sigma$ denote the restriction to $\mathcal{L}^\perp \times \mathcal{L}$, and take

$$Q = Z\Sigma = |((\mathcal{L} \times \mathcal{L}^\perp) + (\mathcal{L}^\perp \times \mathcal{L})|.$$

We view $Z$, $\Sigma$, and $Q$ as bicharacters on $\mathcal{G}$ by letting them vanish on all other factors of $\mathcal{G} \times \mathcal{G}$. We also let $\Omega_L$ denote the restriction $\Omega(\mathcal{L} \times \mathcal{L})$.

**Definition 3.5.** We define $A^2_L$ and $A^2_R$ the be the linear endomorphisms of $u_+ \otimes u_-$ defined by

$$A^2_L(\xi) = (T_+ \otimes 1)(R\xi Q)Q^{-1}\Omega^{-1}_L, \quad A^2_R(\xi) = (1 \otimes T_-)(R\xi Q)Q^{-1}\Omega^{-1}_R.$$

The left and right 2-component ABRR equations are the equations $A^2_L(X) = X$ and $A^2_R(X) = X$ respectively.

We note that $Q$ can be replaced with $\Sigma$ and $Z$ in the expressions for $A^2_L$ and $A^2_R$ respectively. These alternate expressions are preferable for some calculations.

Since $R$ decomposes as a sum $R = \Omega + R_+$, where $R_+$ is in the nilpotent ideal $I_+ \otimes I_-$, we get a corresponding decomposition of $A^2_L$ as

$$A^2_L(\xi) = (T_+ \otimes 1)(\Omega\xi Q)Q^{-1}\Omega^{-1}_L + (T_+ \otimes 1)(R_+\xi Q)Q^{-1}\Omega^{-1}_L.$$

From this one finds that we can solve the left 2-component ABRR equation provided we can solve to the equation $(T \otimes 1)(\Omega X_0 Q)Q^{-1}\Omega^{-1}_L = X_0$ in $\mathbb{C}[G] \otimes \mathbb{C}[G]$. The analogous statement holds for the right ABRR equation. Whence we have the following discrete analog of [14, Cor. 4.1].

**Lemma 3.6** (cf. [14]). Any solution $B \in \mathbb{C}[G] \otimes \mathbb{C}[G]$ to the equation

$$(T \otimes 1)(\Omega X_0 Q)Q^{-1}\Omega^{-1}_L = X_0 \quad (\text{resp. } (1 \otimes T^{-1})(\Omega X_0 Q)Q^{-1}\Omega^{-1}_L = X_0) \tag{3}$$

extends uniquely to a solution $J \in B + I_+ \otimes I_-$ to the ABRR equation $A^2_L(X) = X$ (resp. $A^2_R(X) = X$).

In the proof of the following lemma we use the fact that for any element $K_\mu \in \mathbb{C}[G]$, and $\nu \in \mathcal{G}$, we have

$$(T^{\pm 1}(K_\mu))(\nu) = K_\mu(T^{\mp 1}\nu).$$

This follows from the easy sequence

$$(T^{\pm 1}(K_\mu))(\nu) = K_{T^{\pm 1}_\mu}(\nu) = (T^{\pm 1}_\mu, \nu) = (\mu, T^{\mp 1}_\nu) = K_\mu(T^{\mp 1}_\nu).$$

**Lemma 3.7.** There are unique solutions $J_L, J_R \in S^{-1}\Omega^{-1/2}_L + I_+ \otimes I_-$ to the left and right 2-component ABRR equations, respectively.
Proof. We are claiming first that $S^{-1}\Omega_{L}^{-1/2}$ solves the degree 0 ABRR equations from the previous lemma. Reorganizing, and applying $T^{-1} \otimes 1$, we see that $S^{-1}\Omega_{L}^{-1/2}$ solves ABRR on the left, say, if and only if the equation

$$\Omega\Omega_{L}^{-1/2}(T^{-1} \otimes 1)(\Omega_{L}^{-1/2}) = SQ^{-1}(T^{-1} \otimes 1)(S^{-1}Q)$$

is satisfied. Using the fact that $T|L = id_L$ and $\Omega = \Omega_L\Omega_{L}^\perp$ we reduce to

$$\Omega_{L}^{1/2}(T^{-1} \otimes 1)(\Omega_{L}^{1/2}) = SQ^{-1}(T^{-1} \otimes 1)(S^{-1}Q).$$

Applying to arbitrary elements $\mu, \nu \in G$ gives the equivalent equation

$$\Omega^{1/2}(\mu + T\mu, \nu) = SQ^{-1}(\mu - T\mu, \nu). \quad (4)$$

By writing $\mu$ as a sum of elements in $L$ and $L^\perp$ we see that the above equation holds if and only if it holds when $\mu \in L$, or $\mu \in L^\perp$. When $\mu \in L$ both sides of equation (4) vanish since $T|L = id_L$. Suppose now $\mu \in L^\perp$. When $\nu \in L$ both sides of the equation vanish by the definition of $Q$, and when $\nu \in L^\perp$ the equation reduces to

$$\Omega^{1/2}(\mu + T\mu, \nu) = SQ^{-1}(\mu - T\mu, \nu),$$

which holds by equation (EQ–S). The check on the right is similar. \qed

Lemma 3.8. The elements $J_L$ and $J_R$ from Lemma 3.7 are equal. Rather, there is a unique simultaneous solution $J$ to both the left and right 2-component ABRR equations in $S^{-1}\Omega_{L}^{-1/2} + L \otimes 1$.

Proof. One shows that the operators $A_L^2$ and $A_R^2$ commute, then proceeds as in [14, Cor. 4.1]. \qed

We find now

Lemma 3.9 (cf. [14, Prop. 3.3]). Our proposed twist $J_{T,S}$ solves both the left and right 2-component ABRR equations.

Proof. Let $J$ denote the solution from Lemma 3.8. We have $J = B + J_+$, where $B = S^{-1}\Omega_{L}^{-1/2}$ and $J_+ \in L \otimes 1$. From the appearance of $T_+$ in $A_L^2$, and the fact that $J = A_L^2(J)$, we have

$$J = (A_L^2)^n(J) = (A_L^2)^n(B) + (A_L^2)^n(J_+) = (A_L^2)^n(B),$$

where $n$ is the nilpotence degree of $T_+$. One establishes the equality

$$(A_L^2)^k(B) = (T_+ \otimes 1)(R) \ldots (T_+^k \otimes 1)(R)B(T_+ \otimes 1)(R)^{-1} \ldots (T \otimes 1)(R)^{-1}$$

by induction on $k$, using the fact that $(T \otimes 1)(\Omega B Q)Q^{-1}\Omega_{L}^{-1} = B$. This gives $(A_L^2)^n(B) = J_{T,S}$. \qed
3.3. The 3-component and mixed ABRR equations. For any element \( \xi \in u_q \otimes u_q \) take \( \xi_{12,3} = (\Delta \otimes 1)(\xi) \) and \( \xi_{1,23} = (1 \otimes \Delta)(\xi) \). So the dual cocycle equation for a twist now appears as \( J_{12,3}J_{12} = J_{1,23}J_{23} \), where \( J_{12} \) and \( J_{23} \) are \( J \otimes 1 \) and \( 1 \otimes J \) respectively.

**Definition 3.10.** Take \( A^3_L \) and \( A^3_R \) to be the linear endomorphisms of \( u_+ \otimes u_q \otimes u_- \) defined by

\[
A^3_L(\eta) = (T_+ \otimes 1 \otimes 1)(R_{13}R_{12}\eta Q_{12}Q_{13})Q^{-1}_{13}Q^{-1}_{12}(\Omega_L)^{-1}_{13}(\Omega_L)^{-1}_{12}
\]

\[
A^3_R(\eta) = (1 \otimes 1 \otimes T_-)(R_{13}R_{23}\eta Q_{13}Q_{23})Q^{-1}_{23}Q^{-1}_{13}(\Omega_L)^{-1}_{23}(\Omega_L)^{-1}_{13}.
\]

The left and right 3-component ABRR equations are the equations \( A^3_L(X) = X \) and \( A^3_R(X) = X \).

Let us fix \( J = J_{T,3} \).

**Lemma 3.11.** The elements \( J_{1,23}J_{23} \) and \( J_{12,3}J_{12} \) solve the left and right 3-component ABRR equations respectively.

**Proof.** Take \( T_1 = (T_+ \otimes 1 \otimes 1) \). We claim first that \( A^3_L(J_{1,23}J_{23}) = A^3_L(J_{1,23})J_{23} \).

Note that we may replace \( Q \) with \( \Sigma = s|L^\perp \times L \) in the equation for \( A^3_L \), and that for any bicharacter \( B \) we have \( B_{12}B_{13} = B_{23} \). Also recall that for any comodule element \( h \in H \) we will have \( R\Delta(h) = \Delta(h)R \), and that since \( \Sigma \in \mathbb{C}[L^\perp \times L] \) we will have \( \Sigma_{1,23} = (1 \otimes T_+ \otimes 1)(\Sigma_{1,23}) \) for any nonnegative integer \( k \). Using these facts together, along with the particular form of \( J = J_{T,3} \), one see that

\[
T_1(J_{23}\Sigma_{1,23})\Sigma^{-1}_{1,23}(\Omega_L)^{-1}_{1,23} = J_{23}(T_1(\Sigma))\Sigma^{-1}_{1,23}(\Omega_L)^{-1}_{1,23} = (T_1(\Sigma))\Sigma^{-1}_{1,23}(\Omega_L)^{-1}_{1,23}J_{23},
\]

which implies \( A^3_L(J_{1,23}J_{23}) = A^3_L(J_{1,23})J_{23} \). We now note that

\[
J_{1,23} = (1 \otimes \Delta)(J) = (1 \otimes \Delta)(A^3_L(J)) = A^3_L(J_{1,23})
\]

to see \( A^3_L(J_{1,23}J_{23}) = J_{1,23}J_{23} \). The equality \( A^3_R(J_{12,3}J_{12}) = J_{12,3}J_{12} \) is proved similarly. \( \square \)

As was the case for the 2-component equations, one finds that solutions to the equations \( A^3_L(X) = X \) and \( A^3_R(X) = X \) are uniquely determined by their components in \( \mathbb{C}[G] \otimes u_q \otimes u_- \) and \( u_- \otimes u_q \otimes \mathbb{C}[G] \) respectively. (See also [14, Lem. 4.3].) We also consider the mixed ABRR equation

\[
A^3_LA^3_R(X) = X.
\]

Solutions to this equation are uniquely determined by their component in \( \mathbb{C}[G] \otimes u_q \otimes \mathbb{C}[G] \), which we denote \( X_{0,0} \). Note that

\[
(J_{12,3}J_{12})_{0,0} = (J_{1,23}J_{23})_{0,0} = S_1^{-1}S_{13}^{-1}S_{23}^{-1}(\Omega_L^{-1/2})_{12}(\Omega_L^{-1/2})_{13}(\Omega_L^{-1/2})_{23}.
\]

(5)

So we would like to establish

**Proposition 3.12.** Both \( J_{12,3}J_{12} \) and \( J_{1,23}J_{23} \) solve the mixed ABRR equation \( A^3_LA^3_R(X) = X \).
From this proposition one easily finds the proof of Theorem 3.1. We only prove the proposition for \( J_{1,23} J_{23} \), the situation for \( J_{12,3} J_{12} \) being completely analogous. Let us first give some technical lemmas. Recall \( Z = S | \mathcal{L} \times \mathcal{L}^\perp \).

**Lemma 3.13.** The element \( Z \) solves the following equations:

1. \( (1 \otimes T^{-1})(S^{-1} \Omega^{-1/2}_L)(Z^{-1})Z = S^{-1} \Omega^{-1/2}_L(1 \otimes T^{-1})(Z^{-1})Z \).
2. \( [(\Omega_L)_{13} (1 \otimes 1 \otimes T^{-1})(Z^{-1})_{13}](1 \otimes 1 \otimes T^k)(R_{23})] = 0 \) for all \( k \geq 1 \).

**Proof.** Equation (i) is equivalent to the equation

\[
\Omega^{-1/2}_L(1 \otimes T^{-1})(\Omega^{-1/2}_L) = S^{-1}(1 \otimes T^{-1})(S)(1 \otimes T^{-1})(Z^{-1})Z,
\]

which is seen to hold by (EQ–S), just as in the proof of Lemma 3.7. For (ii) first note that for any \( \mu, \nu \in \mathcal{G} \) we have

\[
(\Omega_L(1 \otimes T^{-1})(Z^{-1})Z)(\mu, \nu) = (\Omega(\bar{\mu}, \nu)\bar{Z}(\mu, \nu - T\nu) = (\Omega(\bar{\mu}, \nu)S(\mu, \nu - T\nu) = (\Omega(\bar{\mu}, \nu)\Omega^{-1/2}(\bar{\mu}, \nu + T\nu) = (\Omega(\bar{\mu}, \nu)\Omega^{-1}(\bar{\mu}, \nu) = 1,
\]

where \( \bar{\mu} \) is the component of \( \mu \) in \( \mathcal{L} \) under the decomposition \( \mathcal{G} = \mathcal{L} \times \mathcal{L}^\perp \). So we see that the bicharacter in question vanishes on \( \mathcal{G} \times \mathcal{G}_1 \), and hence

\[
\Omega_L(1 \otimes T^{-1})(Z^{-1})Z \in \mathbb{C}[G] \otimes \mathbb{C}[G^1].
\]

It follows that all elements in \( \mathbb{C} \otimes u_q \otimes (\mathbb{C}(\mathcal{G}, F_\beta : \beta \in \Gamma_1)) \) centralize \( (\Omega_L)_{13} (1 \otimes 1 \otimes T^{-1})(Z^{-1})_{13} \). Since \( (1 \otimes 1 \otimes T^k)(R_{23}) \) is in this subspace we have (ii). \( \square \)

We can now give the

**Proof of Proposition 3.12.** As noted above, we only prove that \( J_{1,23} J_{23} \) solves the mixed ABRR equation. Since this element already satisfies \( A^2_L(X) = X \) it suffices to show that it also solves \( A^3_R(X) = X \). As in [14, Lem. 4.2], one checks that \( A^2_L \) and \( A^3_R \) commute so that \( A^3_R(J_{1,23} J_{23}) \) solves the left ABRR equation. By uniqueness of solutions we find that \( A^3_R(J_{1,23} J_{23}) = J_{1,23} J_{23} \) if and only if these elements have the same component in \( \mathbb{C}[G] \otimes u_q \otimes u_\perp \). Let \( A^3_R(J_{1,23} J_{23})_0 \) and \( (J_{1,23} J_{23})_0 \) denote these components. Take \( (T^-)_3 = (1 \otimes 1 \otimes T^-) \) and \( B = S^{-1} \Omega^{-1/2}_L \).

Since \( J \) is in the subalgebra \( \mathbb{C}(\mathcal{G} \times G, E_\alpha \otimes F_\beta : \alpha, \beta \in \Gamma) \) we see that \( (J_{1,23} J_{23})_0 = B_{1,23} J_{23} \), and we need to establish

\[
A^3_R(J_{1,23} J_{23})_0 = B_{1,23} J_{23}.
\]

We have

\[
A^3_R(J_{1,23} J_{23})_0 = (T^-)_3(\Omega_{13} R_{23} B_{1,23} J_{23} Z_{12,3}^{-1}(\Omega_L)_{12,3}^{-1})Z_{12,3}^{-1}(\Omega_L)_{12,3}^{-1}.
\]
Use the equality $B\Omega_{L\perp} = S^{-1}\Omega_{L\perp}^{1/2}$ and Lemma 3.13 (i) to get

$$A^3_R(J_{1,23}J_{23})_0 = (T_-)_3((\Omega_L)_{13}B_{12}B_{13}(\Omega_{L\perp})_{13}R_{23}J_{23}Z_{12,3})Z_{12,3}^{-1}(\Omega_L)_{12,3}^{-1} = (\Omega_L)_{13}B_{12}(T_-)_3(B_{13}(\Omega_{L\perp})_{13}R_{23}J_{23}Z_{12,3})Z_{12,3}^{-1}(\Omega_L)_{12,3}^{-1} = (\Omega_L)_{13}B_{12}B_{13}(T_-)_3(Z_{13}^{-1})Z_{13}(T_-)_3(R_{23}J_{23}Z_{12,3})Z_{12,3}^{-1}(\Omega_L)_{12,3}^{-1}.$$ 

Since $J$ solves the 2-component ABRR equations this final expression reduces to

$$A^3_R(J_{1,23}J_{23})_0 = (\Omega_L)_{13}B_{12,3}(T_-)_3(Z_{13}^{-1})Z_{13}(T_-)_3(Z_{13}^{-1}(\Omega_L)_{13}^{-1}.$$ 

By Lemma 3.13 (ii) this final equation reduces to the desired equality

$$A^3_R(J_{1,23}J_{23})_0 = B_{12,3}J_{23} = (J_{1,23}J_{23})_0.$$

This implies that $J_{1,23}J_{23}$ solves the right 3-component ABRR equation, and hence the mixed ABRR equation $A^3_LA^3_R(J_{1,23}J_{23}) = J_{1,23}J_{23}.$

**Proof of Theorem 3.1.** By uniqueness of solutions to the mixed ABRR equation, Proposition 3.12, and (5), we see that $J_{12,3}J_{12} = J_{1,23}J_{23}$. The remaining identity $(\epsilon \otimes 1)(J) = (1 \otimes \epsilon)(J) = 1$ follows from the identity $(\epsilon \otimes 1)(R) = (1 \otimes \epsilon)(R) = 1$ and the fact that $\epsilon$ commutes with $T_\pm$. □

### 4. Subalgebras from the $R$-matrix

We recall here some information from [24]. We will let $D(H)$ denote the Drinfeld double of a Hopf algebra $H$. Recall that this is a quasitriangular Hopf algebra which, as a coalgebra, is simply the tensor coalgebra $D(H) = H \otimes (H^*)^{\text{cop}}$. Recall also that the two inclusions $H \to D(H)$ and $(H^*)^{\text{cop}} \to D(H)$ are Hopf algebra maps. This is all the information we will need about the Drinfeld double, and we invite the reader to see [21, Sect. 10.3] for more information.

#### 4.1. The right and left subalgebras from $R$.

Let $H = (H, R)$ be a quasitriangular Hopf algebra. We can consider for any $Q \subseteq H \otimes H$ the functions $t_Q : H^* \to H, f \mapsto (f \otimes 1)(Q)$ and $t'_Q : H^* \to H, f \mapsto (1 \otimes f)(Q)$. Indeed, for any $H_1, H_2 \subset H$ with $Q \in H_1 \otimes H_2$ we can restrict these functions to $t_Q : H_1^* \to H_2, t'_Q : H_2^* \to H_1$. For the $R$-matrix we have the right and left subspaces in $H$ defined as follows.

**Definition 4.1.** For a quasitriangular Hopf algebra $H = (H, R)$ we take $R_{(r)} = t_R(H^*)$ and $R_{(l)} = t'_R(H^*)$.

We may refer to these subalgebras as the Radford subalgebras associated to $R$. Using the properties of the $R$-matrix one shows

**Proposition 4.2 ([24, Prop. 2]).** The subspaces $R_{(l)}$ and $R_{(r)}$ are Hopf subalgebras in $H$. Furthermore, the maps $t_R$ and $t'_R$ provide Hopf morphisms $(H^*)^{\text{cop}} \to H$ and $(H^*)^{\text{op}} \to H$, and Hopf isomorphisms $(R_{(l)})^{\text{cop}} \cong R_{(r)}$ and $(R_{(r)})^{\text{op}} \cong R_{(l)}$. 
Take $H_R = R(l)R(r)$. It turns out that this is a Hopf subalgebra in $H$, and that it is the minimal Hopf subalgebra in $H$ with $R \in H_R \otimes H_R$. A quasitriangular Hopf algebra is called \textit{minimal} if $H = H_R$. Strictly speaking, we will not be needing the following result. It does, however, inform the approach of the current work, and so we repeat it here.

\textbf{Theorem 4.3 ([24, Thm. 2])}. For a minimal Hopf algebra $H$ there is a (unique) surjective map of quasitriangular Hopf algebras $Y : D(R(l)) \to H$ with $Y|_{R(r)}$ the inclusion and $Y|(R^*_l)^{\text{cop}} = t_R$.

Taking the dual of $Y$, we see that there is a algebra inclusion

\[ H^* \to R^*_l \otimes R^*_l \cong R(r) \otimes R^*_l \]

given as the composite $H^* \xrightarrow{\Delta} H^* \otimes H^* \xrightarrow{t_R \otimes t_R} R(r) \otimes R^*_l$.

We note that although minimality is not preserved under twists, a stronger condition called factorizability is preserved under twists. Indeed, a finite dimensional quasitriangular Hopf algebra $H$ is factorizable if and only if the M"uger center of $\text{rep}(H)$ is trivial [15]. Small quantum groups are examples of factorizable Hopf algebras, and so the twists $u^J_q$ will be factorizable and hence minimal.

4.2. Bicharacters as twists on abelian groups.

\textbf{Lemma 4.4.} Let $\Lambda$ be a finite abelian group. Any bicharacter $B \in \mathbb{C}[\Lambda] \otimes \mathbb{C}[\Lambda]$ is an $R$-matrix for $\mathbb{C}[\Lambda]$.

\textbf{Proof.} We need to check the equations $(\Delta \otimes 1)(B) = B_{13}B_{23}$, $(1 \otimes \Delta)(B) = B_{13}B_{12}$, and $B\Delta(\lambda)B^{-1} = \Delta(\lambda)$ for each $\lambda \in \Lambda$. The first two equations follow from the fact that $B$ is a bicharacter, and the final equation follows from the fact that $\Lambda$ is abelian. \hfill \Box

In the case of a bicharacter $B$ giving an $R$-matrix for $\mathbb{C}[\Lambda]$, the two maps $t_B$ and $t'_B$ restrict to, and are specified by, the standard group maps $\Lambda^\vee \to \Lambda$ induced by $B$.

\textbf{Definition 4.5.} Given a bicharacter $B \in \mathbb{C}[\Lambda] \otimes \mathbb{C}[\Lambda]$, for $\Lambda$ a finite abelian group, let $\Lambda_{(r)}$ and $\Lambda_{(l)}$ denote the images $t_B(\Lambda^\vee)$ and $t'_B(\Lambda^\vee)$ in $\Lambda$ respectively.

We have $B_{(r)} = \mathbb{C}[\Lambda_{(r)}]$ and $B_{(l)} = \mathbb{C}[\Lambda_{(l)}]$.

5. Parabolic subalgebras in $u_q(\mathfrak{g})^J$ and Radford’s subalgebras

For this section fix a Belavin-Drinfeld triple $(\Gamma_1, \Gamma_2, T)$ and solution $S$ to (EQ–S). Fix also $J = J_{T,S}$ from Theorem 3.1.

We saw in Section 4.1 that there are algebra surjections $t_{R^J} : (u_q^J)^* \to R^J_{(r)}$ and $t'_{R^J} : (u_q^J)^* \to (R^J_{(l)})^\text{cop}$. Our main goal is to show that the map

\[ \text{Irrep}(R^J_{(r)}) \to \text{Irrep} ((u_q^J)^*) \]
induced by restriction is a bijection, modulo the action of a finite character group. Both to understand the irreps of \( R_J^{(r)} \) and to establish this proposed bijection we need to understand the subalgebra \( R_J^{(r)} \), which we study here.

For any root \( \alpha \) we will take \( \bar{\alpha} \) to be the component of \( \alpha \) in \( \mathcal{L} \), under the decomposition \( \mathcal{G} = \mathcal{L} \times \mathcal{L}^\perp \).

5.1. **A preemptive change of coordinates.** Let us take

\[
E_\alpha = q^{-\frac{1}{4}(\bar{\alpha},\bar{\alpha})} K_\bar{\alpha}^{1/2} E_\alpha \quad \text{and} \quad F_\beta = q^{-\frac{1}{4}(\bar{\beta},\bar{\beta})} K_\bar{\beta}^{-1/2} F_\beta.
\]

These new generators satisfy the appropriate relations so that we have an algebra automorphism

\[
\text{change of coord's: } u_q \xrightarrow{2\pi i} u_q, \quad \begin{cases} E_\alpha \mapsto E_\alpha \\ F_\beta \mapsto F_\beta \\ K_\mu \mapsto K_\mu. \end{cases}
\]

Recall that each \( E_\mu, \mu \in \Phi^+ \), is a linear combination of permutations of the monomial \( E_{\alpha_{i_1}} \cdots E_{\alpha_{i_m}} \), where \( \mu = \alpha_{i_1} + \cdots + \alpha_{i_m} \) with the \( \alpha_{i_k} \) simple. So each \( E_\mu \) is sent to \( q^{-\frac{1}{4}(\bar{\mu},\bar{\mu})} K_{\bar{\mu}}^{1/2} E_\mu \) under the above change of coordinates. A similar statement holds for the \( F_\nu \), and we may adopt a consistent notation

\[
E_\mu = q^{-\frac{1}{4}(\bar{\mu},\bar{\mu})} K_{\bar{\mu}}^{1/2} E_\mu \quad \text{and} \quad F_\nu = q^{-\frac{1}{4}(\bar{\nu},\bar{\nu})} K_{\bar{\nu}}^{-1/2} F_\nu,
\]

for \( \mu, \nu \in \Phi^+ \). These bold elements produce a \( \mathbb{C}[G] \)-basis for \( u_q = u_q^\Gamma \) just as in Lemma 1.1.

5.2. **The quantum parabolics in \( u_q(\mathfrak{g})^f \) and the \( R \)-matrix.** For a fixed subset \( \Sigma \subset \Gamma \) we let \( p_+ = p_+(\Sigma) \) denote the corresponding positive parabolic in \( \mathfrak{g} \) and \( u_q(p_+) \) denote the Hopf subalgebra

\[
u_q(p_+) = \mathbb{C}\langle G, E_\alpha, F_\beta : \alpha \in \Gamma, \beta \in \Sigma \rangle \subset u_q(\mathfrak{g}).
\]

We have the negative analog

\[
u_q(p_-) = \mathbb{C}\langle G, E_\beta, F_\alpha : \beta \in \Sigma, \alpha \in \Gamma \rangle \subset u_q(\mathfrak{g}).
\]

We let \( u_q(p^{ss}) \) denote the small quantum group associated to the (union of) Dynkin diagram(s) \( \Sigma \) in \( \Gamma \), and suppose that the perpendicular \( G^{\perp}_\Sigma \) to the subgroup \( G_\Sigma = \mathbb{Z}/l\mathbb{Z} \cdot \{K_\beta : \beta \in \Sigma\} \) in \( G \) is a complement to \( G_\Sigma \).

**Lemma 5.1.** Let \( \Sigma \) be a subset in \( \Gamma \) and \( p \) denote the corresponding positive (resp. negative) parabolic. There is an algebra surjection

\[
u_q(p) \twoheadrightarrow \mathbb{C}[G^{\perp}_\Sigma] \otimes u_q(p^{ss}), \quad \begin{cases} E_\beta \mapsto E_\beta & \text{when } \beta \in \Sigma \\ F_\beta \mapsto F_\beta & \text{when } \beta \in \Sigma \\ E_\alpha : \alpha \in \Gamma \setminus \Sigma, \quad F_\alpha : \alpha \in \Gamma \setminus \Sigma \end{cases} (6)
\]

with kernel equal to the nilpotent ideal \( N = \langle E_\alpha : \alpha \in \Gamma \setminus \Sigma \rangle \) (resp. \( N' = \langle F_\alpha : \alpha \in \Gamma \setminus \Sigma \rangle \)).
Proof. It suffices to prove the result for the positive parabolic. We arrive at the result for the negative quantum parabolic by considering the automorphism of \( u_q(g) \) which exchanges the \( E_\alpha \) and \( F_\alpha \), and inverts the \( K_\gamma \), and hence exchanges the positive and negative parabolics. Simply by checking relations we see that there is a surjective algebra map \( \mathbb{C}[G_{S_k}^\vee] \otimes u_q(p^{ss}) \to u_q(p)/N \) defined on the generators in the obvious way. We will show that this map is injective by counting dimensions.

We have that the nonnegative part of \( u_q(p) \) is all of \( u_+ \), and we see for grading reasons that \( u_q(p)_- \) is free over \( \mathbb{C}[G] \) with basis given by ordered monomials in \( E \). By the commutativity relation between the \( E \) and \( F \) we see that the restriction of the multiplication map \( \theta : u(p)_+ \otimes \mathbb{C}[G] u(p)_- \to u_q(p) \) is surjective. Since this map is given by restricting the isomorphism \( u_q(g)_+ \otimes \mathbb{C}[G] u_q(g)_- \to u_q(g) \), and since all modules are flat over \( \mathbb{C}[G] \), we see that \( \theta \) is injective as well. It follows that \( u_q(p) \) has the obvious basis consisting of ordered monomials in \( E\nu \) and \( F\nu \), where \( \nu \) is as above.

When we take the quotient we now see that \( u_q(p)/N \) has a \( \mathbb{C}[G] \)-basis of ordered monomials in the \( E\nu' \) and \( F\nu \), with \( \nu' \in \Phi^+ \cap (\mathbb{Z} \cdot \Sigma) \). Since \( \Psi = \Phi \cap (\mathbb{Z} \cdot \Sigma) \) is the root system for \( p^{ss} \), we find by Lusztig’s basis for \( u_q(p^{ss}) \) that \( u_q(p)/N \) and \( \mathbb{C}[G]_{S_k}^\vee \otimes u_q(p^{ss}) \) have the same dimension. Whence our surjection is an isomorphism. The inverse is given by the same formulas as \( (6) \), and implies the existence of \( (6) \).

As for nilpotence of \( N \), when we grade by the group \( \mathbb{Z}\{\Gamma - \Sigma\} \) we see that \( N^k \) lay in degrees \( \mathbb{Z}_{\geq k}\{\Gamma - \Sigma\} \). Since \( u_q(p) \) is finite dimensional it has no nonzero elements in degrees \( \mathbb{Z}_{\geq k}\{\Gamma - \Sigma\} \) for large \( k \). \( \square \)

5.3. Quantum parabolics and BD triples.

Definition 5.2. For any Belavin-Drinfeld triple \( (\Gamma_1, \Gamma_2, T) \) we let \( u_q(p_1) \) and \( u_q(p_2) \) denote the positive and negative quantum parabolics in \( u_q(g) \) corresponding to \( \Gamma_1 \) and \( \Gamma_2 \) respectively.

So \( u_q(p_1) \) contains only the \( E_\alpha \) with \( \alpha \in \Gamma_1 \), and \( u_q(p_2) \) contains only the \( E_\beta \) with \( \beta \in \Gamma_2 \). Recall our twist \( J = J_{T,S} \) and the definition \( R^J = J_{21}^{-1} R J \). We have also

\[
J_{21}^{-1} = (1 \otimes T)(\Omega) \cdots (1 \otimes T^n)(\Omega) S^{-1} T_{L_+}^{1/2}(S^{-1} \otimes T_+)(R_{21}) \cdots (S^{-1} \otimes T_+)(R_{21})
\]

\[
= (T^{-1} \otimes 1)(\Omega) \cdots (T^{-n} \otimes 1)(\Omega) S^{-1} T_{L_+}^{1/2}(T_+ \otimes S)(R_{21}) \cdots (T_+ \otimes S)(R_{21}).
\]

(7)

Lemma 5.3. There are containments \( R^J_{(i)} \subseteq u_q(p_2) \) and \( R^J_{(i')} \subseteq u_q(p_2) \).

Proof. This is immediate from the form of \( J \) and \( R \), and the fact that \( (T^k_+ \otimes 1)(R) = (1 \otimes T^k)(R) \). \( \square \)

We consider \( \mathbb{C}[G] \) as a quasitriangular Hopf algebra with \( R \)-matrix \( \Omega \). Then \( S^{-1} \) provides a twist for \( \mathbb{C}[G] \) and the new \( R \)-matrix \( S_{21} \Omega S^{-1} = S^{-2} \Omega \).
We take $G_{(r)}$ and $G_{(l)}$ the right and left subgroups associated to $S^{-2}\Omega$, as in Section 4.2. Note that by the duality $t_{\Omega^{-1}}: G_{(l)}^{\vee} \xrightarrow{\cong} G_{(r)}$ we know that these two groups have the same order. We want to prove

**Proposition 5.4.** The inclusions $R_{(r)}^J \subset u_q(p_2)$ and $R_{(l)}^J \subset u_q(p_1)$ are equalities exactly when $G_{(r)} = G_{(l)} = G$. In general, we have that

$$R_{(r)}^J = C(G_{(r)}, E_\beta, F_\gamma : \beta \in \Gamma_2, \gamma \in \Gamma)$$

and

$$R_{(l)}^J = C(G_{(l)}, E_\gamma, F_\alpha : \alpha \in \Gamma_1, \gamma \in \Gamma)$$

in $u_q(g)$.

Section 6 is dedicated to a proof of Proposition 5.4. As a corollary we will have

**Corollary 5.5.** Let $N \subset R_{(r)}^J$ denote the preimage of the ideal $N = (F_\beta : \beta \in \Gamma - \Gamma_2)$ in $u_q(p_2)$ along the inclusion $R_{(r)}^J \to u_q(p_2)$. Take also $\Lambda = G_{(r)} \cap G_{(l)}^\perp$. Then we have a canonical algebra isomorphism

$$R_{(r)}^J/N \xrightarrow{\cong} \mathbb{C}[\Lambda] \otimes u_q(p_2^{ss}).$$

For the analogously defined $N' \subset R_{(l)}^J$ and $N \subset G_{(l)}$ we have also

$$R_{(l)}^J/N' \xrightarrow{\cong} \mathbb{C}[\Lambda'] \otimes u_q(p_1^{ss}).$$

**Proof.** The isomorphisms come from restricting the isomorphisms of Lemma 5.1 along the inclusion $R_{(s)}^J \to u_q(p_*)$. \hfill \Box

### 5.4. An example.

It seems, from considering examples, that the subalgebra $R_{(r)}^J$ will often be the full parabolic $u_q(p_2)$. This will always be the case, for example, when considering twists associated to maximal triples $(\Gamma_1, \Gamma_2, T)$ on $A_3$ (see the discussion following Lemma 3.2). To construct an example in which the containment $R_{(r)}^J \subset u_q(p_2)$ is proper we need only construct an example for which the containment $G_{(r)} \subset G$ is proper.

We claim that in the following example we will have $G_{(r)} \subset G$ and $R_{(r)}^J \subset u_q(p_2)$: Take $l = 3$ and consider the tiple on $A_3$

$$\alpha_1 \quad \bullet \quad \alpha_2 \quad \bullet \quad \alpha_3$$

with $\Gamma_1 = \{\alpha_1\}$, $\Gamma_2 = \{\alpha_3\}$, $T(\alpha_1) = \alpha_3$. We have here

$$L = \mathbb{Z}/3\mathbb{Z} \cdot \{\alpha_2, \alpha_1 + \alpha_3\}, \quad G_{(r)}^\perp = \mathbb{Z}/3\mathbb{Z} \cdot \{\alpha_1, \alpha_2 + \frac{1}{2}\alpha_3\}$$

and the form on $G_{(r)}^\perp$ is given (in multiplicative notation) by

$$(\alpha_1, \alpha_2) = q^2, \quad (\alpha_1, \alpha_2 + \frac{1}{2}\alpha_3) = q^{-1}, \quad (\alpha_2 + \frac{1}{2}\alpha_3, \alpha_2 + \frac{1}{2}\alpha_3) = 1.$$
Since 2 is a unit in \( \mathbb{Z} \) and hence splits as \( G \). Note that \( S \) exchanges \( E \) be deduced from the fact that the algebra automorphism \( \Gamma \) fixes with \( \gamma \). We have the \( (\gamma, \) with \( \gamma \) and sends \( K_\gamma \) to \( K_\gamma^{-1} \), is such that \( \phi(R_r^{J}) = R_r^{J^\prime}. \)

### 6. A Proof of Proposition 5.4

Fix \( J = J_{T,S} \). We will prove by direct calculation that \( R_r^{J} \) is as proposed in Proposition 5.4. Let \( J^\prime \) be the twist associated to the triple \( \Gamma_1^r = \Gamma_2, \Gamma_2^r = \Gamma_1, T^r = T^{-1} \), and solution \( S^r = S^{-1} \). The result for \( R_r^{J} \) can be subsequently be deduced from the fact that the algebra automorphism \( \phi : u_q \to u_q \), which exchanges \( E_\alpha \) with \( F_\alpha \) and sends \( K_\gamma \) to \( K_\gamma^{-1} \), is such that \( \phi(R_r^{J}) = R_r^{J^\prime}. \)

6.1. Some supporting results.

**Lemma 6.1.** \( G_r \subset R_r^{J} \) and \( G_2 \subset G_r. \)

*Proof.* We have the \((u_-, u_+)\)-bimodule isomorphism \( u_- \otimes_{\mathbb{C}[G]} u_+ \to u_q \) given by multiplication. The two projections \( u_+ \to \mathbb{C}[G] \otimes \mathbb{C}[G] \) and hence a bimodule projection \( \Pi : u_q \to \mathbb{C}[G]. \) This gives an embedding \( \Pi^* : \mathbb{C}[G]^* \to u_q. \) We have

\[
(\Pi \otimes 1)(R_r^{J}) = S^{-1} \Omega_{L_+}^{1/2} \Omega S^{-1} \Omega_{L_-}^{-1/2} = S^{-2} \Omega \in \mathbb{C}[G] \otimes u_q.
\]

Note that \( S^{-2} \Omega = (S_2^{-1})^{-1} \Omega S^{-1} \) so that for any character \( \mu \in \mathbb{G} \) we have

\[
(\mu \Pi \otimes 1)(R_r^{J}) = t_{\Omega^{\mu^{-1}}}(\mu)
\]

and hence \( t_{R_r^{J}(\mu \Pi)} = G_r \). This gives the proposed inclusion \( G_r \subset R_r^{J}. \)

As for the inclusion \( G_2 \subset G_r \) note that for \( \alpha \in \Gamma_1 \) we have

\[
S^{-2} \Omega(\alpha - T_\alpha, ?) = \Omega^{-1}(\alpha - T_\alpha, ?) \Omega(\alpha - T_\alpha, ?) = \Omega^{-2}(T_\alpha, ?) = K_{-T_\alpha}^{-2}.
\]

Since 2 is a unit in \( \mathbb{Z}/l\mathbb{Z} \) we see that each \( K_\alpha \in G_r \) and hence \( G_2 \subset G_r. \)

The inclusion \( G_2 \subset G_r \) and splitting \( G = G_2^\perp \times G_2 \) implies that \( G_r \) splits as \( G_r = \Lambda \times G_2 \), where \( \Lambda = G_2^\perp \cap G_r \).

In the following lemma we use the fact that for any bicharacter \( B \in \mathbb{C}[G] \otimes \mathbb{C}[G] \) we have

\[
B = \sum_{\mu, \nu \in \mathbb{G}} B(\mu, \nu)P_\mu \otimes P_\nu,
\]
where \( P_\mu = |G|^{-1} \sum_{\gamma \in G} \mu(K^{-1}_\gamma)K_\gamma \) is the idempotent associated to \( \mu \). Note that \( P_\mu P_\nu = \delta_{\mu,\nu} P_\mu \) and \( \mu(P_\nu) = \delta_{\mu,\nu} \). For any bicharacter \( B \) and \( \mu \in \mathcal{G} \) we take

\[
B(\mu) = \text{the unique element in } \mathcal{G} \text{ with } B(\mu, \nu) = \Omega(B(\mu), \nu) \forall \nu \in \mathcal{G}.
\]

**Lemma 6.2.** For any bicharacter \( B \), and \( \alpha, \beta \in \Gamma \), we have

\[
(E_\alpha \otimes F_\beta)B = B(K_{B21(\beta)} \otimes K_{B^{-1}(\alpha)})(E_\alpha \otimes F_\beta)
\]

and

\[
(F_\beta \otimes E_\alpha)B = B(K_{B21^{-1}(\alpha)} \otimes K_{B(\beta)})(F_\beta \otimes E_\alpha).
\]

**Proof.** We have

\[
E_\alpha K_\gamma = q^{-(\alpha, \gamma)}K_\gamma E_\alpha \Rightarrow E_\alpha P_\mu = P_{\mu + \alpha}E_\alpha
\]

and \( F_\beta P_\nu = P_{\nu - \beta}F_\beta \). So for any bicharacter \( B \) we have

\[
(E_\alpha \otimes F_\beta)B = (\sum_{\mu, \nu} B(\mu, \nu)P_{\mu + \alpha} \otimes P_{\nu - \beta})(E_\alpha \otimes F_\beta)
\]

\[
= (\sum_{\mu, \nu} B(\mu - \alpha, \nu + \beta)P_{\mu} \otimes P_{\nu})(E_\alpha \otimes F_\beta)
\]

\[
= B(\sum_{\mu, \nu} B(\mu, \beta)B^{-1}(\alpha, \nu)P_{\mu} \otimes P_{\nu})(E_\alpha \otimes F_\beta)
\]

\[
= B(K_{B21(\beta)} \otimes K_{B^{-1}(\alpha)})(E_\alpha \otimes F_\beta).
\]

We arrive at the equation for \( F_\beta \otimes E_\alpha \) similarly. \( \square \)

Considering the case \( B = \Omega_L^{1/2} \), for each \( \beta \in \Gamma_2 \) we have

\[
(E_\beta \otimes F_{T^{-k}\beta})\Omega_L^{1/2} = \Omega_L^{1/2}(K_{T^{-k}\beta}^{1/2}E_\beta \otimes K_{T^{-k}\beta}^{-1/2}F_{T^{-k}\beta})
\]

\[
= \Omega_L^{1/2}(K_{\beta}^{1/2}E_\beta \otimes K_{\beta}^{-1/2}F_{\beta})
\]

\[
= q^{1/2(\beta, \beta)}\Omega_L^{1/2}(E_\beta \otimes F_{\beta}).
\]

Similarly \( (F_\alpha \otimes E_{T^k\alpha})\Omega_L^{1/2} = q^{1/2(\alpha, \alpha)}\Omega_L^{1/2}(F_\alpha \otimes E_{T^k\alpha}) \) for \( \alpha \in \Gamma_1 \).

6.2. **Proof of Proposition 5.4.** As explained in the beginning of the section, we need only prove the proposition for \( R^J_{(\nu)} \). We prove the proposition in two parts. First we establish the containment \( C\langle G(\nu), E_\alpha, F_\beta : \alpha \in \Gamma_2, \beta \in \Gamma \rangle \subset R^J_{(\nu)} \), then we establish the opposite containment.

**Proof of Proposition 5.4. Part I:** Take

\[
\Omega(k, m) = \prod_{k \leq i \leq m} (T^i \otimes 1)(\Omega) \quad \text{and} \quad \Omega'(k, m) = \prod_{k \leq i \leq m} (1 \otimes T^i)(\Omega),
\]

with the empty product equal to 1. Now \( J \) appears as

\[
(1 \otimes T_-)(R) \ldots (1 \otimes T_-^m)(R)S^{-1}\Omega_L^{-1/2}\Omega(1, n)^{-1}
\]

and \( J_{21}^{-1} \) appears as

\[
\Omega'(1, n)S^{-1}\Omega_L'^{-1/2}(S^{-1} \otimes T_+^m)(R_{21}) \ldots (S^{-1} \otimes T_+)(R_{21}).
\]

It suffices to prove that each of the \( E_\alpha \) and \( F_\beta \) are in \( R^J_{(\nu)} \), by Lemma 6.1.
From our $\mathbb{C}[G]$-basis for $u_q$ we have the $\mathbb{C}[G]$-linear projection
\[
\pi^{F}_{\beta} : u_q \to \mathbb{C}[G]E_{\beta}
\]
which annihilates each of the basis elements from Theorem 1.1, save for $E_{\beta}$. More specifically, we take $\pi^{F}_{\beta}$ to be the obvious projection composed with the scaling by $q(1 - q^2)^{-1}$. Then we have
\[
(\pi^{F}_{\beta} \otimes 1)(R^J) = \sum_{k=0}^{m(\beta)} S^{-1/2} \Omega L_1 \Omega(0, k-1)(E_{\beta} \otimes F_{T^{-k}\beta}) \Omega(0, n) - 1 S^{-1/2} \Omega 1(n)^{-1}
\]
\[
= \sum_{k} S^{-1/2} \Omega L_1 \Omega(0, k-1)(E_{\beta} \otimes F_{T^{-k}\beta}) \Omega(1, k-1)^{-1} S^{-1} \Omega L_1^{-1/2}
\]
\[
= \sum_{k} S^{-1} \Omega L_1^{-1/2} \Omega(0, k-1)(E_{\beta} \otimes F_{T^{-k}\beta}) \Omega(0, k-1)^{-1} S^{-1/2} \Omega L_1^{1/2},
\]
where $m(\beta) = 0$ when $\beta \notin \Gamma_2$ and otherwise $m(\beta)$ is minimal with $T^{-m(\beta)} \beta \notin \Gamma_2$ and $T^{-i}\beta \in \Gamma_2$ for $0 \leq i < m(\beta)$. We have
\[
\Omega(0, k-1)_{2i}(T^{-k}\beta) = - \frac{1}{k} \sum_{i=1}^{k} T^{-i}\beta \quad \text{and} \quad \Omega(0, k-1)(\alpha) = \sum_{j=0}^{k-1} T^{j}\beta
\]
so that the final expression reduces to
\[
\sum_{k} S^{-1} \Omega L_1^{-1/2} (K_{T^{-i}\beta} \otimes K_{T^{-j}\beta})(E_{\beta} \otimes F_{T^{-k}\beta}) S^{-1/2} \Omega L_1^{1/2}
\]
\[
= \sum_{k} S^{-2} \Omega L_1 \Omega L_2^{-1/2} (K_{T^{-1}\beta} \otimes K_{T^{-2}\beta} \gamma (K_{T^{-1}\beta} \otimes K_{T^{-2}\beta} \gamma ) \Omega L_2^{-1/2}
\]
\[
= q^{1/2}(\beta, \beta) \sum_{k} S^{-2} \Omega (K_{T^{-1}\beta} \otimes K_{T^{-2}\beta} \gamma ) \Omega L_2^{-1/2}
\]
For $\epsilon : \mathbb{C}[G]E_{\beta} \to \mathbb{C}$, $gE_{\beta} \mapsto q^{-1/2}(\beta, \beta)$, we then have
\[
(\epsilon \pi^{F}_{\beta} \otimes 1)(R^J) = \sum_{k=0}^{m(\beta)} K_{\Omega(\beta)}^{-1/2} K_{\sum_{j=0}^{k-1} T^{j}\beta} F_{T^{-k}\beta} \in R^J(\tau), \quad (8)
\]
Note that the coefficients $K_{\Omega(\beta)}^{-1/2} K_{\sum_{j=0}^{k-1} T^{j}\beta}$ are all in $G(\tau)$.

When $m(\beta) = 0$, i.e. when $\beta \in \Gamma - \Gamma_2$, the sum (8) is just the element $K_{\Omega(\beta)}^{-1/2} F_{\beta}$. Since $K_{\Omega(\beta)}^{-1/2} \in G(\tau) \subset R^J(\tau)$ this implies $F_{\beta} \in R^J(\tau)$. Since $m(\beta) = m(T^{-1}\beta) + 1$ when $\beta \in \Gamma_2$, it now follows from (8) and induction on $m(\beta)$ that all $F_{\beta} \in R^J(\tau)$.

The computation for the $E_{\beta}$, $\beta \in \Gamma_2$, is quite similar. Namely, one show for $\alpha \in \Gamma_1$ that $(\pi^{F}_{\alpha} \otimes 1)(R^J)$ is the a sum
\[
q^{1/2}(\alpha, \beta) \sum_{k=1}^{m(\alpha)} S^{-2} \Omega (g_k \otimes K_{\Omega(\alpha)}^{-1/2} K_{T^{\alpha} + \sum_{j=1}^{k} T^{j}\alpha}) \Omega(\alpha) \otimes E_{T^{k}\alpha},
\]
where $\pi^{F}_{\alpha}$ is a scaling of the obvious projection and $g_k \in G$, then proceeds by induction on $m(\alpha)$ as just as above.

Part II: We now give the opposite containment $R^J(\tau) \subset \mathbb{C}(G(\tau), E_{\alpha}, F_{\beta} : \alpha \in \Gamma_2, \beta \in \Gamma)$ to complete the proof. We adopt the same notation $\Omega(k, m)$.
and $\Omega'(k, m)$ as above. We have that $R^J$ is a $\mathbb{C}[G] \otimes \mathbb{C}[G_{(r)}]$-linear combination of elements of the form $M_1M_2M_3$ with

$$M_1 = \Omega'(1, n)S^{-1}L^{1/2}(F_{\xi_i} \otimes E_{T^{n_{\xi_i}}})(1 \otimes T^n)(\Omega)^{-1} \ldots (F_{\xi_1} \otimes E_{T^{n_{\xi_1}}})(1 \otimes T)(\Omega)^{-1},$$

$$M_2 = (E_{\zeta} \otimes F_{\zeta})\Omega,$$

$$M_3 = (E_{n_i} \otimes F_{T^{-1}n_i})(T \otimes 1)(\Omega) \ldots (E_{n_n} \otimes F_{T^{-1}n_n})\Omega(0, n - 1)^{-1}S^{-1}L^{1/2}\Omega^{1/2}.$$  

Here the $\xi_k$ are in $\mathbb{Z}_{\geq 0}\Gamma_1$ with $T^i(\xi_k) \in \mathbb{Z}_{\geq 0}\Gamma_1$ for each $0 \leq i < k$. We take a similar restriction for the $\eta_j \in \mathbb{Z}_{\geq 0}\Gamma_2$ and let $\zeta$ be arbitrary in the positive root lattice. For $\tau = \alpha_1 + \cdots + \alpha_m$ with the $\alpha_i$ simple roots, by $E_\tau$ (resp. $F_\tau$) we simply mean some permutation of the monomial $E_{\alpha_1} \ldots E_{\alpha_n}$ (resp. $F_{\alpha_1} \ldots F_{\alpha_n}$). So we are deviating from the notation of Theorem 1.1 here.

One simply moves all the bicharacters from the right to left, in order, using Lemma 6.2, to find

$$M_1M_2M_3 = q^*(g_1 \otimes g_2)S^{-2}\Omega \left( (\prod_i F_{\xi_i})E_{\zeta}(\prod_j E_{\eta_j}) \otimes (\prod_i E_{T^{n_i}}F_{\xi_i}(\prod_j F_{T^{-1}n_j}) \right)$$

with $g_1 \in G$ and $g_2 \in G_{(r)}$. Hence for any $f \in u_q^*$ we will have, for some constant $c_f \in \mathbb{C}$,

$$(f \otimes 1)(M_1M_2M_3) = c_f g_2S^{-2}\Omega(f)(\prod_i E_{T^{n_i}}F_{\xi_i}(\prod_j F_{T^{-1}n_j}) \in \mathbb{C}(G_{(r)}, E_{\alpha}, F_{\beta} : \alpha \in \Gamma_2, \beta \in \Gamma).$$

Since $R^J$ is a sum of such monomials $M_1M_2M_3$ we find

$$R^J_{(r)}(u_q^*) \subset \mathbb{C}(G_{(r)}, E_{\alpha}, F_{\beta} : \alpha \in \Gamma_2, \beta \in \Gamma).$$

7. Representation theory of the dual $(u_q(\mathfrak{g})^J)^*$

In this section we describe the irreducible representations of the dual $(u_q(\mathfrak{g})^J)^*$, for $J = J_{T,S}$ as in Theorem 3.1.

7.1. Grouplikes and the parabolic subalgebras.

**Lemma 7.1.** Each $K_\mu \in L$ is grouplike in the twist $u_q^J$.

**Proof.** We claim that $K_\mu \otimes K_\mu$ commutes with $J$, so that $\Delta^J(K_\mu) = J^{-1}(K_\mu \otimes K_\mu)J = K_\mu \otimes K_\mu$. From the particular form on $J$, we see that it suffices to show that $K_\mu \otimes K_\mu$ commutes with $T^\mu T_+ \otimes F_{\nu}$ and $F_{\nu} \otimes T^\mu T_+ \otimes F_{\nu}$ for $\nu$ a positive root in $\mathbb{Z}\Gamma_1$ with $T^\mu \nu \in \mathbb{Z}\Gamma_1$ for all $0 \leq i < k$. But this is clear since $\nu - T^\mu T_+ \otimes F_{\nu} \in \mathcal{L}^\perp$ and $\mu \in \mathcal{L}$. \qed

Take $\triangledown$ equal to either $(r)$ or $(l)$. By Lemma 7.1 we now see that the restriction of the multiplication map $\mathbb{C}[L] \otimes R^J_{(\triangledown)} \to u_q^J$ is a coalgebra map, where we just give $\mathbb{C}[L]$ its usual group ring structure. If we let $L$ act on $R^J_{(\triangledown)}$ by conjugation this gives a Hopf map $\mathbb{C}[L] \otimes R^J_{(\triangledown)} \to u_q^J$. According to the particular form of $R^J_{(\triangledown)}$ given in Proposition 5.4, and Lemma 1.3, we see that
this map has image equal to the corresponding quantum parabolic \( u_q(p_i) \).
So we find

**Lemma 7.2.** The quantum parabolics \( u_q(p_i) \) are both Hopf subalgebras in the twist \( u_q(g)^J \).

From the Hopf map \( \mathbb{C}[L] \rtimes R_{(l)}^J \to u_q^J \) we also get a dual Hopf map

\[
(u_q^J)^* \to \mathbb{C}[\mathcal{L}] \otimes (R_{(l)}^J)^* \xrightarrow{1 \otimes 1} \mathbb{C}[\mathcal{L}] \otimes R_{(r)}^J
\]

which extends to an algebra map

\[
(u_q^J)^* \to \mathbb{C}[\mathcal{L}] \otimes R_{(r)}^J / N \cong \mathbb{C}[\mathcal{L}] \otimes \mathbb{C}[\Lambda] \otimes u_q(p_2^{ss}),
\]

by Corollary 5.5. (Recall our subgroup \( \Lambda = G_{(r)} \cap G_2^+ \) from Corollary 5.5.) In a moment we will also need the following lemma.

**Lemma 7.3.** Take \( \mathcal{C} = L/(G_{(l)} \cap L) \). The subgroup \( \Lambda \) in \( G_{(r)} \) is isomorphic to the dual \( (G_{(l)} \cap L)^\vee \), and we have an exact sequence \( 0 \to \mathcal{C}^\vee \to \mathcal{L} \to \Lambda \to 0 \).

**Proof.** The dual \( \Lambda^\vee \) gives the character group of \( (R_{(l)}^J)^* \), by Corollary 5.5. The character group is identified with the group of grouplikes in \( R_{(l)}^J \). Since the intersection \( G_{(l)} \cap L \) provides exactly \( |\Lambda| = |G_{(r)}/G_2| = |G_{(l)}/G_1| \) grouplike elements in \( R_{(l)}^J \) we see that \( \Lambda = (G_{(l)} \cap L)^\vee \). Whence we have an exact sequence \( 0 \to \mathcal{C}^\vee \to \mathcal{L} \to \Lambda \to 0 \). \( \square \)

### 7.2. Irreducible representations of \( (u_q^J)^* \).

We take \( p^{ss} \) to be either of the (isomorphic) Lie algebras \( p_1^{ss} \) or \( p_2^{ss} \). In this section we prove

**Theorem 7.4.** There is a bijection

\[
\text{Irrep}(\mathbb{C}[\mathcal{L}] \otimes u_q(p^{ss})) \xrightarrow{\cong} \text{Irrep}((u_q^J)^*)
\]

given by restricting along an algebra surjection \( u_q^J)^* \to \mathbb{C}[\mathcal{L}] \otimes u_q(p^{ss}) \).

**Remark 7.5.** In Theorem 7.4 we take advantage of the existence of an abstract algebra isomorphism \( \mathbb{C}[\mathcal{C}^\vee \times \Lambda] \cong \mathbb{C}[\mathcal{L}] \), where \( \mathcal{C} \) is as in Lemma 7.3. Such an isomorphism exists simply because both groups are abelian of the same order, by Lemma 7.3. However, as we’ll see below, the character group of the dual \( (u_q^J)^* \) is naturally identified with \( L \), so that the appearance of \( \mathcal{L} \) is appropriate.

Before giving the proof we establish some background material.

**Lemma 7.6.** The subcoalgebra \( A \) in \( R_{(l)}^J \) dual to the quotient \( R_{(l)}^J / N \), under the Hopf isomorphism \( t_{R^J} : (R_{(l)}^J)^* \to R_{(r)}^J \), is exactly the subalgebra \( \mathbb{C}(G_{(l)}, E_{\alpha}, F_{\beta} : \alpha \in \Gamma_2, \beta \in \Gamma_1) \).

From the statement it is clear that \( A \) is actually a Hopf subalgebra. We are claiming that \( A \) is the minimal subspace in \( R_{(l)}^J \) admitting a factoring \( (R_{(l)}^J)^* \to A^* \to R_{(r)}^J / N \).
Proof. Recall $\mathcal{N}$ is generated by all the $F_\alpha$ with $\alpha \in \Gamma - \Gamma_2$. For $\pi$ the projection $R^I_{(r)} \to R^I_{(r)}/\mathcal{N}$, one sees directly from the form of $R^I$ that $(1 \otimes \pi)(R^I)$ lay in the product $A' \otimes (R^I_{(r)}/\mathcal{N})$ where $A' = \mathbb{C}(G, E_\alpha, F_\beta : \alpha \in \Gamma_2, \beta \in \Gamma_1)$. But $(1 \otimes \pi)(R^I)$ also lay in $R^I_{(l)} \otimes (R^I_{(r)}/\mathcal{N})$ so that

$$(1 \otimes \pi)(R^I) \in \left( R^I_{(l)} \otimes (R^I_{(r)}/\mathcal{N}) \right) \cap \left( A' \otimes (R^I_{(r)}/\mathcal{N}) \right).$$

By flatness of everything over $\mathbb{C}$, this intersection is exactly $A \otimes (R^I_{(r)}/\mathcal{N})$. So the surjective map $(\pi^I)(R^I)$ factors through $A$. Since the dimensions of $A$ and $R^I_{(r)}/\mathcal{N}$ agree we must have that $A$ is in fact dual to $R^I_{(r)}/\mathcal{N}$.

The Hopf subalgebra $A$ is strongly related to the intersection of the quantum parabolics $u_q(p_1) \cap u_q(p_2)$, which we denote $\text{Int}$. From considering bases of the two quantum parabolics, as in Theorem 1.1, one arrives at the presentation

$$\text{Int} = u_q(p_1) \cap u_q(p_2) = \mathbb{C}(G, E_\alpha, F_\beta : \alpha \in \Gamma_2, \beta \in \Gamma_1).$$

Note that since the quantum parabolics are Hopf subalgebras, the intersection will be a Hopf subalgebra as well.

Lemma 7.7. Take $\mathcal{C} = L/(G_{(l)} \cap L)$. There is a coalgebra isomorphism $\mathbb{C}[\mathcal{C}] \otimes A \to \text{Int}$ given by multiplication.

Proof. We have the multiplication map $\mathbb{C}[L] \otimes A \to u_q^I$ which is a surjection onto the intersection $\text{Int}$. Choose for each $\xi \in \mathcal{C}$ a representative $\xi \in L$ and restrict the above multiplication map to get an coalgebra embedding

$$\mathbb{C}[\mathcal{C}] \otimes A = \bigoplus_{\xi \in \mathcal{C}} \mathbb{C}\xi \otimes A \to u_q^I, \quad \xi \otimes a \mapsto \xi \cdot a,$$

with image exactly $\text{Int}$. \hfill $\square$

We have now the

Proof of Theorem 7.4. Let $K$ be the kernel of the projection $(u_q^J)^* \to \text{Int}^*$ dual to the inclusion $\text{Int} \to u_q^J$. Note that $K$ will be a Hopf ideal in the dual. We have the Hopf maps $(u_q^J)^* \to \mathbb{C}[L] \otimes R^J_{\mathcal{C}}$ of (9) which factor

$$(u_q^J)^* \xrightarrow{\Delta} (u_q^J)^* \otimes (u_q^J)^* \xrightarrow{(?)L \otimes R^J} \mathbb{C}[L] \otimes R^J_{\mathcal{C}}.$$ 

We claim that the induced maps

$$F : (u_q^J)^* \to \mathbb{C}[L] \otimes R^I_{(r)}/\mathcal{N} \quad \text{and} \quad F' : (u_q^J)^* \to \mathbb{C}[L] \otimes R^I_{(l)}/\mathcal{N}$$

factor through $\text{Int}^*$. Equivalently, we claim that $K$ is in their kernels. Let $\pi$ and $\pi'$ be the projections $\pi : R^I_{(r)} \to R^I_{(r)}/\mathcal{N}$ and $\pi' : R^I_{(l)} \to R^I_{(l)}/\mathcal{N}'$.

We prove the result for $R^I_{(r)}$. Recall that $\mathcal{N}$ is the ideal generated by all the $F_\alpha$ with $\alpha \in \Gamma - \Gamma_2$, and that $K$ consists of all functions vanishing on
Int. Note that the intersection contains all of $\mathbb{C}[G]$, so that $K|L = 0$. Hence for each $f \in K$ we have

$$F(f) = \sum_i (f_{i_1}|L) \otimes \pi_{R^f}(f_{i_2}) = \sum_i (f_{i_1}|L) \otimes ((f_{i_2} \otimes \pi)(R^f))$$

for some $f_{i_2} \in K$. So it suffices to show $(K \otimes \pi)(R^f) = 0$. However, we have already seen in Lemma 7.6 that $(1 \otimes \pi)(R^f)$ lay in $A \otimes R^f(\omega)/N$, and $A \subset \text{Int}$. Hence $(f \otimes \pi)(R^f) = 0$ for each $f \in K$, and we find $(K \otimes \pi)(R^f) = 0$. So the map $F$ factors through $\text{Int}^*$, and a completely analogous argument shows that $F'$ factors through $\text{Int}^*$ as well.

Since $(u_q^J)^* \rightarrow \text{Int}^*$ is a Hopf map the factorizations of $F$ and $F'$ imply that the map

$$(u_q^J)^* \xrightarrow{\Delta} (u_q^J)^* \otimes (u_q^J)^* \xrightarrow{F \otimes F'} (\mathbb{C}[\mathcal{L}] \otimes R^f(\omega)/N) \otimes (\mathbb{C}[\mathcal{L}] \otimes R^f(\delta)/N')$$

factors

$$(u_q^J)^* \rightarrow \text{Int}^* \rightarrow (\mathbb{C}[\mathcal{L}] \otimes R^f(\omega)/N) \otimes (\mathbb{C}[\mathcal{L}] \otimes R^f(\delta)/N').$$

(10)

We note that that the map

$$(u_q^J)^* \xrightarrow{\Delta} (u_q^J)^* \otimes (u_q^J)^* \rightarrow (\mathbb{C}[\mathcal{L}] \otimes R^f(\omega)) \otimes (\mathbb{C}[\mathcal{L}] \otimes R^f(\delta))$$

is an embedding, since its dual is a surjection, so that the kernels of (10) and (11) are nilpotent. It follows that the kernel of the projection $(u_q^J)^* \rightarrow \text{Int}^*$ must be nilpotent as well.

We have from Lemmas 7.6 and 7.7 that $\text{Int}^* \cong \mathbb{C}[\mathcal{G}^\vee] \otimes R^f(\omega)/N$. Recall from Corollary 5.5 that $R^f(\omega)/N$ is isomorphic to $\mathbb{C}[\Lambda] \otimes u_q(p^{ss})$ and that $\mathbb{C}[\mathcal{G}^\vee \times \Lambda] \cong \mathbb{C}[\mathcal{L}]$, abstractly, to arrive at a surjection $(u_q^J)^* \rightarrow \mathbb{C}[\mathcal{L}] \otimes u_q(p^{ss})$ with nilpotent kernel. Restricting then gives the proposed bijection on irreducible representations. \hfill \Box

**Corollary 7.8.** The set of grouplikes $G(u_q^J)$ is exactly $L$.

**Proof.** Since all the elements in $L$ are grouplike, by Lemma 7.1, we need only know that $|G(u_q^J)| = |L|$. But this just follows from the theorem, since grouplikes in $u_q^J$ are identified with one dimensional representations of $(u_q^J)^*$. \hfill \Box

To compare $u_q$ to $u_q^J$ let us consider a maximal BD triple on $A_n$. In this case there is a unique solution $S$ to (EQ–S) and, as mentioned previously, the algebras $R_{\mathcal{L}}^\mathcal{J}$ will be the full parabolics.\(^3\) We will have, for $n = 2$ and $l = 5$ for example, a following variation in the dimensions of the coradicals:

$$\dim \text{Corad}(u(sl_3)) = 25, \quad \dim \text{Corad}(u(sl_3)^J) = 105.$$  

\(^3\)This basically follows from the fact that $G^\sharp$ will be a free $\mathbb{Z}/1\mathbb{Z}$-module so that $S(\mu, \nu) = 0$ for any $\mu, \nu \in G^\sharp$, by antisymmetry. Thus $S^{-1}\Omega|G^\sharp_x \times G^\sharp_x = \Omega_{G^\sharp}$ and we must have all of $G^\sharp_x$ in $G_{\mathcal{L}}$. \hfill \Box
The difference is made more stark from the fact that corepresentation theory of $u_q(sl_{n+1})$ is essentially trivial, at least when we restrict our attention to the irreducibles and fusion rule, while the corepresentation theory of $u_q(sl_{n+1})^J$ should be at least as complicated as the representation theory of $u_q(sl_n)$.

8. The Drinfeld element and properties of the antipode

Here we discuss preservation of the Drinfeld element under twisting. Basic information on the Drinfeld element in a quasitriangular Hopf algebra, and its relation the antipode, can be found in [21, 15]. We fix a Belavin-Drinfeld triple $(\Gamma_1, \Gamma_2, T)$, solution $S$, and twist $J = J_{T,S}$ of $u_q(g)$.

Let $\rho \in G$ be the sum $\rho = \sum_{\mu \in \Phi^+} \mu$. Then we have $(\rho, \alpha) = 2$ for each simple root $\alpha$ [17, Sect. 10.2]. This gives $S^2 = \text{ad}_{K_{\rho}}$ and the Drinfeld element for $u_q(g)$ thus factors $u = K_{\rho}v$, where $v$ is a central element with $\Delta(v) = (v \otimes v)(R_{21}R)^{-1}$, i.e. a ribbon element. Note that $\rho$ is in $L$ as $(\rho, \alpha - T(\alpha)) = 2 - 2 = 0$ for each $\alpha \in \Gamma$. So $K_{\rho}$ remains grouplike in the twist $u_q(g)^J$.

Recall that under an arbitrary twist $J$ of a quasitriangular Hopf algebra $H$ the Drinfeld element for $H^J$ is the product $u^J = Q^{-1}_J S(Q_J)u$ (see e.g. [11]). In our case this means that the Drinfeld element for $u_q(g)^J$ is given by

$$u^J = Q^{-1}_J S(Q_J)K_{\rho}v.$$  

Centrality of $v$ implies

$$\Delta^J(v) = (v \otimes v)J^{-1}(R_{21}R)^{-1}J = (v \otimes v)(R_{21}R^J)^{-1}.$$  

So $v$ is still a ribbon element for the twist, and $Q^{-1}_J S(Q_J)K_{\rho}$ is grouplike in the twist. Since $K_{\rho}$ itself is grouplike we conclude that $Q^{-1}_J S(Q_J)$ is grouplike as well.

**Proposition 8.1.** The twists $J = J_{T,S}$ are such that $Q^{-1}_J S(Q_J) = 1$.

In the proof of the proposition we employ what we call a $T$-grading on $u_q^J$. We define this as any algebra $\mathbb{Z}$-grading with the following properties:

(a) $\mathbb{C}[G]$ is homogeneous of degree 0.

(b) The $E_\alpha$ are of positive degree and the $F_\alpha$ are of negative degree with $\text{deg}(F_\alpha) = -\text{deg}(E_\alpha)$.

(c) $\text{deg}(E_{T\alpha}) > \text{deg}(E_\alpha)$ for each $\alpha \in \Gamma_1$.

It is easy to construct such a grading. For example, one can construct the acyclic directed graph $\text{Graph}(\Gamma, T)$ with vertices $\Gamma$ and an arrow from $\alpha$ to $T(\alpha)$ for each $\alpha \in \Gamma_1$. One then takes

$$\text{deg}(E_\alpha) = -\text{deg}(F_\alpha) = |\text{Graph}_{\leq \alpha}|,$$

where $\text{Graph}_{\leq \alpha}$ is the collection of all vertices with a path to $\alpha$ in $\text{Graph}(\Gamma, T)$, including $\alpha$. Note that the antipode preserves degree under any $T$-grading.
Proof. Under any $T$-grading on $u_q$ we will have that $J$ and $J^{-1}$ both lay in nonnegative degree in $u_q \otimes u_q$, where $\deg(a \otimes b) = \deg(a) + \deg(b)$ for $a, b \in u_q$. This is clear from the explicit forms of the twist and its inverse given at Theorem 3.1 and (7). We have also $J_0 = S^{-1} \Omega_{L_L}'^{-1/2}$ and $(J^{-1})_0 = \Omega_{L_L}'^{1/2}$. It follows, from the expressions of $Q_J$ and $Q^{-1}_J$ given in Section 2, that both $Q^{-1}_J$ and $S(Q_J)$ lay in nonnegative degree with

$$ (Q^{-1}_J)_0 = m(S^{-1} \Omega_{L_L}'^{-1/2}), \quad S(Q_J)_0 = m(\Omega_{L_L}'^{1/2}), $$

where $m$ is multiplication. We have now $(Q^{-1}_J S(Q_J))_0 = (Q^{-1}_J)_0 S(Q_J)_0$ and since the multiplication map on any commutative algebra, such as $\mathbb{C}[G]$, is a ring map

$$ (Q^{-1}_J)_0 S(Q_J)_0 = m(S^{-1} \Omega_{L_L}'^{-1/2} \Omega_{L_L}'^{1/2}) = 1. $$

Finally we note that since $Q^{-1}_J S(Q_J)$ is grouplike it must lay in degree 0. Therefore $Q^{-1}_J S(Q_J) = (Q^{-1}_J S(Q_J))_0 = 1$. □

As an immediate corollary we have

**Corollary 8.2.** The Drinfeld element for $u_q(g)^J$ is equal to the Drinfeld element for $u_q(g)$.

8.1. **Implications for the antipode.** In [22] the question was posed as to whether or not the order of the antipode and the traces of the powers of the antipode are preserved under twisting. The question was answered positively for Hopf algebras with the Chevalley property. Using the expression of the Chevalley property given in [1, Prop. 4.2, 5] it is relatively easy to see that no small quantum group has the Chevalley property. We can, however, verify the proposed invariance for Belavin-Drinfeld twists.

**Corollary 8.3.** For $S$ the antipode on $u_q(g)$ and $S_J$ the antipode on the twist $u_q(g)^J$, and $J = J_{T,S}$, we have $\text{Tr}(S^n_J) = \text{Tr}(S^n)$ for all $m \in \mathbb{Z}$ and $\text{ord}(S_J) = \text{ord}(S)$.

Proof. Since $Q^{-1}_J S(Q_J) = 1$ the proof of [22, Thm. 4.3] still works to get $\text{Tr}(S^n_J) = \text{Tr}(S^n)$. Since $S$ and $S_J$ are semisimple operators invariance of order follows from invariance of the traces. □

We can also get invariance of the so-called regular object of [26, Sect. 5.4] using the condition of [22, Prop. 7.3 (ii)]. This positively answers [26, Question (5.12)] for the twists $J_{T,S}$ on small quantum groups.

9. **Twisted automorphisms and group actions on $\text{rep}(u_q(g))$**

We use below the notion of a 2-group. A 2-group is simply a monoidal category in which all morphisms are invertible and all objects have a weak inverse, i.e. an inverse up to isomorphism. For a tensor category $\mathcal{C}$ we let $\text{Aut}(\mathcal{C})$ denote the 2-group of autoequivalences of $\mathcal{C}$ as a tensor category, with natural isomorphisms, and $\text{Aut}(\mathcal{C})$ denote the associated group of isoclasses of autoequivalences.
Following Davydov [6], for a Hopf algebra $H$ we call a pair $(\phi, J)$ of a twist and a Hopf isomorphism $\phi : H \to H^J$ a \textit{twisted automorphism} of $H$. Each twisted automorphism can be identified with the tensor autoequivalence of $\text{rep}(H)$ given by composing

$$\text{rep}(H) \xrightarrow{J} \text{rep}(H^J) \xrightarrow{\text{res}_\phi} \text{rep}(H).$$

Indeed, twisted automorphisms form a 2-subgroup in the 2-group of autoequivalences $\text{Aut}(\text{rep}(H))$ with product $(\phi', J') \cdot (\phi, J) = (\phi \phi', J \phi \otimes_2 (J'))$.

The induced isomorphisms between twisted automorphisms are gauge equivalences (see 9.3 below). Furthermore, Ng and Schauenburg have shown that when $H$ is finite dimensional any autoequivalence of $\text{rep}(H)$ will be isomorphic to a twisted automorphism [23, Thm. 2.2].

We take $\mathfrak{g}$ simple and simply laced, $q$ a primitive $l$th root of unity, for $l$ as in Section 1.2, and $u_q = u_q(\mathfrak{g})$. In this final section we introduce twists $J^\lambda_\alpha$ of the small quantum group $u_q$ which are paired with automorphisms $\exp^\lambda_\alpha$ so that each pair $(\exp^\lambda_\alpha, J^\lambda_\alpha)$ provides a twisted automorphism of $u_q$. We then relate a canonical algebraic group action on $\text{rep}(u_q)$ to the twisted automorphisms $(\exp^\lambda_\alpha, J^\lambda_\alpha)$, and propose a question regarding a set of “generators” for the collection of all twists of $u_q$.

9.1. Twists via exponentiation: an extended quantum coadjoint action. Recall that $u_q$ embeds as a Hopf subalgebra in Lusztig’s divided powers quantum group

$$U_q = U_q(\mathfrak{g}) = \mathbb{C}\langle K_\alpha^{\pm 1}, E_\alpha, F_\alpha, E_\alpha^{(l)}, F_\alpha^{(l)} : \alpha \in \Gamma \rangle/\text{(relations)}.$$ 

We do not recall the specific construction of $U_q$ here, and refer the reader instead to [20, 19], and in particular [20, Sect. 6.5], for the details.

According to [19, Lem. 4.5] the commutator

$$\text{ad}_{E_\alpha^{(l)}} : U_q \to U_q, \quad x \mapsto [E_\alpha^{(l)}, x]$$

preserves the subalgebra $u_q$ and the restriction $\text{ad}_{E_\alpha^{(l)}}|u_q$ is a nilpotent operator. The same is true if we scale by any $\lambda \in \mathbb{C}$. Hence we can exponentiate this operator to produce an algebra automorphism

$$\exp_\alpha^\lambda := \exp(\text{ad}_{E_\alpha^{(l)}}),$$

of the small quantum group $u_q$. We can similarly define

$$\exp_{-\alpha}^\lambda := \exp(\text{ad}_{F_\alpha^{(l)}}).$$

If we consider $u_q(\mathfrak{sl}_2)$ for example, and $\exp_+^\alpha$ corresponding to the positive simple root, we have $\exp_+^\lambda(E) = \exp_+^\lambda(K) = 0$ and

$$\exp_+^\lambda(F) = F + \lambda \left( \frac{qK + q^{-1}K^{-1}}{q - q^{-1}} \right) E^{(l-1)}.$$
As $E^{(l)}_\alpha$ fails to be primitive, the automorphism $\exp^\lambda \alpha$ fails to be a Hopf map. We have, in the ambient algebra $U_q$:

$$
\Delta(E^{(l)}_\alpha) = E^{(l)}_\alpha \otimes 1 + 1 \otimes E^{(l)}_\alpha + \sum_{1 \leq i \leq l-1} q^{-i(l-i)} K^i E^{(l-i)} \otimes E^{(i)}
$$

and can define the element

$$\mathcal{O}(E_\alpha) = \Delta(E^{(l)}_\alpha) - (E^{(l)}_\alpha \otimes 1 + 1 \otimes E^{(l)}_\alpha)
$$

in $u_q \otimes u_q$. Note that $\mathcal{O}(E_\alpha)$ is square zero, and hence we can exponentiate any scaling $\lambda \mathcal{O}(E_\alpha)$ to arrive at a unit

$$J^\lambda_\alpha = \exp(\lambda \mathcal{O}(E_\alpha)) \in u_q \otimes u_q.$$

We define similarly $J^{-\lambda}_\alpha = \exp(\lambda \mathcal{O}(F_\alpha))$ for $\mathcal{O}(F_\alpha) = \Delta(F^{(l)}_\alpha) - (F^{(l)}_\alpha \otimes 1 - 1 \otimes F^{(l)}_\alpha)$. One can check easily from the expressions of $\mathcal{O}(E_\alpha)$ and $\mathcal{O}(F_\alpha)$ that

$$(\epsilon \otimes 1)(J^\lambda_\alpha) = (1 \otimes \epsilon)(J^{\lambda}_{-\alpha}) = 1.$$

**Theorem 9.1.** For an arbitrary simple root $\alpha$, and $\lambda \in \mathbb{C}$, the unit $J^\lambda_\alpha$ is a twist for $u_q(g)$. Furthermore, each pair $(\exp^{\lambda}_\pm \alpha, J^\lambda_{\pm \alpha})$ is a twisted automorphism of $u_q(g)$.

We will only prove the result for positive $\alpha$, the computation for $-\alpha$ being completely similar. Let us first give a technical lemma.

**Lemma 9.2.** The elements $(\Delta \otimes 1)(\mathcal{O}(E_\alpha))$ and $\mathcal{O}(E_\alpha) \otimes 1$ commute, as do the elements $(1 \otimes \Delta)(\mathcal{O}(E_\alpha))$ and $1 \otimes \mathcal{O}(E_\alpha)$.

**Proof.** Since $E_\alpha$ lay in the Hopf subalgebra $U_q(sl_2) \subset U_q(g)$ generated by $K_\alpha$ and the $E^{(n)}_\alpha$, $F^{(n)}_\alpha$, we may assume $g = sl_2$. We may further restrict to the positive Borel $U_+$, in which $E^{(l)}$ is central. Take $\mathcal{O} E = \mathcal{O}(E)$ and $p E^{(l)} = E^{(l)} \otimes 1 + 1 \otimes E^{(l)}$. We have now

$$
(\Delta \otimes 1)(\mathcal{O} E)(\mathcal{O} E \otimes 1)
= (\Delta \otimes 1)(\mathcal{O} E)(\Delta E^{(l)} \otimes 1) - (\Delta \otimes 1)(\mathcal{O} E \otimes 1)(p E^{(l)} \otimes 1)
$$

$$
= (\Delta \otimes 1)(\mathcal{O} E)(E^{(l)} \otimes 1) - (\Delta \otimes 1)(\mathcal{O} E \otimes 1)(p E^{(l)} \otimes 1)
$$

$$
= (\Delta \otimes 1)(\mathcal{O} E)(E^{(l)} \otimes 1) - (\Delta \otimes 1)(\mathcal{O} E \otimes 1)(p E^{(l)} \otimes 1)
$$

$$
= (\Delta \otimes 1)(\mathcal{O} E \otimes 1)(\Delta \otimes 1)(\mathcal{O} E).
$$

This gives the first proposed commutativity

$$
(\Delta \otimes 1)(\mathcal{O} E)(\mathcal{O} E \otimes 1) = (\mathcal{O} E \otimes 1)(\Delta \otimes 1)(\mathcal{O} E).
$$

The verification of the relation

$$
(1 \otimes \Delta)(\mathcal{O} E)(1 \otimes \mathcal{O} E) = (1 \otimes \mathcal{O} E)(1 \otimes \Delta)(\mathcal{O} E)
$$

is completely similar. \qed
Since all of the elements in the statement of Lemma 9.2 are nilpotent in \( u_q \otimes u_q \otimes u_q \) we can now exponentiate to get
\[
\exp((\Delta \otimes 1)(\lambda \theta E_\alpha) + (\lambda \theta E_\alpha \otimes 1)) = \exp(\Delta \otimes 1)(\lambda \theta E_\alpha) \exp(1 \otimes \lambda \theta E_\alpha) \\
= (\Delta \otimes 1)(\lambda \theta E_\alpha) \exp(1 \otimes \lambda \theta E_\alpha) \\
= (\Delta \otimes 1)(J^{\lambda}_\alpha \otimes 1)
\]
and
\[
\exp((1 \otimes \Delta)(\lambda \theta E_\alpha) + (1 \otimes \lambda \theta E_\alpha)) = (1 \otimes \Delta)(\lambda \theta E_\alpha) + (1 \otimes \lambda \theta E_\alpha),
\]
for arbitrary \( \lambda \in \mathbb{C} \).

Proof of Theorem 9.1. Again, we may assume \( g = sl_2 \). By the above observations (12, 13) the dual cocycle condition for \( J^{\lambda}_\alpha \) is equivalent to the equality
\[
(\Delta \otimes 1)(\lambda \theta E) + (\lambda \theta E \otimes 1) = (1 \otimes \Delta)(\lambda \theta E) + (1 \otimes \lambda \theta E).
\]
By dividing by \( \lambda \) on both sides we may take \( \lambda = 1 \). We then see directly
\[
(\Delta \otimes 1)(\theta E) + (\theta E \otimes 1) = \sum_{0<i,j,k<l} q^{i+j+k} K^{i+j+k} E^{(i)} \otimes K^{i+j+k} E^{(j)} \otimes E^{(k)} \\
= (1 \otimes \Delta)(\theta E) + (1 \otimes \theta E).
\]
Hence \( J^{\lambda}_\alpha \) is a twist.

As for compatibility with the automorphism \( \exp^{\lambda}_\alpha \), we have the diagram
\[
\begin{array}{ccc}
  u_q & \xrightarrow{ad_{\lambda E_\alpha}} & u_q \\
\Delta \downarrow & & \downarrow \Delta \\
  u_q \otimes u_q & \xrightarrow{ad_{\Delta \lambda E_\alpha}} & u_q \otimes u_q
\end{array}
\]
which implies the diagram
\[
\begin{array}{ccc}
  u_q & \xrightarrow{\exp^{\lambda}} & u_q \\
\Delta \downarrow & & \downarrow \Delta \\
  u_q \otimes u_q & \xrightarrow{\exp(\text{ad}_{\Delta \lambda E_\alpha})} & u_q \otimes u_q
\end{array}
\]
Since \( \Delta E_\alpha^{(l)} = E_\alpha^{(l)} \otimes 1 + 1 \otimes E_\alpha^{(l)} + \theta(E_\alpha), \) we have
\[
\exp(\text{ad}_{\Delta \lambda E_\alpha}) = (\exp^{\lambda}_\alpha \otimes \exp^{\lambda}_\alpha)\text{Ad}_{J^{\lambda}_\alpha},
\]
where \( \text{Ad}_u(x) = uxu^{-1} \), and the above diagram gives on elements
\[
(\exp^{\lambda}_\alpha \otimes \exp^{\lambda}_\alpha)\Delta^{J^{\lambda}_\alpha} \Delta^{J^{\lambda}_\alpha}(x) = \Delta(\exp^{\lambda}_\alpha(x)).
\]
Replace \( x \) with \( \exp_{\alpha}^{-\lambda}(x) \), compose with \( (\exp_{\alpha}^{-\lambda} \otimes \exp_{\alpha}^{-\lambda}) \), and swap \( \lambda \) for \( -\lambda \) to find that \( \Delta J_{\lambda}^{\alpha}(\exp_{\alpha}^{\lambda}(x)) = (\exp_{\alpha}^{\lambda} \otimes \exp_{\alpha}^{\lambda})\Delta(x) \). So we see \( \exp_{\alpha}^{\lambda} : u_q \to u_q^{J_{\lambda}^{\alpha}} \) is a Hopf map.

\( \square \)

One can check easily
\[
(\exp_{\pm \alpha}^{\lambda}, J_{\pm \alpha}^{\lambda}) \cdot (\exp_{\pm \alpha}^{\lambda'}, J_{\pm \alpha}^{\lambda'}) = (\exp_{\pm \alpha}^{\lambda + \lambda'}, J_{\pm \alpha}^{\lambda + \lambda'}).
\]

It follows that the assignment \( \lambda \mapsto (\exp_{\pm \alpha}^{-\lambda}, J_{\pm \alpha}^{-\lambda}) \) gives a 1-parameter subgroup in the 2-group of twisted automorphisms for \( u_q \), and hence a 1-parameter subgroup \( \mathbb{C} \to \text{Aut}(\text{rep}(u_q)) \) into the 2-group of autoequivalences \( \text{Aut}(\text{rep}(u_q)) \). The negation here appears for technical reasons, but intuitively corrects the fact that the multiplication of twisted automorphisms defined above appears to be backwards. We denote this 1-parameter subgroup \( \omega_{\pm \alpha} \).

**Remark 9.3.** The algebra automorphisms appearing in the 1-parameter subgroups \( \omega_{\pm \alpha} \) can be recovered alternatively from the quantum coadjoint action of De Concini and Kac, via the reduction \( U_{q}^{DK} \to u_q \) from the non-divided-powers quantum group [8, Prop. 3.5]. So we are saying above that the induced quantum coadjoint action on \( u_q \) extends naturally to an action on the tensor category \( \text{rep}(u_q) \).

### 9.2. Identification with the Arkhipov-Gaitsgory action

Take \( \Theta \) the connected, simply connected, semisimple algebraic group with Lie algebra \( g \). As a set we identify \( \Theta \) with its \( \mathbb{C} \)-points. Taking the (finite) dual of the exact sequence of Hopf algebras \( \mathbb{C} \to u_q(g) \to U_q(g) \to U(g) \to \mathbb{C} \) produces an exact sequence
\[
\mathbb{C} \to \mathcal{O}(\Theta) \to \mathcal{O}_q(\Theta) \to u_q(g)^* \to \mathbb{C}
\]
with \( \mathcal{O}(\Theta) \) laying in the center of the quantum function algebra [9, Thm. 6.3, Lem. 6.1]. According now to [3, Thm. 2.8] and [2, Prop. 4.1] we have a tensor equivalence between the de-equivariantization \( \text{corep}(\mathcal{O}_q(\Theta)) \) and \( \text{rep}(u_q(g)) \). We take \( \mathcal{O} = \mathcal{O}(\Theta) \) and \( \mathcal{O}_q = \mathcal{O}_q(\Theta) \).

Recall that the de-equivariantization is the category of finitely generated left \( \mathcal{O} \)-modules with a compatible right \( \mathcal{O}_q \)-coaction [2, Def. 3.7]. This category is monoidal under the product \( \otimes_{\mathcal{O}} \). The action of \( \Theta \) on itself by left translation, and pushing forward by the corresponding automorphisms of \( \mathcal{O} \), gives an action of \( \Theta \) on the de-equivariantization by tensor functors. Rather, we have a canonical monoidal functor from \( \Theta \) to the 2-group of tensor autoequivalences of \( \text{corep}(\mathcal{O}_q) \). The equivalence \( \text{corep}(\mathcal{O}_q) \to \text{rep}(u_q) \) of [3] is given by taking the fiber at the identity \( ? |_k = \mathbb{C} \otimes_{\mathcal{O}} ? \), and via this equivalence we get an action of \( \Theta \) on \( \text{rep}(u_q) \).

We let \( \gamma_{\pm \alpha} \) denote the 1-parameter subgroup in \( \Theta \) given by exponentiating the root space \( g_{\pm \alpha} \).
Proposition 9.4. For any simple root $\alpha \in \Gamma$, the composite
\[ \mathbb{C} \xrightarrow{\gamma_{\pm \alpha}} \Theta \to \text{Aut} \left( \text{corep} \left( \Theta_{\text{re}} \right) \right) \xrightarrow{\text{Ad}_{\pm \alpha}^{-1}} \text{Aut} \left( \text{rep} \left( u_q \right) \right) \]
is isomorphic to the 1-parameter subgroup $\omega_{\pm \alpha} : \lambda \mapsto (\exp_{\pm \alpha}^{-1}, J_{\pm \alpha}^{-1})$.

What one should mean by a general isomorphism of 1-parameter subgroups is not exactly clear. From our perspective we would like a family of natural isomorphisms between the two functors $\mathbb{C} \to \text{Aut} \left( \text{rep} \left( u_q \right) \right)$ which satisfy all obvious commutativity and additivity relations. We will focus here only on the production of a natural family of natural isomorphisms which vary with $\lambda$.

Proof. We consider only the positive root $\alpha$. Since high powers of $E_{\alpha}(q)$ annihilate any finite dimensional representation, each function $f$ in the finite dual $\Theta_q$ will vanish on high powers of $E_{\alpha}(q)$ [19, Prop. 5.1]. Hence the exponent
\[ \exp(\lambda E_{\alpha}(q)) : \Theta_q \to \mathbb{C} \]
is a well-defined function. Restricting along the inclusion $\Theta \to \Theta_q$ recovers the point $\gamma_{\alpha}(\lambda) = \exp(\lambda e_\alpha)$ in $\Theta$. Let us fix $x = x^\lambda = \gamma_{\alpha}(\lambda)$ and $v = v^\lambda = \exp(-\lambda E_{\alpha}(q))$.

Take $\text{Ad}_v : \Theta_q \to \Theta_q$ the linear automorphism $f \mapsto v(f_1) f_2 v^{-1}(f_3)$, where $v^{-1} = \exp(\lambda E_{\alpha}(q))$. We note that $\text{Ad}_v$ is a Hopf isomorphism from the cocycle twist of $\Theta_q$ via the 2-cocycle $J_{\alpha}^{-\lambda} : \Theta_q \otimes \Theta_q \to \mathbb{C}$ to $\Theta_q$, and so the sequence
\[ \text{corep}(\Theta_q) \xrightarrow{J_{\alpha}^{-\lambda}} \text{corep}(\Theta_q) \xrightarrow{\text{res}_{\text{Ad}_v}} \text{corep}(\Theta_q) \]
is an equivalence. This equivalence induces an equivalence on the equivariantization, where we additionally restrict the action of $\Theta$ along $\text{Ad}_x$. We denote this autoequivalence by $F^\lambda : \text{corep}(\Theta_q) \to \text{corep}(\Theta_q)$.

We have now an isomorphism of monoidal functors $x^\lambda \cong F^\lambda$ given on objects as the composite
\[ x^\lambda V \xrightarrow{\text{com} \text{ilt}} (x^\lambda V) \otimes \Theta_q \xrightarrow{1 \otimes v^\lambda} F^\lambda V. \]
This equivalence is simply given by multiplying by the function $v^\lambda$, and we denote the isomorphism simply by $v^\lambda$.\footnote{The interested reader can check that the family of isomorphisms $\{v^\lambda\}_v$ satisfies all desired commutativity and additivity relations to give an isomorphism between these two 1-parameter subgroups in $\text{Aut} \left( \text{corep} \left( \Theta_q \right) \right)$.} The quasi-inverse to the reduction $?|_\epsilon : \text{corep} \left( \Theta_q \right) \to \text{rep}(u_q)$ is the induction-like functor $\text{Ind} = (\Theta_q \otimes ?)^{u_q}$, where $u_q$ acts on a product $\Theta_q \otimes V$ diagonally by $h \cdot (f \otimes v) = (f S(h_1) \otimes h v)$. We compose with the 2-group map to $\text{Aut} \left( \text{rep}(u_q) \right)$ to get an induced isomorphism of 1-parameter subgroups
\[ v^\lambda = (?|_\epsilon) \circ v^\lambda \circ \text{Ind} : (?|_\epsilon) \circ x^\lambda \circ \text{Ind} \rightarrow (?|_\epsilon) \circ F^\lambda \circ \text{Ind}. \]
Since \( \text{Ad}_q : \mathcal{O} \to \mathcal{O} \) preserves the counit, and \( \text{Ad}_q \) induces the automorphism \( \exp^{-\lambda}_\alpha \) on the quotient \( u_q^* \) (or rather its dual), taking the fiber at the identity gives
\[
(F^\lambda \text{Ind}(V))|_\epsilon = (\exp^{-\lambda}_\alpha u_q^* \otimes V)^u_q = \exp^{-\lambda}_\alpha (u_q^* \otimes V)^u_q
\]
for each \( u_q \)-representation \( V \). The subscript of \( \exp^{-\lambda}_\alpha \) here means that we are restricting the action of \( u_q \) along this automorphism. But now the natural isomorphism of \( u_q \)-modules \( \text{ev}_1 \otimes 1 : (u_q^* \otimes V)^u_q \to V \) given by the counit of \( u_q^* \) produces the desired family of natural isomorphism
\[
\tilde{v}^\lambda : (?|_\epsilon) \circ x^\lambda_\alpha \circ \text{Ind} \xrightarrow{\phi^\lambda} (?|_\epsilon) \circ F^\lambda \circ \text{Ind} \xrightarrow{\text{ev}_1 \otimes 1} (\exp^{-\lambda}_\alpha, J^\lambda).
\]
\[\square\]

Recall that \( \Theta \) is generated by the 1-parameter subgroups \( \gamma_{\pm \alpha} \) [18, Thm. 27.5]. Hence, after taking isoclasses, the group map \( \Theta \to \text{Aut}(\text{rep}(u_q)) \) is determined completely by its value on these 1-parameter subgroups. One can also show that the action of \( G \) on \( \text{rep}(u_q) \) is determined up to unique isomorphism by these 1-parameter subgroups, but this more limited information is enough for us to formulate Question 9.5 below, which proposes a set of generators for the groupoid of twists of \( u_q(g) \).

9.3. Autoequivalences of \( \text{rep}(u_q(g)) \) and the classification of twists.
Question 9.5 below refers to gauge equivalence of twists. We say two twists \( J \) and \( J' \) are gauge equivalent if there is a unit \( v \) in \( H \) with \( J' = \Delta(v)J(v^{-1} \otimes v^{-1}) \). We let \( \text{TW}(H) \) denote the groupoid of twists of \( H \), with morphisms given by gauge equivalences.

We note that the information of an isomorphism from the autoequivalence specified by a twisted automorphisms \( (\phi, J) \) to that of \( (\phi', J') \) is exactly the data of a unit \( v \in H \) so that \( J' = \Delta(v)J(v^{-1} \otimes v^{-1}) \) and \( \phi' = \text{Ad}_v \phi \). We call such a unit a gauge equivalence of twisted automorphisms. Hence we have naturally a 2-group of twisted automorphisms with gauge equivalences.

As is noted in [6], the groupoid \( \text{TW}(H) \) admits a well-defined right action of the 2-group of twisted automorphisms. This action is defined simply by \( J' \cdot (\phi, J) = J\phi^{\otimes 2}(J') \). Take
\[
\tilde{\Theta} = \left\{ \begin{array}{l}
\text{The 2-subgroup of all twisted automorphisms which are isomorph to an element in the image of } \Theta, \text{ in } \text{Aut}(\text{rep}(u_q))
\end{array} \right\}
\]
\[
= \left\{ \begin{array}{l}
\text{The 2-subgroup of twisted automorphisms which are}
\text{gauge equivalent to a product of the } (\exp^{-\lambda}_{\pm \alpha}, J^\lambda_{\pm \alpha})
\end{array} \right\}.
\]

Take also \( \text{BD}(u_q) \subset \text{TW}(u_q) \) the full subcategory of Belavin-Drinfeld twists \( \{J_{T,S} \}_{T,S} \). The following question is also raised in [7], where the authors investigate the algebraic structure of \( \text{Aut}(\text{rep}(u_q)) \), and autoequivalence groups of finite tensor categories in general.
Question 9.5. Is the groupoid of twists of the small quantum group generated by $BD(u_q)$ and the 1-paramater subgroups $\{\exp_{2\alpha}^\lambda, J_{2\alpha}^\lambda\}\lambda \in \mathbb{R}$? Equivalently, is the inclusion $BD(u_q) \cdot \tilde{\Theta} \to TW(u_q)$ an equivalence?

References


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