
Gauge invariants from the antipode

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Throughout by a Hopf algebra $H$ we mean a finite dimensional(!) Hopf algebra $H$ over a base field $k$ (often $\mathbb{C}$).

We take $S = S_H$ to be the antipode of $H$.

The category of finite dimensional modules over $H$ is denoted $\text{rep}(H)$.

Contents:

Introduction

Larson and Radford: some motivation

Main question

Some answers

Some details
Let’s recall...

Let $H$ be a (finite dimensional) Hopf algebra. Recall the following results of Larson and Radford.

Theorem ([8, 4, 3])

*The antipode of $H$ has finite order. We have that $H$ is semisimple and cosemisimple if and only if $\text{Tr}(S^2)$ is nonzero in $k$. Further, in characteristic 0, $H$ is semisimple (and cosemisimple) if and only if $S^2 = \text{id}_H$."

So we see that, at least in a very coarse sense, the traces of the powers of the antipode, and the order of the antipode, act as measures of (non)semisimplicity. By ”in a very coarse sense” we at least find a binary:

$$\text{Is } H \text{ semisimple? } \Leftrightarrow \text{Is } \text{Tr}(S^2) \neq 0? \quad (k=\mathbb{C}) \quad \Leftrightarrow \quad \text{Is } \text{ord}(S^2) = 1?$$
Let’s think about the higher powers: example

Consider the Taft algebra:

$$T_n(q) = \mathbb{C}\langle g, x \rangle / (x^n, g^n - 1, gx - qxg)$$

with $q$ a primitive $n$th root of unity. The Hopf structure is such that $x$ is skew primitive and $g$ is grouplike, and $S(g) = g^{-1}$, $S(x) = -gx$. So

$$S^2(g) = g, \quad S^2(x) = qx \implies \text{ord}(S^2) = n = \text{nilp. deg of } \text{Jac}(T_n(q)).$$

Can also think of small quantum group $u_q(sl_2) =$ copies of $T_n(q)$ and $T_n(q^{-1})$ glued together. Know $\text{ord}(S^2) = n$ and that

$$\frac{\dim(u_q(sl_2))}{\dim(\text{Jac}(u_q(sl_2)))} = \frac{n^4}{n^4 - \frac{1}{6}n(n + 1)(2n + 1)} \xrightarrow{n \to \infty} 1$$
As for the traces of the powers of the antipode, for \( T_n(q) \) with \( n \) odd

\[
\text{Tr}(S^{2r}) = n \left( \sum_{m=0}^{n-1} q^{rm} \right) = 0 \text{ for all } r \in \mathbb{Z},
\]

and

\[
\text{Tr}(S^{2r+1}) = \sum_{m=0}^{n-1} (-1)^m q^{(r+\frac{1}{2})m}.
\]

E.g. taking \( n = 3 \), and \( r = 1 \) gives \( \text{Tr}(S) = 1 \).\(^1\)

The first calculation is reflective of a general result.

**Theorem (Radford-Schneider [9])**

*For pointed \( H \), we have \( \text{Tr}(S^{2r}) = 0 \) for all \( r \in \mathbb{Z} \).*

\(^1\)This calculation could be wrong. I just did it once by hand.
Question

What are some legitimate measures of non-semisimplicity to compare this to, besides global dimension...? (Will have \( \text{gldim}(H) = \infty \) for non-semisimple \( H \).)

Could possibly consider the dimension of

\[
\text{Proj}(\text{Ext}_H^{\text{even}}(k, k)),
\]

if we could prove the finite generation conjecture, which we can’t at the moment. What else is possible? Can we pull something out of the Grothendieck ring of \( \text{rep}(H) \)? At the moment don’t know. Would be interesting.
Main question

Question

Do the traces of the powers of the antipode, and the order of the antipode \( \text{ord}(S^2) \), act as categorial measures of non-semisimplicity? That is, if we have an equivalence of tensor categories

\[ F : \text{rep}(H) \xrightarrow{\sim} \text{rep}(K) \]

for Hopf algebras \( H \) and \( K \), do we then get \( \text{Tr}(S_H^m) = \text{Tr}(S_K^m) \) for all \( m \in \mathbb{Z} \), and \( \text{ord}(S_H^2) = \text{ord}(S_K^2) \)?

Behind this question is another question: Do the traces of the powers of the antipode, and \( \text{ord}(S^2) \), generalize to invariants for finite tensor categories?
Ah! (an aside)

Note: If the order of antipode for $H$ is odd then $H$ is the exceedingly trivial Hopf algebra

$$H = \mathbb{C}[\mathbb{Z}/2\mathbb{Z}]$$

We will ignore this hiccup, which is in any case easy to account for.
Relation to Known Invariants

Some examples of Hopf invariants which have the proposed categorial invariance are the quasi-exponent $\text{qexp}(H)$ [1], and the indicators $\nu_n(H)$ [2]. The quasi-exponent is like the exponent of a group

$$\exp(G) = \min \{ r : g^r = 1 \text{ for all } g \in G \}.$$ 

We will always have $\text{ord}(S^2) | \text{qexp}(H)$, so this value can’t really change that much... There’s also a strong relation between $\text{ord}(S^2)$ and the indicators, which is outlined in [6], but which I won’t talk about here.

In some general sense, we are looking for a package of invariants for non-semisimple Hopf algebras (and tensor categories) with which to attack outstanding problems from a categorial perspective.
Let us now fix $k = \mathbb{C}$!

**Theorem (N-Ng [6])**

Suppose $H$ and $K$ are Hopf algebras with the Chevalley property. If we have a tensor equivalence $F : \text{rep}(H) \sim \rightarrow \text{rep}(K)$ then

$$\text{Tr}(S_H^m) = \text{Tr}(S_K^m) \text{ for all } m \in \mathbb{Z}, \text{ and } \text{ord}(S_H^2) = \text{ord}(S_K^2).$$

We will discuss this “Chevalley property” in a second. Whatever it is, I will tell you that it is preserved under tensor equivalence, and that small quantum groups $u_q(g)$ do not have the Chevalley property.
The small quantum group

Take $q$ a prim. $n$th root of 1. Recall

$$u_q(sl_2) = \mathbb{C}\langle K, E, F \rangle / (\text{rels})$$

where rels is the set of relations

$$K^n - 1, \ E^n = F^n = 0, \ KE = q^2EK, \KF = q^{-2}FK,$$

$$[E, F] = (K - K^{-1})/(q - q^{-1}).$$

For general simple $g$,

$$u_q(g) = \mathbb{C}\langle E_\alpha, F_\alpha, K_\alpha : \alpha \text{ simp root} \rangle / (\text{rels})$$

will be a non-semisimple, finite dimensional, (quasitriangular, factorizable, ribbon, pointed) Hopf algebra which is built out of a number of quantum $sl_2$’s, $u_q(sl_2) \to u_q(g)$. 

The point is that $u_q(g)$ is a Lie theoretic object, and can be studied using approaches from Lie theory. As mentioned above no $u_q(g)$ has the Chevalley property.
More small quantum group

One way in which $u_q(g)$ “acts like a Lie algebra” is as follows: Have the “Cartan subgroup”

$$G = G(u_q(g)) = \langle K_\alpha : \alpha \text{ simp} \rangle \cong (\mathbb{Z}/l\mathbb{Z})^{\text{rank}(g)}.$$

For each char $\mu$ of $G$ have a simple rep $V(\mu)$. This gives a bijection

$$G^\vee \to \text{Irrep}(u_q(g)), \quad \mu \mapsto V(\mu).$$

One constructs the $V(\mu)$ just as one constructs irreps in rep$(g)$. (Make a Verma module, find a maximal submodule, etc.)
Results for the small quantum group

We have, vaguely,

Theorem (N [5])

For small quantum groups $u_q(g)$ (of type $A$, $D$, $E$) we can construct a number of Hopf algebras $u_q(g)^D$, from some combinatorial data $D$ on the Dynkin diagram, which admit equivalences

$$F : \text{rep}(u_q(g)) \sim \rightarrow \text{rep}(u_q(g)^D).$$

The traces of the powers of the antipode, and the order of the antipode, are invariant under these equivalences.

This gives some positive information in the non-Chevalley case.\(^2\)

\(^2\)The paper [5] is mostly concerned with analyzing the general structure of the $u_q(g)^D$ and it’s dual, but in any case we do have this result.
**The Chevalley property**

Let’s go back now: A Hopf algebra $H$ is said to have the **Chevalley property** if the Jacobson radical $\text{Jac}(H)$ is a Hopf ideal. Equivalently, $H$ is said to have the Chevalley property if the abelian subcategory of semisimple objects in $\text{rep}(H)$ is a tensor subcategory.

From the second description it is clear that this property is invariant under equivalence of tensor categories. I think of the Chevalley property as saying $\text{rep}(H)$ is “generated” by a **fusion subcategory**, i.e. semisimple tensor category. (So we can “induct” results from the fusion setting.)

**Examples:**

0. Semisimple Hopf algebras.
1. Taft algebras.
2. Duals of pointed Hopf algebras. (See the classification program of Andruskiewitsch and Schneider.)
Twists (standard stuff)

To prove that a property is invariant under tensor equivalence it suffices to prove it is invariant under “twisting”, by a result of Ng and Schauenburg [7].

A twist for a Hopf algebra $H$ is a unit $F$ in $H \otimes H$ which satisfies the dual cocycle condition

$$(F \otimes 1)(\Delta \otimes 1)(F) = (1 \otimes F)(1 \otimes \Delta)(F)$$

and

$$(\epsilon \otimes 1)(F) = (1 \otimes \epsilon)(F) = 1.$$ 

E.g. any bicharacter $F : G^\vee \times G^\vee \to \mathbb{C}^\times$ on the dual of an abelian group $G$ produces a twist for $\mathbb{C}[G]$. 
Given a twist $F$ for $H$ we produce a new Hopf algebra $H^F$, which is equal to $H$ as an algebra and has new comult $\Delta^F(h) = F\Delta(h)F^{-1}$ and antipode

$$S_F(h) = \beta_F S(h) \beta_F^{-1}, \quad \beta_F = m(1 \otimes S)(F).$$

We have a standard tensor equivalence rep$(H) \to$ rep$(H^F)$, which I’ll just call $F$.

**Theorem (Ng-Schauenburg [7])**

Any equivalence $F : \text{rep}(H) \sim \text{rep}(K)$ of tensor categories is given by such a standard equivalence, modulo Hopf isomorphism.
A method

Fix a twist $F$ of a Hopf algebra $H$.

Take $\gamma_F = \beta_F S(\beta_F)^{-1}$, so that $S_F^2(h) = \gamma_F S^2(h) \gamma_F^{-1}$.

The game: Show $\gamma_F = 1$. Or really! show that the reduction $\gamma_{\overline{F}}$ of $\gamma_F$ in $H/\text{Jac}(H)$ is 1! If you can show this then you win the game! That is, you’ve verified invariance of the traces (by [6]).
Sketch proofs in the Chevalley case

Proof.
In the Chevalley case the equivalence $F : \text{rep}(H) \xrightarrow{\sim} \text{rep}(H^F)$ induces an equivalence $\bar{F} : \text{rep}(H/Jac) \xrightarrow{\sim} \text{rep}(H^F/Jac)$. Now we have an equivalence of fusion categories, and can use all our knowledge of the fusion setting to conclude $\gamma_F = 1$, as desired. This gives invariance.

What we use from the fusion situation is the following: For semisimple $\bar{H}$ and $\bar{K}$, the categories $\text{rep}(\bar{H})$ and $\text{rep}(\bar{K})$ will be pseudo-unitary. This means they have a canonical pivotal structure, whose canonicalness implies preservation under tensor equivalence. Preservation of the pivotal structure forces immediately $\gamma_F = 1$. 
Sketch proof for $u_q(g)$

For the small quantum groups, the combinatorial data $\mathcal{D}$ on the Dynkin diagram of $g$ produces a twist $F = F(\mathcal{D})$ for $u_q(g)$. We take $u_q(g)^{\mathcal{D}} = u_q(g)^F$.

Proof.
For the small quantum group you can just look at the element $\gamma_F$ directly. It turns out that $\gamma_F = 1$. 

Remark: The Hopf algebras $u_q(g)^{\mathcal{D}}$ will really not look like $u_q(g)$ at all. For example, $u_q(g)^{\mathcal{D}}$ will no longer be pointed for a non-trivial data $\mathcal{D}$. 

**TAKEAWAY**

(i) **We wanted to produce a categorical measure of semisimplicity which actually works (unlike the global dimension).**

(ii) **We look to the order of the antipode, or for a more refined invariant to the traces of the powers of the antipode.**

(iii) **We can’t prove categorical invariance of these values in general (the problem is still open), but can prove it in the Chevalley instance and for some twists of the small quantum group.**

It would be nice to have more explicit examples of twists of finite dim’l Hopf algebras so that we could find evidence/counterexamples for this problem (and others). This is *not* a trivial task.

**Goodbye! (thanks!)**
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Y. Kashina, S. Montgomery, and S.-H. Ng.
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Finite-dimensional cosemisimple Hopf algebras in characteristic 0 are semisimple.

Semisimple cosemisimple Hopf algebras.

C. Negron.
Small quantum groups associated to belavin-drinfeld triples.

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Central invariants and higher indicators for semisimple quasi-Hopf algebras.

D. E. Radford.
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On the even powers of the antipode of a finite-dimensional Hopf algebra.