COHOMOLOGY OF FINITE TENSOR CATEGORIES: DUALITY AND DRINFELD CENTERS

CRIS NEGRON AND JULIA YAE PLAVNIK

Abstract. This work concerns the finite generation conjecture for finite tensor categories (Etingof and Ostrik [25]), which proposes that for such a category $\mathcal{C}$, the self-extension algebra of the unit $\text{Ext}^*_\mathcal{C}(1,1)$ is a finitely generated algebra and that, for each object $V$ in $\mathcal{C}$, the graded extension group $\text{Ext}^*_\mathcal{C}(1,V)$ is a finitely generated module over the aforementioned algebra. We prove that this finite generation property for cohomology is preserved under duality (with respect to exact module categories) and taking the Drinfeld center, under suitable restrictions on $\mathcal{C}$. For example, the stated result holds when $\mathcal{C}$ is a braided tensor category of odd Frobenius-Perron dimension. By applying our general results, we obtain a number of new examples of finite tensor categories with finitely generated cohomology. In characteristic 0, we show that dynamical quantum groups at roots of unity have finitely generated cohomology. We also provide a new class of examples in finite characteristic which are constructed via infinitesimal group schemes.

1. Introduction

We are interested in the following conjecture, which first appeared formally in work of Etingof and Ostrik [25].

Conjecture 1.1 (Finite generation). For any finite tensor category $\mathcal{C}$, the cohomology $H^*(\mathcal{C},1)$ is a finitely generated algebra and, for any object $V$ in $\mathcal{C}$, the cohomology $H^*(\mathcal{C},V)$ is a finitely generated $H^*(\mathcal{C},1)$-module.

Here by the cohomology $H^*(\mathcal{C},V)$ we mean the graded group of extensions from the unit $H^*(\mathcal{C},V) := \text{Ext}^*_\mathcal{C}(1,V)$. For us, a tensor category means a finite tensor category. We fix a base field $k$ which is generally of arbitrary characteristic, although some results will be specific to characteristic 0.

Definition. We say a tensor category $\mathcal{C}$ is of finite type (over $k$) if its cohomology $H^*(\mathcal{C},1)$ is a finitely generated algebra and $H^*(\mathcal{C},V)$ is a finitely generated $H^*(\mathcal{C},1)$-module for each $V$ in $\mathcal{C}$. For $\mathcal{C}$ of finite type, we define the Krull dimension of $\mathcal{C}$ as $\text{Kdim} \mathcal{C} := \text{Kdim} H^{**}(\mathcal{C},1)$.

Much of the progress on the finite generation conjecture to date has focused on examples. For a short historical account one can see the introduction to [29], or consult the primary sources [30, 36, 31, 43, 18, 6, 58]. Although in Lie algebraic settings authors have managed to leverage the ambient Lie theory to gain an

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understanding of cohomology—notably the category of polynomial functors in [31]—progress has been hampered by a dearth of tools with which to analyze the behavior of cohomology in any generality. Motivated by the current situation we propose the following related conjectures.

**Conjecture 1.2** (Cohomological stability). If $\mathcal{C}$ is a tensor category which is of finite type and $\mathcal{M}$ is an exact $\mathcal{C}$-module category, then the corresponding dual category $\mathcal{C}^\ast_{\mathcal{M}}$ is also of finite type. Furthermore, in characteristic $0$, the Krull dimension is invariant under duality

$$K\dim \mathcal{C}^\ast_{\mathcal{M}} = K\dim \mathcal{C}.$$  

In general, it is quite difficult to establish a precise equality of Krull dimensions under duality. So here we focus on uniformly bounding the Krull dimensions of the duals $\mathcal{C}^\ast_{\mathcal{M}}$ by a polynomial function in $K\dim \mathcal{C}$. By uniform we mean uniform across all $\mathcal{M}$. Whence we have a relaxation of the above conjecture.

**Conjecture 1.3** (Weak cohomological stability). The finite type property is preserved under categorical duality, as in Conjecture 1.2. Furthermore, in characteristic $0$, there is a polynomial $P_{\text{univ}} \in \mathbb{R}_{\geq 0}[X]$ which provides a uniform bound on the Krull dimensions of the duals

$$K\dim \mathcal{C}^\ast_{\mathcal{M}} \leq P_{\text{univ}}(K\dim \mathcal{C}).$$

In Conjecture 1.3 the polynomial $P_{\text{univ}}$ is to be independent of choice of $\mathcal{C}$ and $\mathcal{M}$. We note that Conjecture 1.3 also implies a lower bound on the Krull dimensions of the duals, as $P_{\text{univ}}$ is an increasing function in $X$ and is therefore invertible.

The notion of categorial duality, also known as categorical Morita equivalence, was introduced by Müger in [47] (cf. [52]) and now plays a fundamental role in the general theory of tensor categories. The precise construction of the dual $\mathcal{C}^\ast_{\mathcal{M}}$ is recalled in Section 4 and a discussion of the specific meaning of Conjectures 1.2 and 1.3 for Hopf algebras is given below, in Subsection 1.1.

In the present paper we prove the weak stability conjecture for “most” braided tensor categories in characteristic $0$; specifically braided tensor categories with semisimple Müger center. The polynomial in this case is simply $P_{\text{braid}}(X) = 2X$. More explicit descriptions of our main results and examples are given below.

Our general method is to prove that the Drinfeld center $Z(\mathcal{C})$ of $\mathcal{C}$ is of finite type provided that $\mathcal{C}$ is of finite type, then to appeal to the fact that the construction of the center is invariant under categorical duality (see Proposition 6.5/6.6). The Drinfeld center approach allows us to obtain some results for categories in finite characteristic as well. In terms of the center, the stability conjecture proposes that $Z(\mathcal{C})$ is of finite type whenever $\mathcal{C}$ is of finite type and that $K\dim Z(\mathcal{C}) = 2K\dim \mathcal{C}$. Similarly, weak stability proposes that the finite type property is preserved under formation of the center and that the Krull dimension of $Z(\mathcal{C})$ grows sub-exponentially as a function of $K\dim \mathcal{C}$.

**Remark 1.4.** Formally speaking, the dual $\mathcal{C}^\ast_{\mathcal{M}}$ will be a “multi-tensor” category when $\mathcal{M}$ is decomposable. Although we are not interested in this case, decomposability of $\mathcal{M}$ causes no problems for us, so we allow $\mathcal{M}$ to be decomposable in general.

**Remark 1.5.** The proposed bound on the Krull dimension is clearly false in finite characteristic. For example, $\text{rep}(\mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}]^{\otimes d})$ has Krull dimension $d$ while...
rep(\mathcal{O}(\mathbb{Z}/p\mathbb{Z})\otimes d) has Krull dimension 0, even though rep(\mathcal{O}(\mathbb{Z}/p\mathbb{Z})\otimes d)^*_\text{vect} is tensor equivalent to rep(\mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}]\otimes d).

1.1. Stability of cohomology for Hopf algebras. Let us explain the situation a bit more clearly for the Hopf algebraically inclined. As with categories, a Hopf algebra always means a finite dimensional Hopf algebra.

In the Hopf setting, Conjecture 1.2 can be seen as a generalization of a theorem of Larson and Radford, which states that a finite dimensional Hopf algebra in characteristic 0 is semisimple if and only if its dual is semisimple [38]. As we will explain, one can obtain vector space duality as a specific instance of categorical duality.

Let us consider some illustrative examples. Take a Drinfeld twist \( J \) of a Hopf algebra \( A \). Then we have a canonical tensor equivalence \( \text{rep}(A) \xrightarrow{\sim} \text{rep}(A^J) \), where \( A^J \) is obtained by altering the comultiplication of \( A \) via \( J \). We also have the forgetful functor \( \text{rep}(A) \to \text{Vect} \) and the alternate functor \( \text{rep}(A) \to \text{rep}(A^J) \to \text{Vect} \). These functors produce \( \text{rep}(A) \)-module categories \( M = \text{Vect} \) and \( \text{Vect}_J \), where \( \text{Vect}_J \) is \( \text{Vect} \) with the \( \text{rep}(A) \)-action "twisted" by \( J \). The dual categories are

\[
\text{rep}(A)^*_\text{vect} = \text{rep}(A^*)^{\text{cop}}, \quad \text{rep}(A)^*_\text{vect}_J = \text{rep}((A^*)_J)^{\text{cop}},
\]

where in the second instance \( J \) alters the multiplicative structure on \( A^* \) as a cocycle twist. The superscript "cop" here means we are taking the opposite tensor product. (Notice that the cop operation does not affect cohomology.) One can also find a module category \( \mathcal{M}(\sigma) \) associated to any 2-cocycle \( \sigma : A \otimes A \to k \) so that \( \text{rep}(A)^*_\mathcal{M}(\sigma) \cong \text{rep}(A_{\sigma}) \).

Whence the (weak) stability conjecture proposes, among other things, that if \( A \) has finitely generated cohomology then any cocycle twist \( A_{\sigma} \) also has finitely generated cohomology, as does its dual \( A^* \) and any cocycle twists of its dual \( A^*_J \). Furthermore, the Krull dimension of cohomology is proposed to be invariant under these operations, or to vary at most as a polynomial in \( \text{Kdim} H^*(A,k) \).

In this work we prove a number of our results in a Hopf theoretic setting before giving more general, categorical, proofs. This is for the practical reason that many mathematicians working on the finite generation conjecture are focused on Hopf algebras. However, such a presentation is not possible for the material of Sections 8–11, and so the interested reader will have to engage with the general theory of tensor categories.

1.2. New examples. Among our contributions herein, there are two explicit classes of new examples. In finite characteristic, we show that if \( G \) is a Frobenius kernel in a smooth algebraic group \( G = G^{(r)} \), then arbitrary duals of the category of representations \( \text{rep}(G)^*_\mathcal{M} \) are of finite type. Furthermore, we uniformly bound the Krull dimensions

\[
\text{Kdim} \text{rep}(G)^*_\mathcal{M} \leq \text{Kdim} \text{rep}(G) + \dim \mathfrak{g},
\]

where \( \mathfrak{g} \) is the Lie algebra of \( G \). These results are obtained as an application of our work here in conjunction with results of Friedlander and the first author [29], and appear in Corollary 5.7 below. We note that exact module categories over \( \text{rep}(G) \) were classified via cohomological data by Gelaki [33].

One can produce concrete examples of new Hopf algebras with finitely generated cohomology via Corollary 5.7. Specifically, one can take cocycle twists of the function algebra \( \mathcal{O}(G) \) to produce Hopf algebras in characteristic \( p \) which are
neither commutative nor cocommutative but are seen to have finitely generated cohomology (see Section 5.3).

In characteristic 0, we prove that the dynamical quantum groups of Etingof and Nikshych [23] (see also [27, 61]) have finitely generated cohomology. More in depth descriptions of dynamical quantum groups are given in Section 7 and Appendix A but let us say here that while usual quantum groups are associated to constant solutions to the Yang-Baxter equation, dynamical quantum groups are associated to parameter dependent solutions to the Yang-Baxter equation. In particular, dynamical quantum groups are not Hopf algebras in the strict sense of the term, although they do have associated tensor categories of representations. These examples are covered in Section 7.

One should also be able to obtain new examples from work of Mastnak, Pevtsova, Schauenburg, and Witherspoon [43]. We give a short discussion of this topic in Section 11.5 and, in particular, Remark 11.13.

1.3. Description of main results. The following result provides the foundation for the present study.

**Theorem (4.9).** Suppose that $\mathcal{D}$ is a tensor category of finite type and that $F : \mathcal{D} \to \mathcal{C}$ is a surjective tensor functor. Then $\mathcal{C}$ is also of finite type and has bounded Krull dimension $Kdim \mathcal{C} \leq Kdim \mathcal{D}$.

When one considers representation categories of Hopf algebras, Theorem 4.9 appears as follows.

**Theorem (3.4).** Suppose that $A \to D$ is a Hopf inclusion and that $D$ has finitely generated cohomology. Then $A$ also has finitely generated cohomology and the Krull dimension is bounded as $Kdim H^{ev}(A, k) \leq Kdim H^{ev}(D, k)$.

The center $Z(\mathcal{C})$ provides the necessary link between the cohomology of $\mathcal{C}$ and the cohomology of its duals $\mathcal{C}^*$. More specifically, any dual $\mathcal{C}^*$ admits a surjective tensor functor from the center $Z(\mathcal{C})$ [28, Proposition 8.5.3], and hence $\mathcal{C}^*$ is of finite type and of Krull dimension $\leq Kdim Z(\mathcal{C})$ whenever the center of $\mathcal{C}$ is of finite type. We use Theorem 4.9 in the braided setting to address (weak) stability of cohomology.

Recall that for a Hopf algebra $A$, a braiding on $rep(A)$ is exactly the information of a quasitriangular structure on $A$. Recall also that the Müger center of a braided tensor category $\mathcal{C}$ is the full tensor subcategory of all objects $V$ in $\mathcal{C}$ which braid “trivially” with every other object in $\mathcal{C}$ (see Section 8).

**Theorem (11.1).** Let $\mathcal{C}$ be a braided tensor category of finite type over a field of characteristic 0. Suppose that the Müger center of $\mathcal{C}$ is semisimple. Then any dual category $\mathcal{C}^*_{\mathcal{M}}$ with respect to an exact $\mathcal{C}$-module category $\mathcal{M}$ is also of finite type. Furthermore, there is a uniform bound on the Krull dimensions

$$Kdim \mathcal{C}^*_{\mathcal{M}} \leq 2 Kdim \mathcal{C}.$$

Basic information on the Müger center, as well as means of determining its semisimplicity, are given in Sections 8.1, 10.3, and 11.5. Theorem 11.1 is more easily understood through a simple corollary. Recall that a tensor category $\mathcal{C}$ is called weakly integral if its Frobenius-Perron dimension is an integer. For example, representation categories of Hopf algebras are weakly integral with $\text{FPdim}(rep(A)) = \dim A$. 
Corollary (11.11). Suppose \( \mathcal{C} \) is of finite type over a field of characteristic 0 and of integral Frobenius-Perron dimension which is not divisible by 4. If \( \mathcal{C} \) admits a braiding, then for every exact module category \( \mathcal{M} \) the corresponding dual \( \mathcal{C}^* \) is also of finite type and, furthermore, \( \text{Kdim} \mathcal{C}^* \leq 2 \text{Kdim} \mathcal{C} \).

For non-degenerate categories, i.e. categories with Müger center equivalent to \( \text{Vect} \), Theorem 11.1 is a fairly straightforward application of Theorem 4.9. We deal with this case independently in Propositions 3.6 and 6.4. When the Müger center of \( \mathcal{C} \) is not trivial, the situation becomes much more dynamic. We must consider here two cases: the case where the Müger center is Tannakian and the case where the Müger center is non-Tannakian (but semisimple). We address the Tannakian case in Section 10 and the non-Tannakian case in Section 11.

In terms of braided categories, there is a final possibility which we do not address here. Namely, when the Müger center of \( \mathcal{C} \) is the representation category of a super group with non-vanishing odd functions (see Section 8.1). At the moment, it seems that this case will require either a different approach than the one taken here, or a much more non-trivial analysis of spectral sequences relating the cohomology of \( \mathcal{C} \) to that of its Drinfeld center.

1.4. Cohomology under (de-)equivariantization and short exact sequences. Our proofs of Theorems 10.2 and 11.1 rely on intermediate results which allow us to trace cohomology through a variety of (de-)equivariantizations and extensions. As these results may be of independent interest, we list them here. In all of the following statements the base \( k \) is assumed to be algebraically closed, although the assumption is inessential (see Lemma 6.2):

- Let char(\( k \)) = 0. Suppose that \( \mathcal{C} \) is a de-equivariantization of \( \mathcal{D} \) by a finite group, or equivalently that \( \mathcal{D} \) is an equivariantization of \( \mathcal{C} \). Then \( \mathcal{C} \) is of finite type if and only if \( \mathcal{D} \) is of finite type (Theorem 5.2). Furthermore, \( Z(\mathcal{C}) \) is of finite type if and only if \( Z(\mathcal{D}) \) is of finite type (Theorem 10.1).

- Suppose that \( \mathcal{B} \to \mathcal{C} \to \mathcal{D} \boxtimes \text{End}_k(\mathcal{M}) \) is an exact sequence relative to an exact \( \mathcal{B} \)-module category \( \mathcal{M} \), as defined in [22]. Then there is a relative invariants functor \( \mathcal{H}^0(\mathcal{D},-) : \mathcal{C} \to \mathcal{B} \) and a multiplicative spectral sequence

\[
H^i(\mathcal{B}, \mathcal{H}^j_{\mathcal{C}}(\mathcal{D},1)) \Rightarrow H^{i+j}(\mathcal{C},1). \tag{1}
\]

Furthermore, for any object \( V \) in \( \mathcal{C} \) there is a spectral sequence

\[
H^i(\mathcal{B}, \mathcal{H}^j_{\mathcal{C}}(\mathcal{D},V)) \Rightarrow H^{i+j}(\mathcal{C},V),
\]

which is equipped with a compatible action of (1) (Proposition 9.12).

We note that the spectral sequence (1) reduces to the familiar Lyndon-Hochschild-Serre spectral sequence in the case of an exact sequence of finite groups, or rather when one applies \( \text{rep}(-) \) to such a sequence.

We discuss the behavior of cohomology under \( G \)-extensions, for a finite group \( G \), in Section 11.1.

1.5. General organization. The present paper has two main portions. In the first portion, which consists of Sections 3–7, we give results relating the finite type property for the center \( Z(\mathcal{C}) \) to the finite type property for arbitrary duals \( \mathcal{C}^* \). These materials are punctuated by the examples of Sections 5.3 and 7. For the first portion of the paper \( k \) is a base field of arbitrary characteristic, unless explicitly stated otherwise. In the second portion, which consists of Sections 8–11, we pursue
the (weak) stability conjecture for braided tensor categories more generally. As
relayed above, we provide a proof of Conjecture 1.3 for braided tensor categories
with semisimple Müger center. We always assume \( k \) is of characteristic 0 in this
latter portion of the paper.

There are two appendices. In Appendix A, we discuss relations between dynamical twists and module categories and in Appendix B, we prove a theorem regarding semisimplicity of Müger centers for categories of finite dimensional representation
of pointed Hopf algebras.

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2. Background on tensor categories and cohomology

We assume the reader is well-acquainted with Hopf algebras. We give here some
information about the more general framework of tensor categories.

2.1. Conventions. We fix a base field \( k \) of arbitrary characteristic. We let \( \text{Kdim} \ H^\bullet \)
denote the Krull dimension of a graded commutative algebra \( H^\bullet \), i.e. the supremum
of the lengths of chains of prime ideals in \( H^\bullet \). Since \( H^\bullet \) is graded commutative,
any odd degree element is nilpotent. So the Krull dimension of \( H^\bullet \) is equal to the
Krull dimension of its (commutative) even subalgebra \( \text{Kdim} \ H^\bullet = \text{Kdim} H^{\text{ev}} \).

By a “\( k \)-linear category” we mean an abelian category enriched over \( \text{Vect} \). For a tensor category \( \mathcal{C} \), we let \( \mathcal{C}^{\text{cop}} \) denote the tensor category which is equal to \( \mathcal{C} \) as a \( k \)-linear category along with the opposite tensor product \( V \otimes^{\text{cop}} W = W \otimes V \).
For a Hopf algebra \( A \), for example, \( \text{rep}(A)^{\text{cop}} = \text{rep}(A^{\text{cop}}) \). When \( \mathcal{C} \) is a braided
tensor category, with braiding \( c \), we write \( \mathcal{C}^{\text{rev}} \) for the tensor category \( \mathcal{C} \) equipped
with the reverse braiding, \( c^{\text{rev}}_{V,W} = c^{-1}_{W,V} \).
Standard, categorial, opposites are denoted by a non-roman superscript, \(C^{op}\). We let \(G(C)\) denote the group of (isomorphism classes of) invertible objects in a tensor category \(C\).

2.2. Tensor categories and representations of Hopf algebras. A \(k\)-linear (abelian) category \(C\) is said to be \(finite\) if it is equivalent to the category of modules over a finite dimensional algebra. Rather, \(C\) is finite if it has finitely many simple objects (up to isomorphism), finite dimensional hom spaces, enough projectives, and all objects have finite length.

In this work, by a tensor category we will always mean a finite tensor category.

Definition 2.1 ([28]). A (finite) tensor category \(C\) is a \(k\)-linear, finite, rigid, monoidal category such that the monoidal structure \(\otimes : C \times C \to C\) is \(k\)-linear in each factor. We require additionally that the unit object \(1\) of \(C\) is simple and that \(\text{End}_C(1,1) = k\).

At times we label the unit of \(C\) as \(1_C\). However, when no confusion will arise, we employ the simpler notation \(1\). Of course, such a \(C\) comes equipped with a (generally nontrivial) associator, and some additional compatibilities with the unit. Although these seemingly subtle structures are quite important in general, they do not play a significant role in our study.

The rigid structure refers to the existence of left and right duals for each object \(V\) in \(C\). These are objects \(V^*\) and \(^*V\) respectively which come equipped with evaluation
\[ ev_V^l : V^* \otimes V \to 1, \quad ev_V^r : V \otimes ^*V \to 1, \]
and coevaluation
\[ coev_V^l : 1 \to V \otimes V^*, \quad coev_V^r : 1 \to ^*V \otimes V, \]
maps which satisfy a number of exceedingly useful axioms. We do not list the axioms here, but refer the reader to [28, Section 2.10] for details and basic implications.

The algebraically inclined reader is free to think of \(C\) as the category of (finite dimensional) representations of a finite dimensional Hopf algebra \(A\). In this case \(\text{rep}(A)\) has the usual monoidal structure \(\otimes = \otimes_k\) and the duals are give by
\[ V^* = \text{Hom}_k(V,k) \text{ with } A\text{-action } a \cdot f = (v \mapsto f(S(a)v)), \]
\[ ^*V = \text{Hom}_k(V,k) \text{ with } A\text{-action } a \cdot f = (v \mapsto f(S^{-1}(a)v)). \]
The evaluation maps are the usual ones, \(f \otimes v \mapsto f(v)\) and \(v \otimes f \mapsto f(v)\), respectively. If we choose dual bases \(\{v_i, f_i\}\), for a representation and its linear dual then the coevaluation maps are given by \(1 \mapsto \sum_i v_i \otimes f_i\) and \(1 \mapsto \sum_i f_i \otimes v_i\), respectively.

2.3. The Yoneda-product on cohomology, a quick review. Recall that the bounded derived category \(D^b(C)\) of an abelian category \(C\) has objects given by complexes with bounded cohomology and morphisms given by equivalence classes of pairs \(X \xrightarrow{s} Y \xleftarrow{t} X'\), where \(s\) is a quasi-isomorphism. We denote such an equivalence class by \(fs^{-1} : X \to X'\). The composition of two morphisms \(X \leftarrow Y \to X'\) and
Remark 2.2. Since the algebra opposite to the composition operation. One can check that this action agrees with the usual action of $V$, the difference between $H^\bullet(\mathcal{C}, 1)$ and $H^\bullet(\mathcal{C}, 1)^{op}$ is essentially negligible, we will however keep track of the distinction for this subsection.

We have for any $V$ in $\mathcal{C}$ the exact functor $- \otimes V$ and subsequent algebra morphism

$$- \otimes V : H^\bullet(\mathcal{C}, 1)^{op} \to \text{Ext}_F^\bullet(V, V).$$

The map $- \otimes V$ takes an extension $1 \xleftarrow{e} X \xrightarrow{f} \Sigma^n 1$ of 1 to the extension $V \xleftarrow{s \otimes V} X \otimes V \xrightarrow{f \otimes V} \Sigma^n V$

of $V$. By an argument similar to the one employed in [59] one finds

Lemma 2.3. The algebra map $- \otimes V : H^\bullet(\mathcal{C}, 1)^{op} \to \text{Ext}_F^\bullet(V, V)$ has image in the (graded) center of $\text{Ext}_F^\bullet(V, V)$.

We define a left action of $H^\bullet(\mathcal{C}, 1)^{op}$ on $H^\bullet(\mathcal{C}, V)$ via the tensor product. Namely, we take

$$\left( 1 \xleftarrow{s} X \xrightarrow{f} \Sigma^n 1 \right) \cdot \left( 1 \xleftarrow{t} Y \xrightarrow{g} \Sigma^n V \right)$$

$$= \left( 1 \xleftarrow{s \otimes t} X \otimes Y \xrightarrow{f \otimes g} \Sigma^n V \right).$$

One can check that this action agrees with the usual action $H^\bullet(\mathcal{C}, V) \otimes H^\bullet(\mathcal{C}, 1) \to H^\bullet(\mathcal{C}, V)$ given by composing morphisms in $D^b(\mathcal{C})$.

If we consider the object $W \otimes V^*$, we have the adjunction

$$H^\bullet(\mathcal{C}, W \otimes V^*) = \text{Ext}_F^\bullet(V, W)$$

which explicitly sends a map $f s^{-1} : k \xleftarrow{} X \to \Sigma^n W \otimes V^*$ to $V \xleftarrow{s \otimes id} X \otimes V \xrightarrow{f^*} \Sigma^n W$ where $f^*$ is $f \otimes V$ composed with the evaluation $id_{\Sigma^n W} \otimes ev_{V}$ (see [28, Proposition 2.10.8]).

Lemma 2.4. The adjunction $H^\bullet(\mathcal{C}, W \otimes V^*) = \text{Ext}_F^\bullet(V, W)$ is an identification of $H^\bullet(\mathcal{C}, 1)^{op}$-modules, where $H^\bullet(\mathcal{C}, 1)^{op}$ acts on $\text{Ext}_F^\bullet(V, W)$ via the above algebra map $- \otimes V$ to $\text{Ext}_F^\bullet(V, V)$.
Proof. For the class of a homogenous map \( f s^{-1} \) in \( H^\bullet(\mathcal{C}, 1) \), and \( g t^{-1} \in H^\bullet(\mathcal{C}, W \otimes V^*) \) with corresponding \( g'(t^{-1}) \in \text{Ext}^\bullet_{\mathcal{C}}(V, W) \), one simply needs to compare \( f s^{-1} \cdot g t^{-1} \) to \( f s^{-1} \cdot g'(t \otimes id_V)^{-1} \) under the adjunction. We have directly

\[
fs^{-1} \cdot gt^{-1} = (f \otimes g)(s \otimes t)^{-1} \quad \text{adj} \quad (id \otimes ev_V)(f \otimes g \otimes id_V)(s \otimes t \otimes id_V)^{-1} = (f \otimes g')(s \otimes t \otimes id_V)^{-1}.
\]

By a direct comparison, as in (2), one see that this last element is \( g'(t \otimes id_V)^{-1} \circ ((fs^{-1}) \otimes V) \), as desired. \( \square \)

Lemma 2.5. The association

\[
\text{Ext}^\bullet_{\mathcal{C}} : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow H^\bullet(\mathcal{C}, 1)\text{-mod}, \quad (W, V) \mapsto \text{Ext}^\bullet_{\mathcal{C}}(W, V),
\]

is functorial in both \( V \) and \( W \).

Proof. The assignment \( (W, V) \mapsto H^\bullet(\mathcal{C}, V \otimes W^*) \) is clearly a bifunctor. So the result follows from the natural isomorphism \( \text{Ext}^\bullet_{\mathcal{C}}(W, V) \cong H^\bullet(\mathcal{C}, V \otimes W^*) \) and Lemma 2.4. \( \square \)

Remark 2.6. One can deduce from Lemma 2.4 that for any finite type tensor category \( \mathcal{C} \) we have the equivalent global definition for the Krull dimension

\[
\text{Kdim} \mathcal{C} = \max \{ \text{GKdim} \text{Ext}^\bullet_{\mathcal{C}}(V, V) : V \in \text{Ob} \mathcal{C} \}.
\]

In the sections that follow we simply refer to \( H^\bullet(\mathcal{C}, 1) \) as acting on the right of \( H^\bullet(\mathcal{C}, V) \), rather than using the left \( H^\bullet(\mathcal{C}, 1)^{\text{op}} \)-action.

3. Transfer of cohomology for Hopf algebras

We say that a Hopf algebra \( A \) has finitely generated cohomology if \( \text{rep}(A) \) is of finite type. Given a Hopf inclusion \( A \to D \), we show that \( A \) has finitely generated cohomology whenever \( D \) has finitely generated cohomology. We show further that the Krull dimension of cohomology for \( A \) is bounded by that of \( D \) in this case. In this way we “transfer” cohomological properties from \( D \) to \( A \). We consider the case in which \( D = D(A) \) is the Drinfeld double of \( A \) in order to speak to stability of cohomology under cocycle twisting and duality for \( A \).

3.1. Finite generation for subalgebras. Fix a Hopf inclusion \( A \to D \). Let \( F : \text{rep}(D) \to \text{rep}(A) \) denote the restriction functor and \( I : \text{rep}(A) \to \text{rep}(D) \) denote the induction functor. Explicitly, \( I(V) = D \otimes_A V \) with left \( D \)-action \( \theta : (\theta' \otimes v) = (\theta' \otimes v) \). By the Nichols-Zoeller theorem, \( D \) is projective over \( A \) and also injective, since \( A \) is Frobenius \([51, 45]\). So induction is exact.

For any \( A \)-representation \( V \), we have the natural map \( V \to I(V) = D \otimes_A V \), \( v \mapsto 1 \otimes v \). Indeed, this gives a natural transformation of functors \( id_{\text{rep}(A)} \to F \circ I \), which one sees is the unit of the adjunction between \( F \) and \( I \). We take the right dual to get a natural transformation \( F \circ (-)^* \circ I = (-)^* \circ F \circ I \to (-)^* \), and precompose with the left dual to get a natural transformation

\[
\tau : F \circ (-)^* \circ I \circ (-)^* \to (-)^* \circ id_{\text{rep}(A)}.
\]

Definition 3.1. We define the dual induction functor \( I^* : \text{rep}(A) \to \text{rep}(D) \) as the composite \( I^* = (-)^* \circ I \circ (-)^* \). On objects, \( I^*(V) = (D \otimes_A V)^* \).
As above, we let

$$\tau : F \circ I^* \to id_{\text{rep}(A)}$$

denote the natural transformation given by the dual of the unit $id_{\text{rep}(A)} \to F \circ I$.

From the counit of $D$ we get a $D$-linear projection $I(k) \to k$, $\theta \otimes 1 \mapsto \epsilon(\theta)$, and subsequent inclusion $i_k : k \to I^*(k)$. One sees directly that the sequence $k \to I(k) \to k$ is the identity, and hence so is the dual sequence $\tau_k i_k : k \to I^*(k) \to k$.

One can deduce from the above information a natural linear map on cohomology

$$\text{Mres} : \text{Ext}^*_D(I^*(k), I^*(V)) \to \text{Ext}^*_A(k, V) = H^*(A, V),$$

(4)

which we call the mixed restriction. Explicitly, for any $A$-representation $V$ we take an injective resolution $V \to Q$, then apply the (exact) functor $I^*$ to get a quasi-isomorphism $I^*(V) \to I^*(Q)$. Since induction preserves projectives, and $D$ and $A$ are Frobenius, we see that $I^*(Q)$ is an injective resolution of $I^*(V)$. So we have

$$\text{Ext}^*_D(I^*(k), I^*(V)) = H^* \left( \text{Hom}_D(I^*(k), I^*(Q)) \right).$$

At the cochain level we now have a map

$$\text{Mres} : \text{Hom}_D(I^*(k), I^*(Q)) \to \text{Hom}_A(k, Q), \ f \mapsto \tau_Q f i_k.$$ (Note that naturality of $\tau$ implies that $\tau_Q$ is a chain map.) Take cohomology of $\text{Mres}$ to arrive at (4).

**Definition 3.2.** For a Hopf inclusion $A \to D$, and a representation $V$ over $A$, the mixed restriction for $V$ is the map

$$\text{Mres} : \text{Ext}^*_D(I^*(k), I^*(V)) \to H^*(A, V)$$

defined as above.

In the following proposition we consider $H^*(A, V)$ to be a $H^*(D, k)$-module by way of the (usual) restriction $H^*(D, k) \to H^*(A, k)$.

**Proposition 3.3.** For any representation $V$ over $A$, the mixed restriction is a surjective morphism of $H^*(D, k)$-modules.

**Proof.** Let $V \to Q$ be an injective resolution. For any $A$-linear map $f : k \to Q$ we have a diagram

$$\begin{array}{ccc}
k & \xrightarrow{i_k} & I^*(k) \\
\downarrow{\tau_k} & & \downarrow{\tau_Q} \\
Q & \xrightarrow{f} & Q
\end{array}$$

via naturality of $\tau$. So we see that the composite

$$H^*(A, V) \xrightarrow{i^*} \text{Ext}^*_D(I^*(k), I^*(V)) \xrightarrow{\text{Mres}} H^*(A, V)$$

is the identity. Hence the mixed restriction is surjective.

As for $H^*(D, k)$-linearity, one can decompose the mixed restriction as the composite

$$\text{Ext}^*_D(I^*(k), I^*(V)) \xrightarrow{\text{res}} \text{Ext}^*_A(I^*(k), I^*(V)) \xrightarrow{(i_k)\tau} \text{Ext}^*_A(k, I^*(V)) \xrightarrow{(\tau_Q)} \text{Ext}^*_A(k, V) = H^*(A, V).$$

Each of the maps in the above composite are $H^*(D, k)$-linear. Hence the composite is $H^*(D, k)$-linear. $\square$
Theorem 3.4. Suppose we have a Hopf inclusion $A \to D$, and that $D$ has finitely generated cohomology. Then $A$ has finitely generated cohomology as well, and the Krull dimension for $A$ is bounded by that of $D$,

$$\text{Kdim } H^\bullet(A, k) \leq \text{Kdim } H^\bullet(D, k).$$

Proof. Supposing that $D$ has finitely generated cohomology, we have that each $\text{Ext}^*_D(I^*(k), I^*(V))$ is a finitely generated $H^\bullet(D, k)$-module. Surjectivity, and $H^\bullet(D, k)$-linearity, of the mixed restriction now implies that each $H^\bullet(A, V)$ is finitely generated over the image of $H^\bullet(D, k)$ in $H^\bullet(A, k)$. Thus each $H^\bullet(A, V)$ is finite over $H^\bullet(A, k)$ itself. If we consider the case $V = k$, this tells us that $H^\bullet(A, k)$ is finite over $H^\bullet(D, k)$, and hence a finitely generated algebra of Krull dimension less than or equal to that of $H^\bullet(D, k)$. \hfill $\square$

3.2. Applications via the Drinfeld double. One applies Theorem 3.4 to obtain

Corollary 3.5. If the Drinfeld double $D(A)$ of a Hopf algebra $A$ has finitely generated cohomology then

- any cocycle twist $A_{\sigma}$ of $A$ has finitely generated cohomology;
- the dual Hopf algebra $A^*$ has finitely generated cohomology;
- any cocycle twist of the dual $(A^*)_J$ has finitely generated cohomology;

and the Krull dimensions are uniformly bounded

$$\text{Kdim } H^\bullet(A_{\sigma}, k), \text{Kdim } H^\bullet(A^*, k), \text{Kdim } H^\bullet((A^*)_J, k) \leq \text{Kdim } H^\bullet(D(A), k).$$

Proof. If we consider $D = D(A)$, in Theorem 3.4, then our claims follow from the isomorphism $D(A^*)^{\text{cop}} \cong D(A)$ and general equivalence $\text{rep}(D(B_0)) \cong \text{rep}(D(B))$ for any Hopf algebra $B$ and 2-cocycle $\sigma$ \cite{[42, 7]}.

The proof of the following proposition uses some material of Section 6.1, which concerns cohomology of Deligne products of tensor categories. All of the results of Section 6.1 are imminently provable for Hopf algebras, and so we simply refer to the results of Section 6.1 when necessary, rather than delay the proposition.

Recall that a Hopf algebra $A$ is called factorizable if it is quasitriangular and the associated algebra map $f_R : D(A) \to A \otimes A$ is an isomorphism \cite[Theorem 2.9]{[54]} (see also \cite{[53, 28]}). The most prominent examples of factorizable Hopf algebras are Lusztig’s small quantum groups.

Proposition 3.6. Let $A$ be a factorizable Hopf algebra, and suppose $A$ has finitely generated cohomology. Then the following hold:

- Any cocycle twist $A_{\sigma}$ of $A$ has finitely generated cohomology;
- the dual Hopf algebra $A^*$ has finitely generated cohomology;
- any cocycle twist of the dual $(A^*)_J$ has finitely generated cohomology.

Furthermore, the Krull dimensions are uniformly bounded

$$\text{Kdim } H^\bullet(A_{\sigma}, k), \text{Kdim } H^\bullet(A^*, k), \text{Kdim } H^\bullet((A^*)_J, k) \leq 2 \text{Kdim } H^\bullet(A, k).$$

Proof. Since $A$ is factorizable there is an isomorphism $D(A) \cong A \otimes A$ of augmented algebras \cite{[53]}. So it suffices to show that $A \otimes A$ has finitely generated cohomology of Krull dimension $2 \text{Kdim } H^\bullet(A, k)$. We have $H^\bullet(A \otimes A, k) = H^\bullet(A, k) \otimes H^\bullet(A, k)$, which implies that $H^\bullet(A \otimes A, k)$ is a finitely generated algebra of Krull dimension twice that of $H^\bullet(A)$. (One uses Noether normalization to find that $\text{Kdim } H^\bullet \otimes H^\bullet = \ldots$)
2Kdim $H^{ullet}$.) The fact that $H^{ullet}(A \otimes A, V)$ is a finitely generated $H^{ullet}(A \otimes A, k)$-module, for each $A$-representation $V$, follows by Lemma 6.3, applied to $\mathcal{C} = \mathcal{C}' = \text{rep}(A)$. □

We consider the specific example of the small quantum groups in Section 7.

4. Transfer of cohomology for tensor categories

In this section we prove a version of Theorem 3.4 for tensor categories. We show that if $F : \mathcal{D} \to \mathcal{C}$ is a surjective tensor functor, and $\mathcal{D}$ is of finite type, then $\mathcal{C}$ is also of finite type. We also bound the Krull dimension of $\mathcal{C}$ by the Krull dimension of $\mathcal{D}$ in this case. Our methods of proof are direct generalizations of those employed in Section 3. Specific applications are given in Section 5.

4.1. (Surjective) tensor functors. Following the conventions of [25], a tensor functor $F : \mathcal{D} \to \mathcal{C}$ is exact by definition. We are free to assume that any tensor functor $F$ is such that $F(1_\mathcal{D}) = 1_\mathcal{C}$, and make this assumption here (see [28, Remark 2.4.6]). We recall the following basic definition.

Definition 4.1. A tensor functor $F : \mathcal{D} \to \mathcal{C}$ is called surjective if every object of $\mathcal{C}$ is a subquotient of $F(X)$ for some $X$ in $\mathcal{C}$.

Restriction $\text{res}_f : \text{rep}(B) \to \text{rep}(A)$ along a Hopf map $f : A \to B$, for example, is surjective if and only if $f$ is injective. Dually, if we employ corepresentations, restriction $\text{res}_w : \text{corep}(\Sigma) \to \text{corep}(\Lambda)$ along a Hopf map $w : \Sigma \to \Lambda$ is surjective if and only if $w$ is surjective. To complete the analogy, we also expect a surjective tensor functor to be faithful. We include the following lemma simply for aesthetic purposes (it is technically unnecessary).

Lemma 4.2. If $\mathcal{D}$ is a tensor category, as opposed to a multi-tensor category, then any surjective tensor functor $F : \mathcal{D} \to \mathcal{C}$ is faithful.

Proof. Since $F$ is exact by definition, it suffices to show that the only object mapped to 0 under $F$ is the zero object in $\mathcal{D}$. Take any $X$ in $\mathcal{D}$ with $F(X) = 0$, and suppose $X$ is nonzero. Since $\mathcal{D}$ is a tensor category it is indecomposable and exact as a left module category over itself. Hence the module functor $- \otimes X : \mathcal{D} \to \mathcal{D}$ is surjective [25, Proposition 3.9]. Since the composite $F \circ (- \otimes X)$ is 0, surjectivity of $- \otimes X$ and exactness of $F$ implies $F = 0$. But $F(1_\mathcal{D}) \cong 1_\mathcal{C}$, and we reach a contradiction. Thus $F$ is faithful. □

Remark 4.3. If $\mathcal{D}$ is a multi-tensor category, it is not necessarily the case that $\mathcal{D}$ is an indecomposable exact left $\mathcal{D}$-module category. This was a key part of the proof of Lemma 4.2.

4.2. Dual induction for tensor categories. Let $F : \mathcal{D} \to \mathcal{C}$ be a tensor functor and $I : \mathcal{C} \to \mathcal{D}$ be the left adjoint functor to $F$, which we refer to as induction. Note that $I$ exists since $F$ is exact. Recall that the unit of the adjoint pair $(I, F)$ is the image of the identity under the adjunction

$$\text{Hom}_\mathcal{D}(I(V), I(V)) \cong \text{Hom}_{\mathcal{C}}(V, FI(V)).$$

The unit provides a natural map $id_{\mathcal{C}} \to FI$. As in the Hopf case, we are interested in a dual induction.
Definition 4.4. Let \( F : \mathcal{D} \to \mathcal{C} \) be a tensor functor. We define the dual induction functor \( I^* : \mathcal{C} \to \mathcal{D} \) to be the composite \( I^* = (-)^* \circ I \circ (-) \).

We let \( \tau : F I^* \to \text{id}_\mathcal{C} \) be the natural isomorphism induced by the unit of the \((I,F)\)-adjunction. On objects, this isomorphism is given by the sequence

\[
\tau_V : F I^*(V) = F(I(V^*)) \cong (FI(V^*))^* \xrightarrow{\text{unit}^*} (V^*)^* \cong V.
\]

For the unit object \( V = 1 \), we have \( 1_\mathcal{C} = F(1_\mathcal{D}) \) and the counit of the adjunction gives a map \( I(1_\mathcal{C}) = IF(1_\mathcal{D}) \to 1_\mathcal{C} \). Apply \( F \) to get \( F(I(1_\mathcal{C})) \to 1_\mathcal{C} \). Taking the dual of this morphism gives a morphism \( i_1 : 1_\mathcal{C} \to F I^*(1_\mathcal{C}) \).

Lemma 4.5. The composition \( 1_\mathcal{C} \xrightarrow{i_1} F I^*(1_\mathcal{C}) \xrightarrow{\tau} 1_\mathcal{C} \) is the identity on \( 1_\mathcal{C} \).

Proof. It suffices to show that the sequence \( 1_\mathcal{C} \to FI(1_\mathcal{C}) \to 1_\mathcal{C} \) is the identity before taking the dual. If we write \( 1_\mathcal{C} = F(1_\mathcal{D}) \) this composition is just the composition of the unit \( F(1_\mathcal{D}) \to FIF(1_\mathcal{D}) \) with the counit \( FIF(1_\mathcal{D}) \to F(1_\mathcal{D}) \). Hence our result follows from the fact that the composition of the unit of any adjunction with the counit is always the identity [41]. \( \square \)

4.3. Transfer of cohomology along surjective tensor functors. In order to define a mixed restriction in the categorical setting (see Section 3), we need to know first that the left adjoint \( I : \mathcal{C} \to \mathcal{D} \) to \( F : \mathcal{D} \to \mathcal{C} \) is exact.

Lemma 4.6. If \( F : \mathcal{D} \to \mathcal{C} \) is a surjective tensor functor then the left adjoint \( I : \mathcal{C} \to \mathcal{D} \) is exact, as is \( I^* : \mathcal{C} \to \mathcal{D} \).

Proof. Let \( R \) be the sum of all minimal projectives in \( \mathcal{D} \). Since \( \mathcal{D} \) is Frobenius, \( R \) is also the sum of all minimal injectives. It follows that a sequence \( 0 \to L \to M \to N \to 0 \) in \( \mathcal{D} \) is exact if and only if the sequence

\[
0 \to \text{Hom}_\mathcal{D}(N, R) \to \text{Hom}_\mathcal{D}(M, R) \to \text{Hom}_\mathcal{D}(L, R) \to 0
\]

is exact. To see this clearly note that \( \mathcal{D} \) is equivalent to the category of representations over a finite dimensional algebra, and that for such an algebra the sum of all minimal projectives \( R \) is such that \( \text{Hom}_\mathcal{D}(R, -) \) is faithfully exact. By applying the equivalence \((-)^*, \) and noting that \( R^* \cong R, \) we see that \( \text{Hom}_\mathcal{D}(-, R) \) is faithfully exact.

Consider an exact sequence \( 0 \to X \to Y \to Z \to 0 \) in \( \mathcal{C}. \) To see if the sequence \( 0 \to I(X) \to I(Y) \to I(Z) \to 0 \) is exact we map into \( R \) to arrive at a sequence

\[
0 \to \text{Hom}_\mathcal{D}(I(Z), R) \to \text{Hom}_\mathcal{D}(I(Y), R) \to \text{Hom}_\mathcal{D}(I(X), R) \to 0. \tag{5}
\]

By [25, Theorem 2.5] the object \( FR \) is injective. Hence the adjoint sequence

\[
0 \to \text{Hom}_\mathcal{D}(Z, FR) \to \text{Hom}_\mathcal{D}(X, FR) \to \text{Hom}_\mathcal{D}(Y, FR) \to 0
\]

is exact. It follows that the original sequence (5) is exact, and hence that the sequence \( 0 \to I(X) \to I(Y) \to I(Z) \to 0 \) is exact. Exactness of \( I^* \) follows from exactness of \( I, \) since the dual functors are anti-equivalences. \( \square \)

Suppose \( F : \mathcal{D} \to \mathcal{C} \) is surjective. For an injective resolution \( V \to Q \) of \( V \) in \( \mathcal{C} \) we apply \( I^* \) to arrive at an injective resolution \( I^*(V) \to I^*(Q), \) by exactness of \( I^* \) and preservation of injectives/projectives under induction. Whence the extensions \( \text{Ext}_\mathcal{\mathcal{D}}^\bullet(W, I^*(V)) \) can be given as the cohomology of the hom complex \( \text{Hom}_\mathcal{\mathcal{D}}(W, I^*(Q)), \) for any object \( W \) in \( \mathcal{D}. \)
Definition 4.7. Let $F : \mathcal{D} \rightarrow \mathcal{C}$ be a surjective tensor functor with left adjoint $I$. For any object $V$ in $\mathcal{C}$, define the mixed restriction

$$\text{Mres}_V : \text{Ext}^\bullet_{\mathcal{D}}(I^*(1), I^*(V)) \rightarrow H^\bullet(\mathcal{C}, V)$$

by taking the class of a map $f : I^*(1) \rightarrow I^*(Q)$ to the class of the composite $\tau_Q F(f)i_1$.

Given a tensor functor $F : \mathcal{D} \rightarrow \mathcal{C}$, and $V$ in $\mathcal{C}$, we let $H^\bullet(\mathcal{D}, 1)$ act on the cohomology $H^\bullet(\mathcal{C}, V)$ via the algebra map $\text{res}_F : H^\bullet(\mathcal{D}, 1) \rightarrow H^\bullet(\mathcal{C}, 1)$ induced by $F$.

Proposition 4.8. For any object $V$ in $\mathcal{C}$, the mixed restriction (6) is a surjective map of $H^\bullet(\mathcal{D}, 1)$-modules.

Proof. One employs Lemma 4.5 and follows exactly the proof of Proposition 3.3. □

Theorem 4.9. If $F : \mathcal{D} \rightarrow \mathcal{C}$ is a surjective tensor functor, and $\mathcal{D}$ is of finite type, then $\mathcal{C}$ is also of finite type. Furthermore, in this case

(i) $\text{res}_F : H^\bullet(\mathcal{D}, 1) \rightarrow H^\bullet(\mathcal{C}, 1)$ is a finite algebra map and

(ii) the Krull dimension of $\mathcal{C}$ is bounded $K\text{dim}\mathcal{C} \leq K\text{dim}\mathcal{D}$.

Proof. As in the proof of Theorem 3.4, surjectivity of the mixed restriction for $V = 1_\mathcal{C}$ implies that the algebra $H^\bullet(\mathcal{C}, 1)$ is finite over the image of $\text{res}_F$, and hence that $H^\bullet(\mathcal{C}, 1)$ is a finitely generated algebra of Krull dimension less than or equal to that of $H^\bullet(\mathcal{D}, 1)$. Surjectivity of the mixed restriction in general implies that $H^\bullet(\mathcal{C}, V)$ is finitely generated over $H^\bullet(\mathcal{D}, 1)$, and hence over $H^\bullet(\mathcal{C}, 1)$. □

5. The Drinfeld center, (de-)equivariantization, and finite group schemes

We apply Theorem 4.9 to relate the cohomology of dual categories $\mathcal{C}^\ast_{\mathcal{M}}$ to the cohomology of the Drinfeld center $Z(\mathcal{C})$. We also discuss behavior of cohomology under (de-)equivariantization, and give a number of new examples of finite type tensor categories in characteristic $p$. In Section 10, we return to the subject of (de-)equivariantization and give a much deeper result regarding the behavior of cohomology under the combined processes of (de-)equivariantization and taking the Drinfeld center.

5.1. The Drinfeld center and stability of cohomology under duality. We apply Theorem 4.9 to the forgetful functor $F : Z(\mathcal{C}) \rightarrow \mathcal{C}$ to obtain the following result.

Corollary 5.1. If the Drinfeld center $Z(\mathcal{C})$ of $\mathcal{C}$ is of finite type then $\mathcal{C}$ is of finite type, as is any dual $\mathcal{C}^\ast_{\mathcal{M}}$ of $\mathcal{C}$ with respect to an exact module category $\mathcal{M}$. Furthermore, the Krull dimensions are uniformly bounded

$$K\text{dim}\mathcal{C}^\ast_{\mathcal{M}} \leq K\text{dim} Z(\mathcal{C}).$$

In a more concise language, the corollary says that the entire categorical Morita equivalence class of $\mathcal{C}$ is of finite type whenever the center $Z(\mathcal{C})$ is of finite type.

Proof. For any exact $\mathcal{C}$-module category $\mathcal{M}$, we have an equivalence of braided tensor categories $Z(\mathcal{C}^\ast_{\mathcal{M}}) \cong Z(\mathcal{C}^\ast_{\mathcal{M}})^\text{rev}$ [25, Corollary 3.35]. Hence there is a surjective tensor functor $F : Z(\mathcal{C}) \rightarrow \mathcal{C}^\ast_{\mathcal{M}}$ by [25, Proposition 3.39]. So we apply Theorem 4.9 to see that $\mathcal{C}^\ast_{\mathcal{M}}$ is of finite type and with bounded Krull dimension as proposed. □
5.2. **Equivariantization and de-equivariantization.** For this subsection we assume \( \text{char}(k) = 0 \). This is to avoid a situation in which \( \text{char}(k) \mid |G| \), for a given finite group \( G \).

Let \( \mathcal{C} \) be a braided tensor category. A central embedding into a tensor category \( \mathcal{D} \) is defined as a fully faithful tensor functor \( Q : \mathcal{C} \to \mathcal{D} \) along with a choice of braided lifting \( Q : \mathcal{C} \to \mathcal{Z}(\mathcal{D}) \).

Recall that for any central embedding \( \text{rep}(G) \to \mathcal{D} \) we can define the de-equivariantization \( \mathcal{D}_G \), which is the tensor category of \( \mathcal{O}(G) \)-modules in \( \mathcal{D} \), where \( \mathcal{O}(G) \) is the linear dual of the group algebra. As an inverse operation to de-equivariantization, we can form the equivariantization \( \mathcal{C}^G \) of any tensor category \( \mathcal{C} \) which is equipped with an action of a finite group \( G \) \[17\]. More precisely, we consider \( \rho : G \to \text{Aut}_\mathcal{C}(\mathcal{C}) \) a group action on \( \mathcal{C} \), where \( \text{Aut}_\mathcal{C}(\mathcal{C}) \) denotes the 2-group of tensor autoequivalences of \( \mathcal{C} \). Then the equivariantization \( \mathcal{C}^G \) consists of objects \( V \) in \( \mathcal{C} \) equipped with compatible isomorphisms \( g^V : \rho(g)V \to V \), for all \( g \in G \). Morphisms \( f : V \to W \) between objects in \( \mathcal{C}^G \) are exactly those maps in \( \mathcal{C} \) which commute with the isomorphisms \( g^V \).

**Theorem 5.2.** Suppose \( F : \mathcal{D} \to \mathcal{C} \) is a de-equivariantization of \( \mathcal{D} \) with respect to a central embedding \( \text{rep}(G) \to \mathcal{D} \), or equivalently a equivariantization of \( \mathcal{C} \) with respect to a \( G \)-action. Then \( \mathcal{D} \) is of finite type if and only if \( \mathcal{C} \) is of finite type, and in this case the Krull dimensions agree \( \text{Kdim} \mathcal{D} = \text{Kdim} \mathcal{C} \).

We prove Theorem 5.2 after establishing some background materials. Recall from \[17\] that for the equivariantization \( F : \mathcal{C}^G \to \mathcal{C} \), the induction \( I : \mathcal{C} \to \mathcal{C}^G \) sends each object in \( \mathcal{C} \) to its orbit under the \( G \)-action. In particular, each \( V \) in \( \mathcal{C} \) is a summand of \( FI(X) \). It follows that the forgetful functor \( F \) is surjective. By writing instead \( \mathcal{C}^G = \mathcal{D}, \mathcal{C} = \mathcal{D}_G \), we have that any de-equivariantization \( F : \mathcal{D} \to \mathcal{D}_G \) is surjective. Whence we apply Theorem 4.9 to arrive at the following result.

**Lemma 5.3.** Whenever \( \mathcal{D} \) is of finite type, the de-equivariantization \( \mathcal{D}_G \) is of finite type.

Note that for the equivariantization \( \mathcal{C}^G \) there is an identification \( \text{Hom}_{\mathcal{C}^G}(V,W) = \text{Hom}_{\mathcal{C}}(V,W)^G \), where \( G \) acts by \( g \cdot f = g^K(g(f))g^{-1}(V) \). By exactness of the invariants functor \( (-)^G \) we have \( \text{Ext}^*(\mathcal{C}^G,V) = \text{Ext}^*(\mathcal{C},V)^G \), and in particular \( H^*(\mathcal{C}^G,V) = H^*(\mathcal{C},V)^G \). We now prove the theorem.

**Proof of Theorem 5.2.** One direction follows directly from Lemma 5.3. That is, \( \mathcal{C}^G \) is of finite type whenever \( \mathcal{D} \) is.

For the converse, suppose \( \mathcal{C}^G \) is of finite type. We have \( H^*(\mathcal{D},V) = H^*(\mathcal{C},FV)^G \) for each \( V \) in \( \mathcal{D} \). Since \( H^*(\mathcal{C},1) \) is finitely generated, the invariants \( H^*(\mathcal{C},1)^G = H^*(\mathcal{D},1) \) are also finitely generated, and \( H^*(\mathcal{D},1) \to H^*(\mathcal{C},1) \) is a finitely generated module over \( H^*(\mathcal{D},1) \).

Consider any object \( V \) in \( \mathcal{D} \). Since \( H^*(\mathcal{D},1) \) is finite over \( H^*(\mathcal{D},1) \), the finite type property for \( \mathcal{C}^G \) implies that each \( H^*(\mathcal{D},V) \) is finite over \( H^*(\mathcal{D},1) \). Since \( H^*(\mathcal{D},1) \) is Noetherian, the submodule \( H^*(\mathcal{D},V) = H^*(\mathcal{C},FV)^G \) is therefore finite over \( H^*(\mathcal{D},1) \) as well. Hence \( \mathcal{D} \) is of finite type. \( \Box \)
Remark 5.4. As the familiar reader may recognize, Theorem 5.2 is also provable by simply considering the identification $H^*(\mathcal{D}, V) = H^*(\mathcal{G}, FV)^G$ and applying basic commutative algebra.

5.3. Examples in finite characteristic: Frobenius kernels. We give here some examples in finite characteristic. Examples in characteristic 0 are given in Section 7.

Let $k$ be a field of (finite) odd characteristic $p$. Let $G$ be a smooth algebraic group over $k$, and $G_{(r)}$ denote the $r$-th Frobenius kernel in $G$. Rather, $G_{(r)}$ is the group scheme theoretic kernel of the $r$-th Frobenius map $G \to G^{(r)}_r$. We let $\mathcal{O} = \mathcal{O}(G_{(r)})$ denote the (commutative) algebra of global functions on $G_{(r)}$, and $kG_{(r)}$ denote the vector space dual $kG_{(r)}^* = \mathcal{O}^*$.

In [29], Friedlander and the first author show that the double Frobenius kernel in a smooth algebraic group $G$ can be understood in terms of cohomological data [33]. We present his result in the particular case of a finite group $G$. Gelaki has classified exact module categories for arbitrary finite group schemes in [33]. We present his result in the particular case of a Frobenius kernel in a smooth algebraic group $G$.

Theorem 5.5 ([29, Theorem 5.3]). For an arbitrary smooth algebraic group $G$ in characteristic $p$, the Drinfeld double $D(kG_{(r)})$ has finitely generated cohomology. Furthermore, we calculate the Krull dimension

$$K\dim H^*(D(kG_{(r)}), k) = K\dim H^*(kG_{(r)}, k) + \dim G.$$  

Here $\dim G$ is the dimension of $G$ as a variety. As another point of interest, Gelaki has classified exact module categories for arbitrary finite group schemes in terms of cohomological data [33]. We present his result in the particular case of a Frobenius kernel in a smooth algebraic group $G$.

Theorem 5.6 ([33, Theorem 3.9]). Exact module categories over $\text{Coh}(G_{(r)}) = \text{rep}(\mathcal{O})$ are classified by pairs consisting of a choice of closed subgroup $H \subset G_{(r)}$ and 2-cocycle $\psi : kH \otimes kH \to k$. For any pair $(H, \psi)$ the associated module category $\mathcal{M}(H, \psi)$ is the category of $H$-equivariant sheaves on $G_{(r)}$, under the translation action $G_{(r)} \times H \to G_{(r)}$, with associativity given by $\psi$.

In the case $\psi = 1$ the module category $\mathcal{M}(H, 1)$ is equivalent to the category of coherent sheaves on the quotient $G/H = \text{Spec}(\mathcal{O}(G)^H)$. This equivalence is derived from the fact that the quotient $G \to G/H$ is $H$-Galois, so that we may apply descent to find $\text{Coh}(G)^H \cong \text{Coh}(G/H)$. The module structure on $\text{Coh}(G/H)$ is induced by the pushforward functor $\text{Coh}(G) \to \text{Coh}(G/H)$.

One applies Theorems 4.9 and 5.5 to arrive at the following corollary.

Corollary 5.7. Let $G$ be a smooth algebraic group in characteristic $p$. For any pair $(H, \psi)$ consisting of a closed subgroup $H$ of $G_{(r)}$, and 2-cocycle $\psi$ for $H$, the corresponding dual category $\text{rep}(\mathcal{O})^*_{\mathcal{M}(H, \psi)}$ is of finite type. Furthermore, the Krull dimensions of the duals are uniformly bounded

$$K\dim \text{rep}(\mathcal{O})^*_{\mathcal{M}(H, \psi)} \leq K\dim \text{rep}(G_{(r)}) + \dim G.$$  

Of course, it is already very interesting to consider cocycle twists of $\mathcal{O}$. We recall that $\text{rep}D(\mathcal{O}) \cong \text{rep}D(\mathcal{O}_\sigma)$ for any 2-cocycle $\sigma$ [42, 7].

Corollary 5.8. For any smooth algebraic group $G$, Frobenius kernel $G_{(r)}$, and 2-cocycle $\sigma$ for the coordinate algebra $\mathcal{O} = \mathcal{O}(G_{(r)})$, the twisted algebra $\mathcal{O}_\sigma$ has finitely generated cohomology. Furthermore,

$$K\dim \text{rep}(\mathcal{O}_\sigma) \leq K\dim \text{rep}(G_{(r)}) + \dim G.$$  

\footnote{This assumption on the characteristic is not optimal. See [29, Section 4.3].}
We note that the twisted algebra $\mathcal{O}_\sigma$ is neither commutative nor cocommutative in general. Gelaki provides in [33, Example 6.14] an explicit example of a cocycle twist $\sigma$ of the algebra of functions for the semi-direct product $(G_a \rtimes G_m)_{(1)}$ for which the corresponding cocycle twisted algebra is neither commutative nor cocommutative. As an algebra, one has explicitly

$$\mathcal{O}((G_a \rtimes G_m)_{(1)})_\sigma = k[x,t]/(x^p,t^p-1,[x,t]-t^2+2t-1),$$

while the coalgebra structure is unchanged, and hence remains non-cocommutative.

This single cocycle proliferates to produce an infinite class of such Hopf algebras. Indeed, given any embedding $(G_a \rtimes G_m)_{(1)} \to \mathcal{G}_{(r)}$, for arbitrary smooth $G$, we can restrict this cocycle to $\mathcal{G}_{(r)}$ to arrive at a Hopf algebra in characteristic $p$ which is neither commutative nor cocommutative. Of course, there should be many more interesting examples of cycle twisted algebras which are not coming from the above small example as well.

We can also complete some work from [29, Section 5.3] by applying Theorem 3.4. For smooth $G$, any $r > 0$, and group scheme quotient $\mathcal{G}_{(r)} \to G'$, one can form the relative double $D(G', \mathcal{G}_{(r)})$, which as an algebra is the smash product $\mathcal{O}(G') \# k\mathcal{G}_{(r)}$ under the adjoint action of $\mathcal{G}_{(r)}$ on $G'$. This algebra has representation category equivalent to the relative center of the corresponding module category $\text{rep}(G')$ over $\text{rep}(\mathcal{G}_{(r)})$ (see Section 8). The following result was proved for $G'$ of the form $\mathcal{G}_{(r)}/\mathcal{G}_{(r-i)}$ in [29].

**Corollary 5.9** (cf. [29, Theorem 5.7]). For any smooth algebraic group $G$, Frobenius kernel $\mathcal{G}_{(r)}$, and quotient map $\mathcal{G}_{(r)} \to G'$ of group schemes, the relative double $D(G', \mathcal{G}_{(r)})$ has finitely generated cohomology. Additionally, the Krull dimension is bounded

$$\text{Kdim } H^* (D(G', \mathcal{G}_{(r)}), k) \leq \text{Kdim } \text{rep}(\mathcal{G}_{(r)}) + \dim G.$$

**Proof.** We have the Hopf inclusion $D(G', \mathcal{G}_{(r)}) \to D(\mathcal{G}_{(r)})$ and apply Theorem 3.4 in conjunction with Theorem 5.5. \qed

**Remark 5.10.** To our knowledge, the cocycle twisted algebras $\mathcal{O}(\mathcal{G}_{(r)})_\sigma$, in addition to the doubles of [29], provide the first examples of noncommutative and noncocommutative Hopf algebras in characteristic $p$ for which the cohomology is known to be finitely generated (see also [50, 19], where the authors consider cohomology with trivial coefficients).

**Remark 5.11.** The material of this subsection focuses on infinitesimal group schemes. One could, at the other extreme, consider finite (discrete) groups in characteristic $p$. In this setting, again, module categories have been classified [25, Section 4]. Indeed, the work of [25] precedes and motivates the work of [33]. It is easily checked that the double $D(kG)$ of a finite group $G$ has finitely generated cohomology and $\text{Kdim } \text{rep}(D(kG)) = \text{Kdim } \text{rep}(G)$. So we see that all duals $\text{rep}(G)^\Delta$ are of finite type for an arbitrary finite group $G$.

6. Deligne products and non-degenerate categories

In this section we show that if $\mathcal{C}$ is braided, non-degenerate, and of finite type, then every dual category $\mathcal{C}^\Delta$ with respect to an exact module category $\mathcal{M}$ is also of finite type. That means that the finite type property is preserved by Morita equivalences in the non-degenerate case. At times in this section it will be convenient to employ an equivalence $\mathcal{C} \simeq \text{rep}(A)$ to express an arbitrary tensor category as the
category of representations over a Hopf algebroid $A$. We recall the notion of a Hopf algebroid in Appendix A, and reconstruct a standard equivalence $\mathcal{C} \simeq \text{rep}(A)$ at Lemma A.1 therin.

### 6.1. Some standard lemmas

Let us collect a few standard results, which seem not to have appeared in an organized manner in the literature.

**Lemma 6.1** (cf. [55]). Suppose the cohomology $H^\bullet(\mathcal{C}, 1)$ is a Noetherian algebra. Then the following are equivalent:

1. The cohomology $H^\bullet(\mathcal{C}, V)$ is finitely generated as a $H^\bullet(\mathcal{C}, 1)$-module for each simple object $V$ in $\mathcal{C}$.
2. The cohomology $H^\bullet(\mathcal{C}, W)$ is finite generated as a $H^\bullet(\mathcal{C}, 1)$-module for arbitrary $W$ in $\mathcal{C}$.

**Proof.** Obviously (ii) implies (i). Suppose now that (i) holds. We proceed by induction on the length of $W$. As our base case, (i) tells us the cohomology $H^\bullet(\mathcal{C}, W)$ is finitely generated whenever $W$ is length 1. Now suppose the cohomology is finitely generated for each object of length $< l$, and consider a length $l$ object $W$. We place $W$ in an exact sequence $0 \to W' \to W \to V \to 0$, with $W'$ and $V$ of length $< l$, and get an exact sequence on cohomology $H^\bullet(\mathcal{C}, W') \to H^\bullet(\mathcal{C}, W) \to H^\bullet(\mathcal{C}, V)$. (Exactness at the middle term follows from the standard long exact sequence on cohomology.)

Since $H^\bullet(\mathcal{C}, 1)$ is Noetherian, and $H^\bullet(\mathcal{C}, V)$ is a finitely generated module, the image $N$ of $H^\bullet(\mathcal{C}, W)$ in $H^\bullet(\mathcal{C}, V)$ is finitely generated, as is the image $M$ of $H^\bullet(\mathcal{C}, W')$ in $H^\bullet(\mathcal{C}, W)$. Hence we have an exact sequence $0 \to M \to H^\bullet(\mathcal{C}, W) \to N \to 0$ of $H^\bullet(\mathcal{C}, 1)$-modules, with $M$ and $N$ finitely generated. It follows that $H^\bullet(\mathcal{C}, W)$ is finitely generated. Now (ii) follows by induction. \[\Box\]

For a Hopf algebroid $A$ over $R$ we can change base along any field extension $k \to k'$ to produce a new weak Hopf algebroid $A_{k'}$ over $R_{k'}$. For any tensor category $\mathcal{C}$ we define $\mathcal{C}_{k'}$ by choosing an equivalence $\mathcal{C} \cong \text{rep}(A)$ with $A$ a Hopf algebroid and taking $\mathcal{C}_{k'} = \text{rep}(A_{k'})$. One can check that the base change is unique up to tensor equivalence.

**Lemma 6.2.** Let $k'$ be an arbitrary field extension of $k$. Then $\mathcal{C}$ is of finite type over $k$ if and only if $\mathcal{C}_{k'}$ is of finite type over $k'$. In this case $\text{Kdim} \mathcal{C} = \text{Kdim} \mathcal{C}_{k'}$.

**Proof.** Let $(-)_{k'} : \mathcal{C} \to \mathcal{C}_{k'}$ denote the base change functor. We have an identification of algebras $H^\bullet(\mathcal{C}_{k'}, 1_{k'}) = k' \otimes H^\bullet(\mathcal{C}, 1)$, and an identification of $H^\bullet(\mathcal{C}_{k'}, 1_{k'})$-modules $H^\bullet(\mathcal{C}_{k'}, V_{k'}) = k' \otimes H^\bullet(\mathcal{C}, V)$ for every object $V$ in $\mathcal{C}$. Since any simple object in the base change $\mathcal{C}_{k'}$ is a summand of some $V_{k'}$, for a simple $V$ in $\mathcal{C}$, one can apply Lemma 6.1 to see that $\mathcal{C}_{k'}$ is of finite type whenever $\mathcal{C}$ is of finite type.

Suppose now that $\mathcal{C}_{k'}$ is of finite type. If $H^\bullet(\mathcal{C}_{k'}, 1_{k'})$ is finitely generated then we can choose $N > 0$ so that $H^\bullet(\mathcal{C}_{k'}, 1_{k'})$ is generated in degrees $\leq N$. Hence the natural map $\text{Sym}(H^{\leq N}(\mathcal{C}_{k'}, 1_{k'})) \to H^\bullet(\mathcal{C}_{k'}, 1_{k'})$ is surjective. By the diagram

\[
\begin{align*}
& k' \otimes \text{Sym}(H^{\leq N}(\mathcal{C}, 1)) \quad \longrightarrow \quad k' \otimes H^\bullet(\mathcal{C}, 1) \\
& \downarrow \quad \cong \quad \downarrow \quad \cong \\
& \text{Sym}(H^{\leq N}(\mathcal{C}_{k'}, 1_{k'})) \quad \longrightarrow \quad H^\bullet(\mathcal{C}_{k'}, 1_{k'}),
\end{align*}
\]
and faithful flatness of \( k' \) over \( k \), we conclude that the map \( \text{Sym}(H^{\leq N}(\mathcal{C}, 1)) \to H^*(\mathcal{C}, 1) \) is surjective. Since \( H^*(\mathcal{C}, 1) \) is finite dimensional in each degree, it follows that \( H^*(\mathcal{C}, 1) \) is a finitely generated algebra. A similar argument shows that each \( H^*(\mathcal{C}, V) \) is a finitely generated module over \( H^*(\mathcal{C}, 1) \).

As for the Krull dimension, any finite map \( k[X_1, \ldots, X_n] \to H^*(\mathcal{C}, 1) \) changes base to a finite map \( k'[X_1, \ldots, X_n] \to H^*(\mathcal{C}_{k'}, 1) \). Whence \( \text{Kdim} \mathcal{C}_{k'} = \text{Kdim} \mathcal{C} \), by Noether normalization.

\[ \text{Lemma 6.3.} \quad \text{For tensor categories } \mathcal{C} \text{ and } \mathcal{C}' \text{ the following conditions are equivalent:} \]

(i) Both \( \mathcal{C} \) and \( \mathcal{C}' \) are of finite type;

(ii) The Deligne product \( \mathcal{C} \boxtimes \mathcal{C}' \) is of finite type.

When either (i) or (ii) holds we have \( \text{Kdim}(\mathcal{C} \boxtimes \mathcal{C}') = \text{Kdim} \mathcal{C} + \text{Kdim} \mathcal{C}' \).

\[ \text{Proof.} \quad \text{We have for the algebraic closure } k \to \bar{k} \text{ that } (\mathcal{C} \boxtimes \mathcal{C}')(\bar{k}) \simeq \mathcal{C}_k \boxtimes \mathcal{C}_{k'}', \text{ and hence we may assume } k \text{ is algebraically closed, by Lemma 6.2. Notice that the second product is “over } \bar{k} \text{”. In this case the simples of } \mathcal{C} \boxtimes \mathcal{C}' \text{ are exactly the products } V \boxtimes V' \text{ of simples } V \text{ from } \mathcal{C} \text{ and } V' \text{ from } \mathcal{C}' \text{. We have a canonical isomorphism of algebras} \]

\[ H^*(\mathcal{C}, 1) \otimes H^*(\mathcal{C}', 1) \xrightarrow{\cong} H^*(\mathcal{C} \boxtimes \mathcal{C}', 1), \quad [f] \otimes [g] \mapsto [f \otimes g]. \quad (7) \]

Similarly, for simple objects \( V \) and \( V' \) in \( \mathcal{C} \) and \( \mathcal{C}' \) there is an isomorphism of \( H^*(\mathcal{C} \boxtimes \mathcal{C}', 1) \)-modules

\[ H^*(\mathcal{C}, V) \otimes H^*(\mathcal{C}', V') \xrightarrow{\cong} H^*(\mathcal{C} \boxtimes \mathcal{C}', V \boxtimes V'), \quad (8) \]

where \( H^*(\mathcal{C} \boxtimes \mathcal{C}', 1) \) acts on \( H^*(\mathcal{C}, V) \otimes H^*(\mathcal{C}', V') \) via the algebra identification (7).

The fact that these maps are isomorphisms essentially follows from the fact that the product of projective resolutions \( P \to 1_{\mathcal{C}} \) and \( P' \to 1_{\mathcal{C}'} \) produces a projective resolution \( P \boxtimes P' \to 1_{\mathcal{C} \boxtimes \mathcal{C}'} \).

By the above information, and Lemma 6.1, we see that (i) implies (ii). Suppose now that (ii) holds. The identification (7) implies that restriction \( \text{res} : H^*(\mathcal{C} \boxtimes \mathcal{C}', 1) \to H^*(\mathcal{C}, 1) \) is a surjective algebra map, and hence that \( H^*(\mathcal{C}, 1) \) is a finitely generated algebra. From (7) and (8) we see also that for any object \( V \) in \( \mathcal{C} \), the restriction \( \text{res}_V : H^*(\mathcal{C} \boxtimes \mathcal{C}', V \boxtimes 1_{\mathcal{C}'}) \to H^*(\mathcal{C}, V) \) is a surjective \( H^*(\mathcal{C} \boxtimes \mathcal{C}', 1) \)-module map, where we let \( H^*(\mathcal{C} \boxtimes \mathcal{C}', 1) \) act on the codomain via the projection to \( H^*(\mathcal{C}, 1) \). Since \( H^*(\mathcal{C} \boxtimes \mathcal{C}', V \boxtimes 1_{\mathcal{C}'}) \) is finitely generated over \( H^*(\mathcal{C} \boxtimes \mathcal{C}', 1) \), we conclude that \( H^*(\mathcal{C} \boxtimes \mathcal{C}', V) \) is finitely generated over \( H^*(\mathcal{C} \boxtimes \mathcal{C}', 1) \) as well, and hence over \( H^*(\mathcal{C}, 1) \). Hence \( \mathcal{C} \) is of finite type, and the analogous argument shows \( \mathcal{C}' \) is of finite type as well.

The Krull dimension calculation follows by Noether normalization. \[ \square \]

6.2. Results for non-degenerate categories. Recall that a braided tensor category \( \mathcal{C} \) is called non-degenerate if any object \( V \) for which the square of the braiding \( c_{-V, V} : V \otimes - \to V \otimes - \) is the identity is trivial, that is \( V \cong 1^\oplus n \). Non-degeneracy is equivalent to the condition that the canonical braided tensor functor

\[ \mathcal{C} \boxtimes \mathcal{C}^\text{rev} \to Z(\mathcal{C}), \quad X \boxtimes 1 \mapsto (X, c_{X,-}), \quad 1 \boxtimes X \mapsto (X, c_{-X}^{-1}) \]
is an equivalence [28]. For the category $\text{rep}(A)$ of representations for a quasitriangular Hopf algebra $A$, we have that $\text{rep}(A)$ is non-degenerate if and only if $A$ is factorizable. The following proposition is a direct extension of Proposition 3.6.

**Proposition 6.4.** Suppose $\mathcal{C}$ is non-degenerate and of finite type over $k$. Then the center $Z(\mathcal{C})$ is of finite type and of Krull dimension $2K\text{dim }\mathcal{C}$. Furthermore, in this case any dual $\mathcal{C}^*_{\mathcal{M}}$ with respect to an exact module category $\mathcal{M}$ is of finite type, and the Krull dimensions are uniformly bounded $\text{Kdim }\mathcal{C}^*_{\mathcal{M}} \leq 2\text{Kdim }\mathcal{C}$.

**Proof.** Since $\mathcal{C}$ is non-degenerate, we have a tensor equivalence $Z(\mathcal{C}) \cong \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$. Whence we conclude that $Z(\mathcal{C})$ has finitely generated cohomology and is of Krull dimension $2K\text{dim }\mathcal{C}$, by Lemma 6.3. The claim about duals now follows by Corollary 5.1. □

### 6.3. An equivalence of two conjectures

One sees from Lemma 6.3 that the stability conjecture has an equivalent expression via the Drinfeld center.

**Proposition 6.5.** Over an arbitrary base field $k$, the following two conjectures are equivalent:

(A) Suppose that $\mathcal{C}$ is a tensor category of finite type, and that $\mathcal{M}$ is an exact $\mathcal{C}$-module category. Then the dual $\mathcal{C}^*_{\mathcal{M}}$ is also of finite type and $\text{Kdim }\mathcal{C}^*_{\mathcal{M}} = \text{Kdim }\mathcal{C}$.

(B) Suppose that $\mathcal{C}$ is a tensor category of finite type. Then the Drinfeld center $Z(\mathcal{C})$ is also of finite type and $\text{Kdim }Z(\mathcal{C}) = 2\text{Kdim }\mathcal{C}$.

**Proof.** Suppose Conjecture A is true and that $\mathcal{C}$ is a finite type tensor category. Then, by Lemma 6.3, the product $\mathcal{C} \boxtimes \mathcal{C}^{\text{cop}}$ is of finite type. We now apply A, in conjunction with Lemma 6.3, to the tensor equivalence $(\mathcal{C} \boxtimes \mathcal{C}^{\text{cop}})^*_{\mathcal{M}} \cong Z(\mathcal{C})$ [28, Proposition 7.13.8] to find that $Z(\mathcal{C})$ is of finite type and that $\text{Kdim }Z(\mathcal{C}) = 2\text{Kdim }\mathcal{C}$. Hence Conjecture B holds.

Suppose now that Conjecture B holds, i.e. that the finite type property is stable under taking centers. Suppose $\mathcal{C}$ is of finite type and that $\mathcal{M}$ is an exact module category over $\mathcal{C}$. By B and Corollary 5.1 the dual $\mathcal{C}^*_{\mathcal{M}}$ is of finite type, and we apply B again to find

$$\text{Kdim }\mathcal{C}^*_{\mathcal{M}} = \frac{1}{2}\text{Kdim }Z(\mathcal{C}^*_{\mathcal{M}}) = \frac{1}{2}\text{Kdim }Z(\mathcal{C}) = \text{Kdim }\mathcal{C}.$$ Hence we establish Conjecture A. □

A similar argument can be applied to obtain a variant of the above proposition for the weak stability conjecture.

**Proposition 6.6.** Over an arbitrary base field $k$, the following two conjectures are equivalent:

(wA) Suppose that $\mathcal{C}$ is a tensor category of finite type, and that $\mathcal{M}$ is an exact $\mathcal{C}$-module category. Then the dual $\mathcal{C}^*_{\mathcal{M}}$ is also of finite type. Furthermore, there is a polynomial $P \in \mathbb{R}_{\geq 0}[X]$ such that $\text{Kdim }\mathcal{C}^*_{\mathcal{M}} \leq P(\text{Kdim }\mathcal{C})$.

(wB) Suppose that $\mathcal{C}$ is a tensor category of finite type. Then the Drinfeld center $Z(\mathcal{C})$ is also of finite type. Furthermore, there is a polynomial $Q \in \mathbb{R}_{\geq 0}[X]$ such that $\text{Kdim }Z(\mathcal{C}) \leq Q(\text{Kdim }\mathcal{C})$.

In the above statements the polynomials $P, Q \in \mathbb{R}_{\geq 0}[X]$ are to be independent of choice of $\mathcal{C}$ and $\mathcal{M}$. As mentioned in the introduction, Conjecture wB says that
the Krull dimension of the center $Z(C)$ grows sub-exponentially as a function of $Kdim C$.

7. DYNAMICAL QUANTUM GROUPS

In this section we give some examples in characteristic 0. We consider dynamical
twists of small quantum groups. General information on dynamical twists can
be found in [23, 44], as well as Appendix A here. We employ relations between
dynamical twists, module categories, and weak Hopf algebras which we outlined in
detail in the Appendix. These relations should be known to experts, but seemingly
have not appeared explicitly in the literature to this point.

We fix $g$ a simply-laced, simple, Lie algebra over $\mathbb{C}$, i.e. $g$ of type $A, D, E$, and
fix $\Gamma$ a set of simple roots for $g$. By an abuse of notation, we identify $\Gamma$ with the
Dynkin diagram for $g$.

We suppose that $q$ is a root of unity of sufficiently large prime order. Specifi-
cally, the order of $q$ should be larger than the Coxeter number for $g$, in accordance
with [36], and should be coprime to a finite collection of numbers which depends on
the Dynkin type for $g$, in accordance with [23, Section 5.2]. We consider the small
quantum group $u_q(g)$ as defined in [39, 40].

7.1. Small quantum groups and cohomology. Recall that $u_q(g)$ is a finite di-
imensional, factorizable, pointed Hopf algebra. The algebra $u_q(g)$ is generated by
grouplikes $K_\alpha$ and skew primitives $E_\alpha$ and $F_\alpha$, for $\alpha \in \Gamma$. The group of grouplikes $G = G(u_q(g))$ is abelian, and there is a group isomorphism $(\mathbb{Z}/l\mathbb{Z})^\Gamma \to G, (n_\alpha) \mapsto \prod_\alpha K_\alpha^{n_\alpha}$. We refer the reader to Lusztig’s original papers [39, 40] for more
information.

By results of Ginzburg-Kumar, the small quantum group $u_q(g)$ has finitely gen-
erated cohomology.

**Theorem 7.1** ([36], cf. [6]). The cohomology $H^\bullet(u_q(g), \mathbb{C})$ is a finitely generated algebra, and for any $u_q(g)$-representation $V$ the cohomology $H^\bullet(u_q(g), V)$ is a finitely generated module over $H^\bullet(u_q(g), \mathbb{C})$. Furthermore, there is an algebra isomorphism

$$H^\bullet(u_q(g), \mathbb{C}) \cong \mathcal{O}(\text{Nil}(g)),$$

where $\text{Nil}(g)$ is the cone of ad-nilpotent elements in $g$.

We note that the above theorem does not require $g$ to be simply-laced. The
remainder of the section is dedicated to an explanation of how one can apply
Proposition 6.4 to find that Etingof-Varchenko style dynamical quantum groups [23]
(cf. [27]) have finitely generated cohomology.

7.2. A review of dynamical quantum groups and Belavin-Drinfeld triples [23].

**Definition 7.2** ([5]). A Belavin-Drinfeld triple for $g$ is a triple $(\Gamma_1, \Gamma_2, T)$, where
$\Gamma_1, \Gamma_2 \subset \Gamma$ are subgraphs and $T : \Gamma_1 \to \Gamma_2$ is a choice of graph isomorphism. We
say a triple $(\Gamma_1, \Gamma_2, T)$ is nilpotent if for each $\alpha \in \Gamma_1$ there is a positive power $N$ of $T$ so that $T^N(\alpha) \in \Gamma \setminus \Gamma_1$.

Consider an arbitrary triple $T = (\Gamma_1, \Gamma_2, T)$ for $g$, i.e. one which is not necessarily
nilpotent. One associates to $T$ a dynamical twist

$$J_T : G^\Gamma \to u_q(g) \otimes u_q(g),$$
which is built out of the $R$-matrix for $u_q(g)$ and an extension of $T$ to a Hopf map between the quantum Borels $T_\pm : u_q(b_\pm) \to u_q(b_-)$, with $T_+(E_\alpha) = E_{T_+(\alpha)}$ when $\alpha \in \Gamma_1$ and $T_+(E_\alpha) = 0$ otherwise. We refer the reader to [23] for the precise construction, and to [49] for an explicit presentation when $T$ is nilpotent (see also [26]).

As with any dynamical twist, $J_T$ induces a non-trivial associator for the natural $\text{rep}(u_q(g))$-action on $\text{rep}(G)$ and we obtain new exact, indecomposable $\text{rep}(u_q(g))$-module category structure $\mathcal{M}_T$ on $\text{rep}(G)$ (see Section A.2). We take the dual to arrive at a tensor category

$$\text{rep}(u_q(g))^{\ast}_{\mathcal{M}_T}.$$ One can explicitly construct from $\mathcal{M}_T$ a weak Hopf algebra $u_q(g, T)$ which is equipped with a canonical equivalence $\text{rep}(u_q(g, T)) \simeq \text{rep}(u_q(g))^{\ast}_{\mathcal{M}_T}$ (see Lemma A.3). The weak Hopf algebra $u_q(g, T)$ is called the (Etingof-Varchenko style) dynamical quantum group associated to the triple $T$ [27, 23].

**Remark 7.3.** In general, the weak Hopf algebra $u_q(g, T)$ realizes solutions to the quantum dynamical, i.e. parameter dependent, Yang-Baxter equation. Furthermore, the dynamical $R$-matrix for the analogously defined dynamical Drinfeld-Jimbo quantum group $U_h(g, T)$ (or rather its dual) was employed to provide explicit quantizations of classical $r$-matrices for Lie algebras produced by Belavin and Drinfeld [26, 5]. One can see [20] for a survey on the topic.

The structure of the category $\text{rep}(u_q(g))^{\ast}_{\mathcal{M}_T}$ is not well understood at general $T$. However, for nilpotent $T$ it is clear that the (quantum) parabolics $u_q(p_i) \subset u_q(g)$ associated to the subgraphs $\Gamma_i$ appearing in the given triple play a significant role in determining the structure of the dual [49]. In the extreme cases of an empty triple $T$ (i.e. when $\Gamma_1 = \Gamma_2 = \emptyset$) or when $T$ an automorphism of the Dynkin diagram the category $\text{rep}(u_q(g))^{\ast}_{\mathcal{M}_T}$ “degenerates” to standard quantum groups, and we can understand the situation clearly.

**Theorem 7.4** ([23, Theorem 5.4.1]). *Let $T$ be a Belavin-Drinfeld triple for $g$.*

(i) When $T$ is empty, there is an equivalence $\text{rep}(u_q(g))^{\ast}_{\mathcal{M}_T} \simeq \text{rep}(u_q(g))^{G \times G^{\op}}$ of $k$-linear categories, where $G$ acts on $\text{rep}(u_q(g)^\ast)$ via left and right translation.

(ii) When $T$ is an automorphism of the Dynkin diagram there is an equivalence of tensor categories $\text{rep}(u_q(g))^{\ast}_{\mathcal{M}_T} \simeq \text{rep}(u_q(g))$.

Note the presence of the linear dual in statement (i).

**Proof.** (i) The dynamical twist in this case is constant of value 1, and hence the dynamical quantum group category is equivalent to the proposed equivariantization of $\text{rep}(u_q(g)^\ast)$, by Proposition A.4. (ii) By [23, Theorem 5.4.1], in the case in which $T$ is an automorphism the dynamical quantum group $u_q(g, T)^\ast$ is isomorphic to the Xu-style dynamical twisted algebra $u_q(g)^{iT}$ (see Appendix A.2). We have then a canonical equivalence of tensor categories

$$\text{Sz}(J_T) : \text{rep}(u_q(g)) \simeq \text{rep}(u_q(g)^{iT})$$

(see again Appendix A.2).\hfill\Box

\footnote{Our weak Hopf algebra $u_q(g, T)$ is isomorphic to the weak Hopf algebra $D_J^T$ of [23]. Its dual is the weak Hopf algebra $u_q(g)^{iT}$ of Appendix A.}
At these two extremes two distinct subspaces in $\mathfrak{g}$ appear as the reduced spectra of cohomology. For the equivariantization $\text{rep}(u_q(\mathfrak{g})^*)^{G \times G^{op}}$ the unit is the equivariant $u_q(\mathfrak{g})^*$-representation $1 = k[G]$, and one can calculate the spectrum of cohomology is the sum of the positive and negative nilpotent subalgebras $n_\pm \subset \mathfrak{g}$,

$$\text{Spec} H^\bullet(\text{rep}(u_q(\mathfrak{g})^*)^{G \times G^{op}}, 1)_{\text{red}} = n_+ \times n_-,$$

using [36]. For $u_q(\mathfrak{g})$, [36] tells us directly that the spectrum is the nilpotent cone $\text{Nil(} \mathfrak{g})$.

7.3. Cohomology of dynamical quantum groups. From Proposition 3.6 we derive finite generation of cohomology for dynamical quantum groups.

**Corollary 7.5.** Given any Belavin-Drinfeld triple $T$ for $\mathfrak{g}$, the associated dynamical quantum group $u_q(\mathfrak{g}, T)$ has finitely generated cohomology. Furthermore, the Krull dimension is bounded as

$$\text{Kdim} \text{rep}(u_q(\mathfrak{g}, T)) \leq 2 \text{dim} \mathfrak{g}/\mathfrak{h}.$$  

**Proof.** We have the equivalence $\text{rep}(u_q(\mathfrak{g}, T)) \simeq \text{rep}(u_q(\mathfrak{g}))^{\#_{\#}}$ and apply Proposition 6.4. Our calculation of dimension comes from the fact that $\text{Kdim} \text{rep}(u_q(\mathfrak{g})) = \text{dim} \text{Nil(} \mathfrak{g}) = \text{dim} \mathfrak{g}/\mathfrak{h}$. □

At the moment we do not have an explicit understanding of the spectrum of cohomology at general $T$. From the results at the extreme ends, where the spectra are $n_+ \times n_-$ and $\text{Nil}(\mathfrak{g})$, we expect that the spectra should admit a general description in terms of purely Lie theoretic data, and should be of constant dimension $\text{dim} \mathfrak{g}/\mathfrak{h}$. In particular, we expect that the parabolics associated to the $\Gamma_i$ will appear in such a formulation.

**Problem 7.6.** Give a uniform description of reduced spectrum of cohomology $\text{Spec} H^\bullet(u_q(\mathfrak{g}, T), 1)_{\text{red}}$ in terms of Lie theoretic data determined by the triple $T$. Furthermore, give an explicit formula for the dimensions of these varieties as a (possibly constant) function in $T$.

**Remark 7.7.** In considering the above examples (or much simpler examples from [21, Theorem 4.1] in conjunction with [16]) one could come to a somewhat pessimistic conclusion. Namely, one sees that basic properties such as smoothness of spectra of cohomology are not preserved under categorical Morita equivalence. However, the singularity at 0 in $\text{Nil}(\mathfrak{g})$ is somewhat artificial. Indeed, the spectrum of cohomology is always a conical variety, and hence will generally have such a singularity. So, this is an indication that one should be considering the projectivization $\text{Proj} H^\bullet(\mathcal{C}, 1)$, rather than the usual spectrum, in order to avoid such artifacts. In the example under consideration, the projective scheme $(\text{Nil(} \mathfrak{g}) - \{0\})/\mathbb{C}^*$ is in fact smooth again.

This preference for projective varieties is also emphasized by considerations of the stable module category, or singularity category, $\underline{\mathcal{C}}$, from which one expects to only be able to recover the projective scheme $\text{Proj} H^\bullet(\mathcal{C}, 1)$ from $\underline{\mathcal{C}}$. Namely, for a well-behaved (braided) category $\mathcal{C}$ one expects to recover the projectivized spectrum of cohomology as the Balmer spectrum of the triangulated tensor category $(\underline{\mathcal{C}}, \otimes)$ [4].
8. Finite type braided categories I: a preliminary discussion

Take \( k \) algebraically closed and of characteristic 0. This assumption will hold generically for the remainder of this work, although in Sections 9 and 11.1 the assumption can be relaxed. By our base change result, Lemma 6.2, the effective restriction here is that \( \mathcal{C} \) should be a tensor category over a base field of characteristic 0.

We now turn our focus from specific occurrences of weak stability to a discussion of the weak stability conjecture in general, with an emphasis on tensor categories which admit a braiding. Recall the following basic definition of Müger [48].

**Definition 8.1.** For a braided tensor category \( \mathcal{C} \), the Müger center \( Z_2(\mathcal{C}) \) is the full subcategory generated by all object \( V \) in \( \mathcal{C} \) for which the square braiding \( c_{V,V} \in \text{End}_{\text{Fun}}(V \otimes -) \) is the identity.

The main purpose of the present section is to outline a plan for addressing the cohomology of braided categories in characteristic 0. We outline our general approach in Section 8.4, after establishing some background material. We carry out the plan outlined here in the remaining sections of the paper.

8.1. Müger centers and (super-)Tannakian categories. By Deligne’s theorem [15, 14], any (finite) symmetric braided tensor category \( \mathcal{E} \) (over an algebraically closed field of characteristic 0) is equivalent to the category \( \text{srep}(SG) \) of super-representations of a finite supergroup \( SG \). In Hopf theoretic terms, \( \text{srep}(SG) \) is the symmetric category of representations of a triangular Hopf algebra of the form \( \text{Wedge}(V) \rtimes G \), where \( G \) is a finite group, \( V \) is a \( G \)-representation of \((z,1)\)-skew primitives, where \( z \) in the center of \( G \) with \( z^2 = 1 \), and the \( R \)-matrix is \( R_z = \frac{1}{2}(1 + z \otimes 1 + 1 \otimes z - z \otimes z) \) (cf. [1]).

If we suppose that \( \mathcal{E} \) is semisimple then \( V \) must vanish, and \( \mathcal{E} \) is equivalent to the symmetric category \( \text{rep}(G, R_z) \).

**Remark 8.2.** The braiding induced by \( R_z \) is just the usual signed super-swap, where the even and odd parity components of a representation are given by the \( \pm 1 \) eigenspaces for the action of \( z \). Note that when \( z = 1 \) this braiding is trivial, as all representations are concentrated in even degree.

When \( z \) is not 1, the existence of objects in \( \text{srep}(SG) \) with self braiding \( c_{VV} = -id_{V \otimes V} \) obstructs the existence of a symmetric fiber functor to \( \text{Vect} \). So if \( \mathcal{E} \) admits a symmetric fiber functor to \( \text{Vect} \) then \( \mathcal{E} \) is isomorphic to \( \text{rep}(G) \) for some finite group \( G \), with the trivial braiding. In particular, \( \mathcal{E} \) must be semisimple in this case.

A symmetric category \( \mathcal{E} \) is called Tannakian if it admits a symmetric fiber functor to \( \text{Vect} \), and non-Tannakian, or super-Tannakian, otherwise. All super-Tannakian categories considered in this work will be, or will be proved to be, semisimple, i.e. of the form \( \mathcal{E} \cong \text{rep}(G, R_z) \).

In the sections that follow we address weak stability of cohomology for braided categories whose Müger centers are semisimple, which we refer to colloquially as categories with a semisimple degeneracy. Our investigations bifurcate into an analysis of categories with Tannakian Müger center, and categories with semisimple, super-Tannakian, Müger center.
8.2. **Relative centers.** Consider a tensor category \( \mathcal{C} \) and a surjective tensor functor \( F : \mathcal{C} \to \mathcal{D} \). In this case \( F \) is faithful by Lemma 4.2. For convenience, we identify \( \mathcal{C} \) with its image in \( \mathcal{D} \).

We have that \( \mathcal{D} \) is an exact module category over both \( \mathcal{C} \) and \( \mathcal{D} \otimes \mathcal{C}^{\text{op}} \). Following [34], we define the **relative center** \( Z_{\mathcal{C}}(\mathcal{D}) \) as follows:

- The objects of \( Z_{\mathcal{C}}(\mathcal{D}) \) are pairs \((V, \gamma_V)\), where \( V \) is an object of \( \mathcal{D} \) and \( \gamma_V : V \otimes - \xrightarrow{\sim} - \otimes V \) is a natural isomorphism between the functors \( V \otimes - , - \otimes V : \mathcal{C} \to \mathcal{D} \) which is compatible with the associators and unit in the expected way. For any such pair \((V, \gamma_V)\), we call \( \gamma_V \) the half-braiding.

- Morphisms in \( Z_{\mathcal{C}}(\mathcal{D}) \) are maps \( f : V \to W \) in \( \mathcal{D} \) which are compatible with the chosen half braiding \( \gamma_V \) and \( \gamma_W \) in the sense that the diagram

\[
\begin{array}{ccc}
V \otimes X & \xrightarrow{\gamma_{V,X}} & X \otimes V \\
\downarrow{f \otimes \text{id}} & & \downarrow{\text{id} \otimes f} \\
W \otimes X & \xrightarrow{\gamma_{W,X}} & X \otimes W
\end{array}
\]

commutes for all \( X \in \mathcal{C} \).

- The tensor product \((V, \gamma_V) \otimes (W, \gamma_W)\) is the object \( V \otimes W \) in \( \mathcal{D} \) with the obvious half braiding.

We note, as in [34], that there is an equivalence \((\mathcal{C} \otimes \mathcal{D}^{\text{op}})_{\mathcal{D}} \xrightarrow{\sim} Z_{\mathcal{C}}(\mathcal{D})\) which sends a \( \mathcal{C} \otimes \mathcal{D}^{\text{op}} \)-module endofunctor \( L \) of \( \mathcal{D} \) to the pair consisting of the object \( V = L(1) \) along with the half-braiding \( \gamma_{V,X} : V \otimes X = L(1) \otimes X \xrightarrow{\sim} L(X) \xrightarrow{\sim} X \otimes L(1) = X \otimes V \), for \( X \in \mathcal{C} \). We have a pair of functors

\[
\begin{array}{ccc}
Z(\mathcal{C}) & \xrightarrow{q} & Z_{\mathcal{C}}(\mathcal{D}) \\
& & \xleftarrow{r} Z(\mathcal{D})
\end{array}
\]

(9)

where \( q \) is the map taking a pair of an object in \( \mathcal{C} \) with a half braiding to the same object in \( \mathcal{D} \) with the same half braiding, and \( r \) restricts the half braiding on an object in \( \mathcal{D} \) to objects in \( \mathcal{C} \).

8.3. **Relative centers and de-equivariantization.** Let \( \mathcal{C} \) be a braided tensor category in characteristic 0 and consider a central embedding \( \text{rep}(G) \to \mathcal{C} \) which has image in the M"uger center of \( \mathcal{C} \). In this case the de-equivariantization \( \mathcal{C}_G \) inherits a braiding from \( \mathcal{C} \).

Since any de-equivariantization functor is surjective, we may consider the relative center \( Z_{\mathcal{C}}(\mathcal{C}_G) \).

**Lemma 8.3.** For any central embedding \( \text{rep}(G) \to \mathcal{C} \) there is a tensor equivalence \( Z_{\mathcal{C}}(\mathcal{C}_G) \simeq Z(\mathcal{C}_G) \), where the de-equivariantization of \( Z(\mathcal{C}) \) is induced by the braided inclusion \( \text{rep}(G) \to \mathcal{C} \) to \( Z(\mathcal{C}) \).

The result should be well-known, and so we only sketch the proof.

**Sketch proof.** There is a tensor functor \( F : Z_{\mathcal{C}}(\mathcal{C}_G) \to Z(\mathcal{C}_G) \) which sends an object \((V, \text{act}_V, \gamma_V)\) in the relative center, where \( \text{act}_V : \mathcal{C} \otimes V \to V \) is the action of \( \mathcal{C} = \mathcal{C}(G) \), to the object \((V, \gamma_V)\) in \( Z(\mathcal{C}) \) with the action \( \text{act}_V \). The fact that \((V, \text{act}_V, \gamma_V)\) is an object in \( Z_{\mathcal{C}}(\mathcal{C}_G) \) exactly implies that \( \text{act}_V : \mathcal{C} \otimes V \to V \) is a
morphism in $Z(\mathcal{C})$. So, we see that $((V,\gamma_V),\text{act}_V)$ is indeed an object in $Z(\mathcal{C})_G$. One similarly constructs the inverse $F^{-1}$ to see that $F$ is an equivalence. □

**Corollary 8.4.** For any braided tensor category $\mathcal{C}$ and any central embedding $\text{rep}(G) \to \mathcal{C}$, the double $Z(\mathcal{C})$ is of finite type if and only if the relative double $Z_G(\mathcal{C})$ is of finite type. Furthermore $\text{Kdim } Z(\mathcal{C}) = \text{Kdim } Z_G(\mathcal{C})$.

**Proof.** Apply Theorem 5.2 and Lemma 8.3. □

8.4. **The approach of Section 10.** Consider a braided tensor category $\mathcal{C}$ and a central embedding $\text{rep}(G) \to \mathcal{C}$ with image equal to the Müger center $Z_2(\mathcal{C})$ in $\mathcal{C}$. Suppose $\mathcal{C}$ is of finite type. Our goal is to show that the Drinfeld center $Z(\mathcal{C})$ is of finite type, from which we deduce the finite type property for arbitrary duals $\mathcal{C}^*_{\mathcal{M}}$. However, we cannot access the center directly here, as was the case in the non-degenerate setting. We therefore employ the relative center $Z_G(\mathcal{C})$ as an intermediary between $\mathcal{C}$ and $Z(\mathcal{C})$, in the manner outlined below.

Recall the following essential theorem from [17], which is implicit in the initial works of [9, 46].

**Theorem 8.5** ([9, 46, 17]). Suppose $\text{rep}(G) \to \mathcal{C}$ is a braided equivalence onto $Z_2(\mathcal{C})$. Then the de-equivariantization $\mathcal{C}_G$ is non-degenerate.

Suppose as before that $\mathcal{C}$ is of finite type. By transfer of cohomology along de-equivariantization, Proposition 6.4, and the above theorem, we have that $Z_G(\mathcal{C})$ is of finite type and of Krull dimension $\text{Kdim } Z_G(\mathcal{C}) = 2 \text{Kdim } \mathcal{C}$.

As in (9), we have the diagram

\[
\begin{array}{ccc}
Z(\mathcal{C}) & \xrightarrow{q} & Z_G(\mathcal{C}) \\
& \downarrow{r} & \downarrow{r} \\
& Z_G(\mathcal{C}) & \\
\end{array}
\]

Recall that we want to verify the finite type property for $Z(\mathcal{C})$. Since $Z_G(\mathcal{C})$ is a de-equivariantization of $Z(\mathcal{C})$, we can transfer cohomology up from $Z_G(\mathcal{C})$ to $Z(\mathcal{C})$. So $Z(\mathcal{C})$ is of finite type whenever the relative center is of finite type. Therefore, we need to find a way to transfer cohomology down from the usual center $Z(\mathcal{C})$ to $Z_G(\mathcal{C})$.

We achieve the desired transfer of cohomology along $r$ by extending to a (relative) exact sequence

\[
Z_G(\mathcal{C}) \xrightarrow{r} Z_G(\mathcal{C}) \to \mathcal{Q},
\]

where $\mathcal{Q}$ is a yet to be defined monoidal category, and employing a generalized Lyndon-Hochschild-Serre spectral sequence to obtain the cohomology of $Z_G(\mathcal{C})$ from the cohomology of $Z(\mathcal{C})$. In Section 9 below we provide the necessary background on short exact sequences, and in Section 10 we realize the above outline to find that $Z(\mathcal{C})$ is of finite type whenever $\mathcal{C}$ is of finite type and has Tannakian Müger center. In Section 11 we enhance the above approach to address categories $\mathcal{C}$ with possibly non-Tannakian, semisimple, Müger center.

9. **The Lyndon-Hochschild-Serre spectral sequence for exact sequences of categories**

For this section, we let $k$ be algebraically closed and of arbitrary characteristic. We will, however, only apply the results of this section to cases in which $\text{char}(k) = 0$. 

We show that any exact sequence of tensor categories
\[ \mathcal{B} \to \mathcal{C} \to \mathcal{D} \boxtimes \text{End}_k(\mathcal{M}), \]
in the generalized sense of Etingof and Gelaki [22] (cf. Bruguières and Natale [10]),
gives rise to a spectral sequence
\[ "H^i(\mathcal{B}, H^j(\mathcal{D}, \mathcal{C}))" \Rightarrow H^{i+j}(\mathcal{C}, \mathcal{V}). \]
In order to do this, we must first specify what exactly is on the left hand side of the
above equation. The issue here is that the \( \mathcal{D} \)-invariants functor is a functor from \( \mathcal{D} \)
to \( \text{Vect} \), not one from \( \mathcal{C} \) to \( \mathcal{B} \). So we define an appropriate "\( \mathcal{D} \)-invariants functor"
in this setting, which we denote \( \mathcal{S}_K^\mathcal{D}(\mathcal{D}, \mathcal{-}) : \mathcal{C} \to \mathcal{B} \). Of course, for exact sequences
of finite groups our spectral sequence reduces to the familiar Lyndon-Hochschild-
Serre spectral sequence.

9.1. Exact sequences (following [22]). Recall that a \( k \)-linear category is, for us,
an abelian category enriched over \( \text{Vect} \). We have the standard definition,

**Definition 9.1.** For a finite \( k \)-linear category \( \mathcal{M} \) we let \( \text{End}_k(\mathcal{M}) \) denote the
\( k \)-linear monoidal category of right exact \( k \)-linear endofunctors of \( \mathcal{M} \).

**Remark 9.2.** The finiteness assumption here is inessential, but appropriate for
our analysis.

The category \( \text{End}_k(\mathcal{M}) \) is monoidal under composition and has unit \( \text{id}_\mathcal{M} \). If we
write \( \mathcal{M} = \text{rep}(R) \), for a finite dimensional algebra \( R \), then the functor \( \text{bimod}(R) \to \text{End}_k(\mathcal{M}) \), \( M \mapsto M \otimes_R - \) is an equivalence of \( k \)-linear monoidal categories, by
classical Morita theory. In particular, \( \text{End}_k(\mathcal{M}) \) is Artinian and Noetherian.

Consider an arbitrary tensor category \( \mathcal{D} \) and the corresponding Deligne product
\( \mathcal{D} \boxtimes \text{End}_k(\mathcal{M}) \). We embed \( \text{End}_k(\mathcal{M}) \) in \( \mathcal{D} \boxtimes \text{End}_k(\mathcal{M}) \) as \( X \mapsto 1 \otimes X \). The
following lemma will help us make sense of the notion of a "normal" map to \( \mathcal{D} \boxtimes \text{End}_k(\mathcal{M}) \).

**Lemma 9.3.** Every object \( W \) in \( \mathcal{D} \boxtimes \text{End}_k(\mathcal{M}) \) admits a unique maximal subobject
\( W' \) in (the image of) \( \text{End}_k(\mathcal{M}) \). Furthermore, \( W' \) can be defined as the maximal subobject in \( W \) which admits a surjective map \( 1 \otimes X \to W \) from an object \( X \) in
\( \text{End}_k(\mathcal{M}) \).

**Proof.** If any such maximal object exists it provides a universal morphism \( 1 \otimes X(W) \to W \) from an object \( X(W) \) in \( \text{End}_k(\mathcal{M}) \). Consider now the maximal
subobject \( W' \subset W \) which is the image of some \( 1 \otimes X \to W \). Such a \( W' \) exists as
\( \mathcal{D} \boxtimes \text{End}_k(\mathcal{M}) \) is Noetherian and for any two \( 1 \otimes X \to W \) and \( 1 \otimes Y \to W \) the
coproduct map is expressible as a map from \( 1 \otimes (X \oplus Y) \).

We claim that the kernel of any projection \( 1 \otimes X \to W' \) is of the form \( 1 \otimes Y \)
for some subobject \( Y \) in \( X \). This is easy to see if we adopt a Morita equivalence
\( \mathcal{D} \cong \text{rep}(\Pi) \) for a basic algebra \( \Pi \), and \( \mathcal{M} = \text{rep}(R) \), so that \( \mathcal{D} \boxtimes \text{End}_k(\mathcal{M}) \cong \text{rep}(\Pi \otimes (R \otimes R^{op})) \). Under such an equivalence \( 1 \) is identified with a 1-dimensional
representation \( 1 = \text{id}_\Pi \) for \( \Pi \), and the kernel of any map \( 1 \otimes X \to W' \) is a span
of vectors \( \sum e \otimes x_i = e \otimes (\sum x_i) \). Hence the kernel is of the form \( 1 \otimes Y \). By
exactness of the tensor product this gives \( W' \cong 1 \otimes X(W) \) for \( X(W) = X/Y \). \( \square \)

The following definitions is due to Etingof and Gelaki [22], (cf. the earlier work
of Bruguières and Natale [10]).
Definition 9.4 ([22]). Let \( \mathcal{M} \) be a finite \( k \)-linear abelian category, and consider tensor categories \( \mathcal{C} \) and \( \mathcal{D} \). We say an exact monoidal functor \( F : \mathcal{C} \rightarrow \mathcal{D} \boxtimes \text{End}_k(\mathcal{M}) \) is normal relative to \( \mathcal{M} \) if for each \( V \) in \( \mathcal{C} \) there is a subobject \( V_{\text{triv}} \subset V \) such that
\[
F(V_{\text{triv}}) \subset F(V)
\]
is the maximal subobject in \( F(V) \) contained in \( \text{End}_k(\mathcal{M}) \).

Definition 9.5 ([22]). Let \( \mathcal{M} \) be a finite \( k \)-linear abelian category. Let
\[
\mathcal{B} \xrightarrow{i} \mathcal{C} \xrightarrow{F} \mathcal{D} \boxtimes \text{End}_k(\mathcal{M})
\]
be a composition of a tensor functor \( i : \mathcal{B} \rightarrow \mathcal{C} \) with an exact monoidal functor \( F : \mathcal{C} \rightarrow \mathcal{D} \boxtimes \text{End}_k(\mathcal{M}) \). We say the sequence (10) is an exact sequence with respect to \( \mathcal{M} \) if the following conditions hold:
(a) The tensor functor \( i \) is fully faithful embedding.
(b) The monoidal functor \( F \) is normal relative to \( \mathcal{M} \), and surjective.
(c) The category \( \mathcal{B} \) is the kernel of \( F \) relative to \( \mathcal{M} \), i.e. \( \mathcal{B} \) is the full subcategory in \( \mathcal{C} \) consisting of all objects which map into \( \text{End}_k(\mathcal{M}) \) under \( F \).
(d) The \( \mathcal{B} \)-action on \( \mathcal{M} \) induced by the functor \( \mathcal{B} \rightarrow \text{End}_k(\mathcal{M}) \) makes \( \mathcal{M} \) into an exact indecomposable \( \mathcal{B} \)-module category.

Remark 9.6. When convenient, we identify \( \mathcal{B} \) with its image \( i(\mathcal{B}) \) in \( \mathcal{C} \). Indeed, we already did this in the previous definition in item (c).

Remark 9.7. It is natural, from the algebraic perspective, to only consider exact sequences with respect to the trivial module category \( \mathcal{M} = \text{Vect} \). This is the perspective taken in the initial work of Bruguières and Natale [10]. However, if we take the dual of such a nice sequence \( \mathcal{D} \rightarrow \mathcal{B} \rightarrow \mathcal{I} \) with respect to an exact indecomposable \( \mathcal{I} \)-module category \( \mathcal{N} \), which is a completely natural thing to do, we arrive at a sequence
\[
\mathcal{I}^*_\mathcal{N} \rightarrow \mathcal{B}^*_\mathcal{N} \rightarrow \mathcal{D}^*_\text{Vect} \boxtimes \text{End}_k(\mathcal{N}).
\]
This new sequence will be exact with respect to \( \mathcal{M} \) [22, Theorem 4.1], but generally not exact in the sense of [10]. This duality property is essential to our study.

9.2. Homological properties of the relative invariants.

Lemma 9.8. If \( \mathcal{B} \xrightarrow{i} \mathcal{C} \xrightarrow{F} \mathcal{D} \boxtimes \text{End}_k(\mathcal{M}) \) is an exact sequence relative to \( \mathcal{M} \), then \( F \) is faithful.

Proof. As is explained in [22], we have
\[
\mathcal{D} \boxtimes \text{End}_k(\mathcal{D}) \cong \text{End}_{\mathcal{D}^{\text{op}}}(\mathcal{D}) \boxtimes \text{End}_k(\mathcal{M}).
\]
It follows that the map \( F : \mathcal{C} \rightarrow \mathcal{D} \boxtimes \text{End}_k(\mathcal{M}) \) produces an action of \( \mathcal{C} \) on \( \mathcal{D} \boxtimes \mathcal{M} \) which commutes with the action of \( \mathcal{D}^{\text{op}} \) on \( \mathcal{D} \). Hence \( F \) is naturally given by restricting the codomain of an action map
\[
\mathcal{D}^{\text{op}} \boxtimes \mathcal{C} \rightarrow \text{End}_k(\mathcal{D} \boxtimes \mathcal{M}).
\]
By [22, Corollary 2.5], \( \mathcal{D} \boxtimes \mathcal{M} \) is exact over \( \mathcal{D}^{\text{op}} \boxtimes \mathcal{C} \), and hence the functor (11) is faithful. As a consequence we find that the restriction \( \mathcal{C} \rightarrow \text{End}_k(\mathcal{D} \boxtimes \mathcal{M}) \) to \( \mathcal{C} \subset \mathcal{D}^{\text{op}} \boxtimes \mathcal{C} \) is faithful, as is its factoring \( F : \mathcal{C} \rightarrow \mathcal{D} \boxtimes \text{End}_k(\mathcal{M}) \) through \( \text{End}_{\mathcal{D}^{\text{op}}}(\mathcal{D}) \boxtimes \text{End}_k(\mathcal{M}) \). \( \square \)
Lemma 9.9. Let $F : \mathcal{C} \to \mathcal{D} \boxtimes \text{End}_k(\mathcal{M})$ be a faithful monoidal functor which is normal relative to $\mathcal{M}$. Then the following properties hold:

(i) Every object $V$ in $\mathcal{C}$ admits a subobject $V_{\mathcal{D}\text{-triv}} \subset V$ such that $F(V_{\mathcal{D}\text{-triv}})$ is the unique maximal subobject in $F(V)$ which lies in $\text{End}_k(\mathcal{M})$. Furthermore, $V_{\mathcal{D}\text{-triv}}$ is determined uniquely by this property.

(ii) For any map $f : V \to W$ in $\mathcal{C}$, the restriction $f|_{V_{\mathcal{D}\text{-triv}}}$ factors (uniquely) through $W_{\mathcal{D}\text{-triv}}$.

That is to say, the operation $V \mapsto V_{\mathcal{D}\text{-triv}}$ is an endofunctor of $\mathcal{C}$ with image in the kernel of $F$.

Proof. (i) Normality ensures that such a $V_{\mathcal{D}\text{-triv}}$ exists. For uniqueness, suppose another submodule $V' \subset V$ is mapped to an object in $\text{End}(\mathcal{M})$ under $F$, and consider the sequence

$$F(V') \to F(V) \to F(V)/F(V_{\mathcal{D}\text{-triv}}).$$

By maximality of $F(V_{\mathcal{D}\text{-triv}})$, the above sequence is 0. By faithfulness of $F$ the sequence $V' \to V \to V/V_{\mathcal{D}\text{-triv}}$ is also 0. Hence $V' \subset V_{\mathcal{D}\text{-triv}}$. This implies uniqueness of $V_{\mathcal{D}\text{-triv}}$. (ii) Functoriality of $(\cdot)_{\mathcal{D}\text{-triv}}$ with respect to morphisms follows by a similar argument. $\square$

Lemmas 9.8 and 9.9 together tell us that any exact sequence

$$\mathcal{B} \xrightarrow{i} \mathcal{C} \xrightarrow{F} \mathcal{D} \boxtimes \text{End}_k(\mathcal{M}) \quad (12)$$

produces a functor $\mathcal{C} \to \mathcal{B}$ which assigns to any object in $\mathcal{C}$ its maximal $\mathcal{D}$-trivial subobject. The fact that this subobject lies in $\mathcal{B}$ follows from the identification $\mathcal{B} = \text{ker}(F)$.

Definition 9.10. For an exact sequence (12), relative to an exact $\mathcal{D}$-module category $\mathcal{M}$, we define the functor $\mathcal{I}_\mathcal{D}^0(\mathcal{D}, -)$ by taking

$$\mathcal{I}_\mathcal{D}^0(\mathcal{D}, -) : \mathcal{C} \to \mathcal{B}, \quad V \mapsto V_{\mathcal{D}\text{-triv}},$$

where $V_{\mathcal{D}\text{-triv}}$ is as in Lemma 9.9. We call $\mathcal{I}_\mathcal{D}^0(\mathcal{D}, -)$ the $\mathcal{D}$-relative invariants.

Lemma 9.11. For an exact sequence (12), the relative invariants functor $\mathcal{I}_\mathcal{D}^0(\mathcal{D}, -)$ has the following properties:

(i) $\mathcal{I}_\mathcal{D}^0(\mathcal{D}, -)$ is left exact.

(ii) $\mathcal{I}_\mathcal{D}^0(\mathcal{D}, -)$ is exact when $\mathcal{D}$ is a fusion category.

(iii) $\mathcal{I}_\mathcal{D}^0(\mathcal{D}, -)$ sends injectives in $\mathcal{C}$ to injectives in $\mathcal{B}$.

(iv) There is a natural identification of functors

$$\text{Hom}_\mathcal{B}(1_\mathcal{B}, \mathcal{I}_\mathcal{D}^0(\mathcal{D}, -)) = \text{Hom}_\mathcal{C}(1_\mathcal{C}, -).$$

Proof. (i) For any exact sequence $0 \to L \to M \to N \to 0$ it is clear that $\mathcal{I}_\mathcal{D}^0(\mathcal{D}, L) \to \mathcal{I}_\mathcal{D}^0(\mathcal{D}, M)$ is injective, via the maximal image interpretation of Lemma 9.3 and the fact that $F$ is faithfully exact. Now, after applying $F$, the kernel of $F\mathcal{I}_\mathcal{D}^0(\mathcal{D}, M) \to F\mathcal{I}_\mathcal{D}^0(\mathcal{D}, N)$ is an object of the form $1 \otimes Z$, which necessarily factors through $F(L)$, and hence lies in $F\mathcal{I}_\mathcal{D}^0(\mathcal{D}, L)$ by Lemma 9.3. So the kernel of $F\mathcal{I}_\mathcal{D}^0(\mathcal{D}, M) \to F\mathcal{I}_\mathcal{D}^0(\mathcal{D}, N)$ is contained in $F\mathcal{I}_\mathcal{D}^0(\mathcal{D}, L)$. The opposite containment is immediate, and we therefore have that $F\mathcal{I}_\mathcal{D}^0(\mathcal{D}, L)$ is mapped isomorphically onto the kernel of $F\mathcal{I}_\mathcal{D}^0(\mathcal{D}, M) \to F\mathcal{I}_\mathcal{D}^0(\mathcal{D}, N)$. Faithfulness of $F$ implies that $\mathcal{I}_\mathcal{D}^0(\mathcal{D}, L)$ is the kernel of $\mathcal{I}_\mathcal{D}^0(\mathcal{D}, M) \to \mathcal{I}_\mathcal{D}^0(\mathcal{D}, N)$, as desired.
(ii) When \( \mathcal{D} \) is fusion, we may enumerate the simples \( \{ V_0, V_1, \ldots, V_n \} \) with \( V_0 = 1 \). In this case all objects of the Deligne product \( \mathcal{D} \otimes \text{End}_k(\mathcal{M}) \) are direct sums of objects of the form \( V_i \otimes X \), and there are no nonzero maps between \( V_i \otimes X \) and \( V_j \otimes Y \) for \( i \neq j \). Therefore taking the trivial summand \( V_{\text{triv}} \otimes X \), which provides the maximal subobject in \( \text{End}_k(\mathcal{M}) \), is an exact operation. It follows that for an exact sequence 0 \( \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \) in \( \mathcal{C} \) the sequence

\[
0 \rightarrow \mathcal{S}_g^0(\mathcal{D}, L) \rightarrow \mathcal{S}_g^0(\mathcal{D}, W) \rightarrow \mathcal{S}_g^0(\mathcal{D}, N) \rightarrow 0
\]

is exact in \( \mathcal{B} \), since its image under \( F \) is exact.

(iii) The functor \( \mathcal{S}_g^0(\mathcal{D}, -) \) can alternatively be defined as the functor which sends an object \( V \) to the universal map \( \mathcal{S}_g^0(\mathcal{D}, V) \rightarrow V \) from an object in \( \mathcal{B} \). So for any object \( L \) in \( \mathcal{B} \) we have \( \text{Hom}_\mathcal{C}(L, V) = \text{Hom}_\mathcal{B}(L, \mathcal{S}_g^0(\mathcal{D}, V)) \).

Suppose that \( V \) is injective in \( \mathcal{C} \) and that 0 \( \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \) is an exact sequence in \( \mathcal{B} \). Then we have the diagram

\[
\begin{align*}
\text{Hom}_\mathcal{B}(N, \mathcal{S}_g^0(\mathcal{D}, V)) & \longrightarrow \text{Hom}_\mathcal{B}(M, \mathcal{S}_g^0(\mathcal{D}, V)) & \longrightarrow \text{Hom}_\mathcal{B}(L, \mathcal{S}_g^0(\mathcal{D}, V)) \\
\downarrow \cong & & \downarrow \cong \\
0 & \longrightarrow \text{Hom}_\mathcal{B}(N, V) & \longrightarrow \text{Hom}_\mathcal{B}(M, V) & \longrightarrow \text{Hom}_\mathcal{B}(L, V) & \longrightarrow 0,
\end{align*}
\]

and conclude that the top row is exact. Whence \( \mathcal{S}_g^0(\mathcal{D}, V) \) is seen to be injective in \( \mathcal{B} \).

(iv) Since \( 1_\mathcal{E} = i(1_\mathcal{B}) \) is an object in (the image of) \( \mathcal{B} \), the universal map perspective employed in (iii) gives a natural identification

\[
\text{Hom}_\mathcal{B}(1_\mathcal{E}, V) = \text{Hom}_\mathcal{B}(1_\mathcal{B}, \mathcal{S}_g^0(\mathcal{D}, V)),
\]

for arbitrary \( V \) in \( \mathcal{C} \).

9.3. **Lyndon-Hochschild-Serre spectral sequence.** Consider as before a relative exact sequence (12). By point (i) of Lemma 9.11 we can derive the functor \( \mathcal{S}_g^0(\mathcal{D}, -) : \mathcal{C} \rightarrow \mathcal{B} \) to get a triangulated functor

\[
\text{R} \mathcal{S}_g^0(\mathcal{D}, -) : \text{D}^+(\mathcal{C}) \rightarrow \text{D}^+(\mathcal{B}).
\]

We denote the corresponding \( i \)-th derived functor

\[
\mathcal{S}_g^0(\mathcal{D}, -) = \text{R}^i \mathcal{S}_g^0(\mathcal{D}, -).
\]

Each \( \mathcal{S}_g^0(\mathcal{D}, -) \) is a functor taking values in the category \( \mathcal{B} \).

The object \( \text{R} \mathcal{S}_g^0(\mathcal{D}, 1) \) admits an algebra structure in \( \text{D}^+(\mathcal{B}) \) via the usual yoga. Namely, one takes an injective resolution \( 1_\mathcal{B} \rightarrow I \), so that \( \text{R} \mathcal{S}_g^0(\mathcal{D}, 1) = \mathcal{S}_g^0(\mathcal{D}, I) \), and chooses a homotopy equivalence \( I \otimes I \rightarrow I \) to get

\[
\text{mult}_{\text{R} \mathcal{S}_g^0(\mathcal{D}, 1)} : \mathcal{S}_g^0(\mathcal{D}, I) \otimes \mathcal{S}_g^0(\mathcal{D}, I) \rightarrow \mathcal{S}_g^0(\mathcal{D}, I \otimes I) \rightarrow \mathcal{S}_g^0(\mathcal{D}, I).
\]

We similarly get an action of \( \text{R} \mathcal{S}_g^0(\mathcal{D}, 1) \) on \( \text{R} \mathcal{S}_g^0(\mathcal{D}, V) \) for each \( V \) in \( \mathcal{C} \).

**Proposition 9.12.** Let \( \mathcal{B} \rightarrow \mathcal{C} \rightarrow \mathcal{D} \otimes \text{End}_k(\mathcal{M}) \) be an exact sequence with respect to \( \mathcal{M} \). There is a convergent multiplicative spectral sequence

\[
E_2^{i,j} = H^i(\mathcal{B}, \mathcal{S}_g^j(\mathcal{D}, 1)) \Rightarrow H^{i+j}(\mathcal{C}, 1),
\]

and for each \( V \) in \( \mathcal{C} \) there is a spectral sequence

\[
V E_2^{i,j} = H^i(\mathcal{B}, \mathcal{S}_g^j(\mathcal{D}, V)) \Rightarrow H^{i+j}(\mathcal{C}, V),
\]

which is a module over \( E_2^{i*,*} \).
Proof. By point (iii) of Lemma 9.11, the proposed spectral sequence arises as a Grothendieck spectral sequence. The multiplicative properties follow by [29]. □

We only use the above spectral sequence in the case in which $D$ is fusion. In this case the spectral sequence degenerates to give a $k$-algebra identification
\[ H^*(B, 1_B) = H^*(B, S^0_\mathcal{E}(D, 1)) = H^*(\mathcal{E}, 1_\mathcal{E}), \]
and an identification of graded $H^*(B, 1)$-modules
\[ H^*(B, S^0_\mathcal{E}(D, V)) = H^*(\mathcal{E}, V). \]
Both of these identifications are induced by the exact embedding $i : B \to C$. We record these findings in a corollary.

Corollary 9.13. Let $B \to \mathcal{E} \to \mathcal{D} \boxtimes \text{End}_k(\mathcal{M})$ be an exact sequence with respect to $\mathcal{M}$ and suppose that $D$ is a fusion category. Then there is a natural identification of $k$-algebras
\[ H^*(B, 1_B) = H^*(C, 1_C), \]
and an identification of graded $H^*(B, 1)-$modules
\[ H^*(B, S^0_\mathcal{E}(D, V)) = H^*(C, V), \]
where the cohomology of $C$ acts on the right hand side of this equality via its identification with the cohomology of $B$.

10. Finite type braided categories II: Tannakian Müger center

We fix $k$ an algebraically closed field of characteristic 0. We follow the outline of Section 8.4 to verify stability of cohomology for finite type braided tensor categories with Tannakian Müger center. We describe some practical methods for determining whether or not the Müger center of a given braided category is Tannakian in Section 10.3.

10.1. De-equivariantization and doubling for braided tensor categories. Let $\text{rep}(G) \to \mathcal{E} \to \mathcal{D} \boxtimes \text{End}_k(\mathcal{M})$ be a central embedding, with $G$ a finite group, and consider the de-equivariantization $\mathcal{E}_G$. We then have the exact sequence
\[ \text{rep}(G) \to \mathcal{E} \to \mathcal{E}_G \]
with respect to $\text{Vect}$ and tensor with $\mathcal{E}_G^{\text{cop}}$ on the right to get another exact sequence
\[ \text{rep}(G) \to \mathcal{E} \boxtimes \mathcal{E}_G^{\text{cop}} \to \mathcal{E}_G \boxtimes \mathcal{E}_G^{\text{cop}} \]
with respect to $\text{Vect}$. By [22, Theorem 4.1] we may take the dual with respect to the indecomposable exact $\mathcal{E}_G$-bimodule category $\mathcal{E}_G$ to get another exact sequence
\[ Z(\mathcal{E}_G) \to Z_{\mathcal{E}}(\mathcal{E}_G) \to \text{Vect} \boxtimes \text{End}_k(\mathcal{E}_G), \]
(13)
now with respect to $\mathcal{E}_G$.

Theorem 10.1. Suppose $k$ is algebraically closed and of characteristic 0. Let $\text{rep}(G) \to \mathcal{E}$ be a central embedding, with $G$ a finite group. Then the center $Z(\mathcal{E})$ is of finite type if and only if the center of the de-equivariantization $Z(\mathcal{E}_G)$ is of finite type. In this case $k\text{dim } Z(\mathcal{E}) = k\text{dim } Z(\mathcal{E}_G)$.

Proof. Via the exact sequence (13) and Corollary 9.13 we see that the relative center $Z_{\mathcal{E}}(\mathcal{E}_G)$ is of finite type if and only if $Z(\mathcal{E}_G)$ is of finite type, in which case the Krull dimensions agree. By Lemma 8.3, $Z_{\mathcal{E}}(\mathcal{E}_G)$ is a de-equivariantization of $Z(\mathcal{E})$. Thus we apply Theorem 5.2 to see that $Z(\mathcal{E})$ is of finite type if and only if $Z_{\mathcal{E}}(\mathcal{E}_G)$ is of finite type, and again the Krull dimensions agree. □
10.2. Weak stability of cohomology under duality and doubling.

**Theorem 10.2.** Suppose \( \mathcal{C} \) is braided and of finite type over an algebraically closed field \( k \) of characteristic 0. Suppose also that the Müger center of \( \mathcal{C} \) is Tannakian. Then, for every exact \( \mathcal{C} \)-module category \( \mathcal{M} \), the dual \( \mathcal{C}^*\mathcal{M} \) is also of finite type. Furthermore, the Krull dimensions are uniformly bounded

\[
K\dim \mathcal{C}^*\mathcal{M} \leq 2 K\dim \mathcal{C}.
\]

**Proof.** It suffices to show that the center \( Z(\mathcal{C}) \) is of finite type and has Krull dimension \( K\dim Z(\mathcal{C}) = 2 K\dim \mathcal{C} \), by Corollary 5.1. We adopt a braided identification \( \text{rep}(G) \cong Z_2(\mathcal{C}) \).

By Theorem 5.2, the de-equivariantization is also of finite type and of the same Krull dimension as \( \mathcal{C} \). Recall that \( \mathcal{C}_G \) is non-degenerate in this case (see Theorem 8.5). It follows that \( Z(\mathcal{C}_G) \) is of finite type and has Krull dimension

\[
K\dim Z(\mathcal{C}_G) = 2 K\dim \mathcal{C}_G = 2 K\dim \mathcal{C},
\]

by Proposition 6.4. Whence \( Z(\mathcal{C}) \) is of finite type and of the prescribed Krull dimension, by Theorem 10.1. \( \Box \)

**Corollary 10.3.** Let \( A \) be a quasitriangular Hopf algebra over an algebraically closed field of characteristic 0. Suppose additionally that the Müger center of \( \text{rep}(A) \) is Tannakian, and that \( A \) has finitely generated cohomology. Then the dual Hopf algebra \( A^* \) also has finitely generated cohomology, as do all cocycle twists of \( A \) and all cocycle twists of the dual \( A^* \).

10.3. Practical checks: Tannakian vs super-Tannakian type. Throughout this subsection we maintain the assumptions \( k = \overline{k}, \text{char}(k) = 0 \). From the material of Section 8.1 one sees that any symmetric tensor category \( \mathcal{C} \) over \( k \) has the Chevalley property. That is to say, the full subcategory \( \overline{\mathcal{C}} \subset \mathcal{C} \) generated by the simples is a tensor subcategory.

**Lemma 10.4.** [17, Corollary 2.50(i)] Let \( \mathcal{C} \) be a symmetric tensor category. If \( \text{FPdim}(\mathcal{C}) \) is odd, then \( \mathcal{C} \) is Tannakian.

**Proof.** Let \( \overline{\mathcal{C}} \) denote the fusion subcategory generated by the simples. We have \( \overline{\mathcal{C}} \cong \text{rep}(G, R_z) \), where \( z \in G \) is such that \( z^2 = 1 \), and \( z = 1 \) if and only if \( \mathcal{C} \) is Tannakian. If \( z \neq 1 \) then \( G \) has an order 2 subgroup, and hence \( 2 \mid |G| = \text{FPdim}(\overline{\mathcal{C}}) \). Since \( \mathcal{C} \) is integral this implies \( 2 \mid \text{FPdim}(\mathcal{C}) \), which is explicitly not the case. Hence \( z = 1 \) and \( \mathcal{C} \) is Tannakian. \( \Box \)

**Corollary 10.5.** Suppose \( \mathcal{C} \) is a braided tensor category of odd Frobenius-Perron dimension. Then the Müger center of \( \mathcal{C} \) is Tannakian.

**Proof.** Let \( \mathcal{C} \) denote the Müger center of \( \mathcal{C} \). Since the categories are weakly integral, we have that \( \text{FPdim}(\mathcal{C}) \mid \text{FPdim}(\mathcal{C}) \). So \( \mathcal{C} \) must be of odd dimension and therefore Tannakian, by Lemma 10.4. \( \Box \)

The following lemma is standard. We repeat the proof for the convenience of the reader.

**Lemma 10.6.** Let \( \mathcal{C} \) be a braided tensor category. Suppose that \( \text{Irr}(Z_2(\mathcal{C})) \) contains no objects \( V \) which solve the equation \( c_{V,V} = -\text{id}_{V \otimes V} \), or more generally that \( \text{Irr}(\mathcal{C}) \) contains no such objects. Then \( Z_2(\mathcal{C}) \) is Tannakian.
Proof. Take $E = Z_2(C)$. Then the fusion category $\bar{E}$ generated by the simple in $E$ is of the form $\text{rep}(G, R_z)$, with $\text{ord}(z) = 2$ whenever $E$ is non-Tannakian. When \( z \neq 1 \) then \( 1 + z \) is not a unit in the group ring $k[G]$, since $(1 - z)(1 + z) = 0$. So the quotient representation $k[G] / k[G](1 + z)$ is non-zero and any simple summand $V$ of this representation is such that $c_{V,V} = -\text{id}_V \otimes V$. Thus, if $\text{Irr}(E)$, or more generally $\text{Irr}(C)$, has no solutions to the equation $c_{V,V} = -\text{id}_V \otimes V$ then $E$ must be Tannakian. \qed

Following [20], we define the (braided) quasiexponent $qexp_{\text{br}}(C)$ of a braided tensor category $C$ is the minimal integer $N$ such that the double braiding $(c_{V,W}^2)^N$ is a unipotent automorphism, at arbitrary $V$ and $W$. This number was shown to be finite at [20, Theorem. 4.1].

Lemma 10.7. If $qexp_{\text{br}}(C)$ is odd then the M"uger center of $C$ is Tannakian.

Proof. For any non-Tannakian symmetric category $E$ we have $qexp_{\text{br}}(E) = 2$. Furthermore, one can readily check that for any braided embedding $E \to C$ we have $qexp_{\text{br}}(E) | qexp_{\text{br}}(C)$. Thus, if the quasiexponent of $C$ is odd then $C$ admits no braided embedding from a super-Tannakian category. \qed

Corollary 10.8. Suppose $C$ is braided and of finite type over a field of characteristic 0. Let $\mathcal{M}$ be an exact $C$-module category and consider the dual $C^*_{\mathcal{M}}$. The category $C^*_{\mathcal{M}}$ is of finite type provided any of the following properties hold for $C$:

- $C$ is odd-dimensional.
- The quasiexponent $qexp_{\text{br}}(C)$ is odd.
- There are no simples in $Z_2(C)$ with $c_{V,V} = -\text{id}_V \otimes V$.
- There are no simples in $C$ with $c_{V,V} = -\text{id}_V \otimes V$.
- $Z_2(C)$ is trivial, that is $C$ is non-degenerate.
- $Z_2(C)$ is Tannakian.

11. Finite type braided categories III: semisimple M"uger center

We fix $k$ algebraically closed and of characteristic 0. We have seen previously that if $C$ of finite type and braided with Tannakian M"uger center, then all duals $C^*_{\mathcal{M}}$ are also of finite type. We extend this result to allow (more generally) for any semisimple M"uger center.

Theorem 11.1. Let $C$ be a braided tensor category of finite type over an algebraically closed field of characteristic 0. Suppose that the M"uger center of $C$ is semisimple. Then, for any exact module category $\mathcal{M}$, the dual category $C^*_{\mathcal{M}}$ is also of finite type and the Krull dimension is bounded as

$$\text{Kdim } C^*_{\mathcal{M}} \leq 2 \text{Kdim } C.$$ 

We first discuss $G$-extensions, and minimal non-degenerate extensions for slightly degenerate categories. We then employ some results of [13] to prove Theorem 11.1. After proving Theorem 11.1, we discuss means of determining semisimplicity of M"uger centers in general.

\footnote{In [20], Etingof shows precisely that the double braiding is unipotent at any given pair of objects. However, unipotency at each pair of simples implies global unipotency. Also, we slightly deviate from [20], where the quasiexponent is defined via the center $Z(C)$, which requires no braiding assumption on $C$. Whence our use of the term “braided” quasiexponent.}
11.1. Cohomology and G-extensions. Fix $G$ a finite group. A $G$-grading on a tensor category $\mathcal{D}$ is a choice of decomposition $\mathcal{D} = \bigoplus_{g \in G} \mathcal{D}_g$ such that $\mathcal{D}_g \otimes \mathcal{D}_h \subseteq \mathcal{D}_{gh}$, where the components are full, orthogonal, $k$-linear subcategories of $\mathcal{D}$. We call such a grading on $\mathcal{D}$ faithful if the $\mathcal{D}_g$ are nonzero for each $g \in G$.

An embedding $i : \mathcal{C} \to \mathcal{D}$ of tensor categories is called a $G$-extension if $\mathcal{D}$ comes equipped with a faithful grading by $G$ under which $\mathcal{C}$ is identified with the neutral component $\mathcal{D}_1$ via $i$.

Lemma 11.2. Let $i : \mathcal{C} \to \mathcal{D}$ be a $G$-extension. Then $\mathcal{C}$ is of finite type if and only if $\mathcal{D}$ is of finite type and, in this case, the Krull dimensions agree.

Proof. Let $p : \mathcal{D} \to \mathcal{D}_1 = \mathcal{C}$ be the $k$-linear projection onto the neutral component. Since $\mathcal{D}$ decomposes as an abelian category $\mathcal{D} = \bigoplus_{g \in G} \mathcal{D}_g$, and since $1$ is an object in $\mathcal{C}$, we see that the inclusion

$$H^\bullet(\mathcal{C}, pV) \xrightarrow{i} H^\bullet(\mathcal{D}, pV) \xrightarrow{\pi} H^\bullet(\mathcal{D}, V) \quad (14)$$

is an isomorphism for each $V$ in $\mathcal{D}$. Indeed, both of the maps in the composition are isomorphisms. In particular, the inclusion $i : \mathcal{C} \to \mathcal{D}$ identifies the algebras $H^\bullet(\mathcal{C}, 1)$ and $H^\bullet(\mathcal{D}, 1)$, and (14) is an isomorphism of $H^\bullet(\mathcal{C}, 1) = H^\bullet(\mathcal{C}, 1)$-modules. So we see that $\mathcal{C}$ is of finite type if and only if $\mathcal{D}$ is of finite type, and that the Krull dimensions agree in this case.

Remark 11.3. One can show furthermore that for any $G$-extension $\mathcal{C} \to \mathcal{D}$, the double $Z(\mathcal{C})$ is of finite type if and only if $Z(\mathcal{D})$ is of finite type. Indeed, there is a $k$-linear decomposition $Z(\mathcal{D}) = Z_1 \oplus Z_2^\perp$, where $Z_1$ is the preimage of $\mathcal{C}$ along the forgetful functor $Z(\mathcal{D}) \to \mathcal{D}$, and one uses [34, Corollary 3.7] to obtain $Z(\mathcal{C})$ as a de-equivariantization of $Z_1$ (cf. [35, Theorem 2.4]).

11.2. Minimal extensions and the fermionic grading [11] (cf. [35, Section 3]). Consider a braided tensor category $\mathcal{C}$ with M"uger center isomorphic to $sVect$. By a minimal non-degenerate extension of $\mathcal{C}$ we mean a braided embedding $\mathcal{C} \to \mathcal{D}$ for which $\mathcal{D}$ is non-degenerate and of Frobenius-Perron dimension $2\text{FPdim}(\mathcal{C})$. The $2$ here comes from the Frobenius-Perron dimension of the degeneracy $sVec \cong Z_2(\mathcal{C})$.

Fix $\mathcal{C}$ a braided tensor category with M"uger center isomorphic to $sVect$ and $\mathcal{C} \to \mathcal{D}$ a minimal non-degenerate extension. Let $f \in G(\mathcal{C})$ be the unique non-trivial simple in $Z_2(\mathcal{C})$. Tensoring by $f$ is an automorphism of $\mathcal{D}$, and the double braiding $c^2_{f,-}$ is a natural automorphism of the functor $f \otimes - : \mathcal{D} \to \mathcal{D}$. We take $c \in \text{Aut}_{\text{Fun}}(id_{\mathcal{D}})$ to be the unique natural automorphism of the identity functor so that $f \otimes c = c^2_{f,-}$.

Lemma 11.4. The automorphism $c$ is such that $c^2 = 1$.

Proof. Let $V$ be an arbitrary object in $\mathcal{D}$. Since $f \otimes f \cong 1$ we have

$$f^2 \otimes id_V = c^2_{f,-} = (f \otimes c_{f,V})(c_{f,V} \otimes f)(f \otimes c_{f,V}) = (f \otimes c_{f,V})c_{f,V} = c_{f,V}^2 = f^2 \otimes c_{f,V}. \quad (15)$$

In the above calculation, we have employed MacLane’s strictness theorem to ignore the associators. Since the above equations hold at all $V$ in $\mathcal{D}$, we find $c^2 = 1$. □
The global operators \((1 \pm c)\) satisfy
\[(1 - c)(1 + c) = 0, \quad (1 + c) + (1 - c) = 2, \quad (1 \pm c)^2 = 2(1 \pm c).
\]
Hence the global operators \(p_{\pm 1} := \frac{1}{2}(1 \pm c)\) provide orthogonal idempotents which sum to 1. We therefore get a \(k\)-linear splitting
\[\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_{-1}, \quad \text{where} \quad \mathcal{D}_{\pm 1} = p_{\pm 1}\mathcal{D}.
\]

**Lemma 11.5** (cf. [11]). The decomposition \(\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_{-1}\) is a faithful \(\mathbb{Z}/2\mathbb{Z}\)-grading. In addition \(\mathcal{C} \subset \mathcal{D}_1\).

**Proof.** The fact that \(p_{\pm 1}\) are orthogonal idempotents implies that there are no maps, and no non-trivial extensions, between objects in \(\mathcal{D}_1\) and objects in \(\mathcal{D}_{-1}\). So the decomposition is a decomposition of \(\mathcal{D}\) as a \(k\)-linear category. The fact that \(\mathcal{D}_a\mathcal{D}_b = \mathcal{D}_{ab}\) follows from the braid relation.

Since \(f\) is in the Müger center of \(\mathcal{C}\) we have \(e_{f,-}^2|_{\mathcal{C}} = f \otimes \text{id}\) and hence \(c|_{\mathcal{C}} = \text{id}_{\mathcal{C}}\). So \(\mathcal{C} \subset \mathcal{D}_1\). To see that the grading is faithful, i.e. that \(\mathcal{D}_{-1} \neq 0\), we note that if \(\mathcal{D}_{-1}\) vanished then \(f\) would provide a non-trivial simple object in the Müger center of \(\mathcal{D}\), which cannot occur since \(\mathcal{D}\) is non-degenerate. \(\square\)

We call the above defined \(\mathbb{Z}/2\mathbb{Z}\)-grading on the minimal non-degenerate extension \(\mathcal{D}\) the fermionic grading. The existence of such a grading in the fusion setting was established first in [11].

**Lemma 11.6** ([24, Proposition 8.20]). If \(\mathcal{D}\) is faithfully graded by a finite group \(G\) then \(\text{FPdim}(\mathcal{D}) = |G| \text{FPdim}(\mathcal{D}_1)\).

**Proof.** The proof is just as in [24], except one uses the regular object of [28, Definition 6.1.6] in place of the element \(R\). \(\square\)

**Lemma 11.7.** The embedding \(\mathcal{C} \to \mathcal{D}_1\) is an equivalence. That is to say, any minimal non-degenerate extension \(\mathcal{C} \to \mathcal{D}\) is a \(\mathbb{Z}/2\mathbb{Z}\)-extension of \(\mathcal{C}\) with respect to the fermionic grading.

**Proof.** By Lemma 11.6, \(\text{FPdim}(\mathcal{D}) = 2 \text{FPdim}(\mathcal{D}_1)\). By our minimality assumption we also have \(\text{FPdim}(\mathcal{D}) = 2 \text{FPdim}(\mathcal{C})\). Hence \(\text{FPdim}(\mathcal{C}) = \text{FPdim}(\mathcal{D}_1)\). Agreement of Frobenius-Perron dimension implies that the embedding \(\mathcal{C} \to \mathcal{D}_1\) is an equivalence [28, Proposition 6.3.3]. \(\square\)

11.3. The Drinfeld center of a slightly degenerate category. Consider a braided tensor category \(\mathcal{C}\) with semisimple Müger center \(\mathcal{E}\). Let \(Z(\mathcal{C}, \mathcal{E})\) be the Müger centralizer of \(\mathcal{E}\) in \(Z(\mathcal{C})\). To be clear, we embed \(\mathcal{E}\) in \(Z(\mathcal{C})\) via the map \(\mathcal{C} \to Z(\mathcal{C})\) induced by the braiding, and we consider the full subcategory \(Z(\mathcal{C}, \mathcal{E})\) of all \(X\) in \(Z(\mathcal{C})\) such that \(\gamma_{X,V}\gamma_{V,X} = \text{id}_{V \otimes X}\) for every \(V\) in \(\mathcal{E}\), where \(\gamma\) denotes the braiding on \(Z(\mathcal{C})\).

The following proposition is a straightforward generalization of [13, Corollary 4.4].

**Proposition 11.8.** Let \(\mathcal{C}\) be a braided tensor category with Müger center \(\mathcal{E}\). Then

(i) The canonical functor \(F: \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}} \to Z(\mathcal{C})\) is a surjection onto \(Z(\mathcal{C}, \mathcal{E})\).

(ii) The inclusion \(\mathcal{E} \to Z(\mathcal{C}, \mathcal{E})\) is an equivalence onto the Müger center of \(Z(\mathcal{C}, \mathcal{E})\).

(iii) When \(\mathcal{E} \simeq \text{sVect}\), the surjection \(\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}} \to Z(\mathcal{C}, \text{sVect})\) extends to an exact sequence \(\text{rep}(\mathbb{Z}/2\mathbb{Z}) \to \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}} \to Z(\mathcal{C}, \text{sVect})\).
Proof. Let \( \mathcal{C}_+ \) and \( \mathcal{C}_- \) denote the images of \( \mathcal{C} \) and \( \mathcal{C}^{rev} \) in \( Z(\mathcal{C}) \) under the embeddings given by the braiding. For a subcategory \( \mathcal{D} \), we let \( \mathcal{D}' \) denote its centralizer.

(i) \& (ii) The image of \( F \) is the tensor subcategory \( \mathcal{C}_+ \vee \mathcal{C}_- \) generated by \( \mathcal{C}_+ \) and \( \mathcal{C}_- \). We note that the centralizer of \( \mathcal{C}_\pm \) in \( Z(\mathcal{C}) \) is \( \mathcal{C}_\mp \), and hence \( \mathcal{C}_+ \vee \mathcal{C}_- \) is equivalent to \( \mathcal{C}_+ \vee \mathcal{C}_- = \mathcal{D}' \). The opposite containment follows from the fact that \( \mathcal{D}' \) centralizes both \( \mathcal{C}_+ \) and \( \mathcal{C}_- \). Hence \( \mathcal{C}_+ \vee \mathcal{C}_- \) is non-degenerate, we have

\[
Z(\mathcal{C}, \mathcal{D}) = \mathcal{D}' = (\mathcal{C}_+ \vee \mathcal{C}_-)^{\prime} = \mathcal{C}_+ \vee \mathcal{C}_-,
\]

by [56, Theorem 4.9]. It follows that \( F \) provides a surjective map onto \( Z(\mathcal{C}, \mathcal{D}) \).

(iii) Let \( f \) denote the fermion in \( s\text{Vect} \subset \mathcal{C} \) and let \( \zeta \) denote the unique non-trivial simple in \( \text{rep}(\mathbb{Z}/2\mathbb{Z}) \). We define a braided embedding \( i : \text{rep}(\mathbb{Z}/2\mathbb{Z}) \to \mathcal{C} \otimes \mathcal{C}^{rev} \) by taking \( i(\zeta) = f \otimes f \). Since \( \zeta \otimes \zeta \cong 1 \), \( F(\text{rep}(\mathbb{Z}/2\mathbb{Z})) \) is the trivial subcategory in \( Z(\mathcal{C}, \mathcal{D}) \). Whence we have the sequence

\[
\text{rep}(\mathbb{Z}/2\mathbb{Z}) \xrightarrow{i} \mathcal{C} \otimes \mathcal{C}^{rev} \xrightarrow{F} Z(\mathcal{C}, \mathcal{D}),
\]

with \( F \) surjective and \( F_i \) factoring through the fiber functor \( \text{rep}(\mathbb{Z}/2\mathbb{Z}) \to \text{Vect} \).

Now, by [56, Lemma 4.8] we have

\[
\text{FPdim}(Z(\mathcal{C}, \mathcal{D})) = \frac{\text{FPdim}(\mathcal{C}_+) \text{FPdim}(\mathcal{C}_-)}{\text{FPdim}(s\text{Vect})} = \frac{\text{FPdim}(\mathcal{C} \otimes \mathcal{C}^{rev})}{\text{FPdim}(\text{rep}(\mathbb{Z}/2\mathbb{Z}))}.
\]

By [22, Theorem 3.4] it follows that the sequence (15) is exact. \( \square \)

For our purposes, one can replace (iii) with the equally useful statement that \( F \) induces an equivalence \( \mathcal{C} \otimes s\text{Vect} \mathcal{C}^{rev} \cong Z(\mathcal{C}, s\text{Vect}) \) [13]. Here we define the product over \( s\text{Vect} \) as the de-equivariantization of \( \mathcal{C} \otimes \mathcal{C} \) by the Tannakian subcategory \( \text{rep}(\mathbb{Z}/2\mathbb{Z}) \).

**Corollary 11.9.** If \( \mathcal{C} \) is of finite type, and the M"uger center of \( \mathcal{C} \) is equivalent to \( s\text{Vect} \), then \( Z(\mathcal{C}, s\text{Vect}) \) is also of finite type and \( \text{Kdim} Z(\mathcal{C}, s\text{Vect}) = 2 \text{Kdim} \mathcal{C} \).

**Proof.** The surjective tensor functor \( F : \mathcal{C} \otimes \mathcal{C}^{rev} \to Z(\mathcal{C}, s\text{Vect}) \) implies that \( Z(\mathcal{C}, s\text{Vect}) \) is of finite type, by Theorem 4.9. Also, the spectral sequence of Proposition 9.12 gives

\[
H^*(\mathcal{C} \otimes \mathcal{C}^{rev}, 1) = H^*(Z(\mathcal{C}, s\text{Vect}), 1)^{\mathbb{Z}/2\mathbb{Z}},
\]

from which we deduce the Krull dimensions. \( \square \)

**Lemma 11.10.** Suppose \( \mathcal{C} \) is a braided tensor category of finite type with M"uger center equivalent to \( s\text{Vect} \). Then the Drinfeld center \( Z(\mathcal{C}) \) is of finite type and \( \text{Kdim} Z(\mathcal{C}) = 2 \text{Kdim} \mathcal{C} \).

**Proof.** By Proposition 11.8 we have that \( Z(\mathcal{C}, s\text{Vect}) \) has M"uger center \( s\text{Vect} \). As we saw in the proof of Proposition 11.8, we also have

\[
\text{FPdim}(Z(\mathcal{C}, s\text{Vect})) = \frac{\text{FPdim}(\mathcal{C})^2}{2}.
\]

Thus the inclusion \( Z(\mathcal{C}, s\text{Vect}) \to Z(\mathcal{C}) \) is a minimal non-degenerate extension and, by Lemma 11.7, \( Z(\mathcal{C}) \) is a \( \mathbb{Z}/2\mathbb{Z} \)-extension of \( Z(\mathcal{C}, s\text{Vect}) \). By Lemma 11.2 and Corollary 11.9 it follows that \( Z(\mathcal{C}) \) is of finite type and of the proposed Krull dimension. \( \square \)
11.4. **Proof of Theorem 11.1.**

*Proof of Theorem 11.1.* By Corollary 5.1, it suffices to show that the Drinfeld center $Z(\mathcal{C})$ is of finite type over $\mathcal{C}$ and that $K\dim Z(\mathcal{C}) = 2 K\dim \mathcal{C}$. By Theorem 10.1, we may de-equivariantizing by the maximal Tannakian part of the Müger center to assume $Z_2(\mathcal{C}) \subseteq sVect$. The result now follows by Proposition 6.4 (non-degenerate case) and Lemma 11.10 (slightly degenerate case).

11.5. **Semisimplicity of the Müger center.** In general, detecting semisimplicity of the Müger center of a given category is a non-trivial problem. Specifically, it can be difficult to tell if the Müger center is semisimple but not Tannakian (cf. Corollary 10.8). We provide a non-exhaustive discussion of the topic here. To begin with, there is a coarse obstruction to non-semisimplicity of the Müger center via the Frobenius-Perron dimension.

**Proposition 11.11.** If $\mathcal{C}$ is a weakly integral braided tensor category and $4 \nmid \text{FPdim}(\mathcal{C})$, then the Müger center of $\mathcal{C}$ is semisimple.

*Proof.* Let $\mathcal{C}$ be a braided tensor category with Müger center $\mathcal{E}$ and suppose that $\mathcal{E}$ is not semisimple. Then $\mathcal{E}$ is the representation category of Hopf algebra of the form $\text{Wedge}(V) \rtimes G$, where $G$ is a finite group with a specified order 2 element $z$ and $V$ nonvanishing, by Deligne [15, 14] (see also [1]). Hence $4 \mid \dim(\text{Wedge}(V) \rtimes G) = \text{FPdim}(\mathcal{E})$. If $\mathcal{C}$ is weakly integral this implies $4 \mid \text{FPdim}(\mathcal{C})$.

Of course, if we demand that $\mathcal{C}$ has certain additional properties, then one can give more refined obstructions to the non-semisimplicity of $Z_2(\mathcal{C})$. We present one such instance here.

Let $A$ be a pointed Hopf algebra and write $\text{Prim}^g(A)$ for the space of $(g, 1)$-skew primitives in $A$. We define the spectrum $\sigma(A)$ of $A$ as the union of the spectra of the operators $\text{Ad}_g$ on the quotient $\text{Prim}^g(A)/k(1-g)$,

$$\sigma(A) = \bigcup_{g \in G(A)} \sigma\left(\text{Ad}_g|_{\text{Prim}^g(A)/k(1-g)}\right).$$

We prove the following result in Appendix B.

**Proposition 11.12.** Suppose that $A$ is pointed with abelian group of grouplikes and that $-1 \notin \sigma(A)$. Then the Müger center of $\text{rep}(A)$ is semisimple for any choice of a quasitriangular structure on $A$. Furthermore, if $|G(A)|$ is odd then the Müger center is Tannakian.

The algebras we have in mind here are the generalized small quantum groups $u(D)$ appearing in the work of Andruskiewitsch and Schneider [2]. These algebras are defined by data $D$ consisting of a collection of Dynkin diagrams and a “linking data” between these diagrams. The spectra for such algebras $\sigma(u(D))$ always avoid $-1$, and it was shown in work of Mastnak, Pevtsova, Schauenburg, and Witherspoon that all $u(D)$ have finitely generated cohomology [43].

**Remark 11.13.** To our knowledge, it is not known what combinatorics on the linking data $D$ ensures that the algebra $u(D)$ admits an $R$-matrix. In the case of “unlinked” data $N$, where $u(N)$ is coradically graded, there are in fact no $R$-matrices for $u(N)$ [8]. (This is a consequence of the fact that $\text{rep}(u(N))$ is pointed and the spectrum for $u(N)$ is assumed to avoid $-1$.) Thus, in order for $u(D)$ to
admit an \( R \)-matrix one requires non-trivial linking between the Dynkin diagrams appearing in \( \mathcal{D} \). Also, the fact that \( u_q(\mathfrak{sl}_2) \) at an even root of unity admits no \( R \)-matrix suggests that there may be meaningful restrictions on the grouplikes as well [37] (cf. [32, 12]).

**Problem 11.14.** Give a combinatorial classification of linking data \( \mathcal{D} \) for which \( u(\mathcal{D}) \) admits a quasitriangular structure.

**Appendix A. Dynamical twists and module categories**

We discuss how one can understand dynamical twists, and dynamical cocycle twists, via module categories. This material will certainly be unsurprising to experts. One can see a light version of the below discussion in [52, Section 4.4], for example. The material of this appendix is used in Section 7.

We first recall, briefly, relations between Hopf algebroids, weak Hopf algebras, and (finite) tensor categories.

**A.1. Hopf algebroids, weak Hopf algebras, and tensor categories.** We only give here a reminder of Hopf algebroids and weak Hopf algebras. We refer the reader to [60, 61, 23] for precise definitions.

A (left) bialgebroid over a base algebra \( R \) is an algebra \( A \) equipped with a structure map \( R \otimes R^{\text{op}} \to A \), coassociative comultiplication \( \Delta : A \to A \otimes_R A \), and (left) counit \( \epsilon : A \to R \). The comultiplication is required to be an algebra map, although one needs to place some restrictions on the image of \( \Delta \) in \( A \otimes_R A \) in order for this to make sense. The structure \( (\Delta, \epsilon) \) on \( R \otimes R^{\text{op}} \to A \) is equivalent to a choice of monoidal structure on \( \text{rep}(A) \) so that the forgetful functor \( \text{rep}(A) \to \text{bimod}(R) \) is strict monoidal.

A Hopf algebroid \( A \) over \( R \) is a bialgebroid for which \( \text{rep}(A) \) is rigid. Since right and left duals are unique, and preserved under monoidal functors, one can conclude that for any Hopf algebroid \( A \) each object in \( \text{rep}(A) \) is projective over \( R \) [28, Exercise 2.10.16]. Of course, the tensor product \( A \otimes A' \) of Hopf algebroids over \( R \) and \( R' \) respectively is a Hopf algebroid over \( R \otimes R' \).

We have the following basic fact, which is apparent from the work of Szlachanyi [60].

**Lemma A.1.** Any tensor category \( \mathcal{C} \) admits a tensor equivalence \( \mathcal{C} \sim \to \text{rep}(A) \) for a Hopf algebroid \( A \).

*Sketch proof.* One considers a finite dimensional algebra \( R \) with an exact \( \mathcal{C} \)-module structure on \( M = \text{rep}(R) \) then uses the corresponding representation \( \rho : \mathcal{C} \to \text{End}(M) = \text{bimod}(R) \) to construct the desired algebroid \( A \) over \( R \), as in [60, Theorem 1.8]. For example, one can take \( R \) such that \( M = \mathcal{C} \cong \text{rep}(R) \) and consider the regular representation for \( \mathcal{C} \). \( \square \)

When \( A \) is a Hopf algebroid over a separable base \( R \), one can also use a bimodule splitting \( R \to R \otimes R \) of the multiplication map to lift the comultiplication for \( A \) to a map \( \hat{\Delta} : A \to A \otimes A \). When the unit in \( \text{rep}(A) \) is simple, with \( \text{End}_A(1) = k \), we furthermore get a canonical map \( \hat{\epsilon} : A \to k \). The resulting structure \( (A, \hat{\Delta}, \hat{\epsilon}) \) is a weak Hopf algebra (see [52, Section 4]).
A.2. Dynamical twists and module categories. We refer the reader to [44, 23] for basic information regarding dynamical twists. We present some relationships between dynamical twists, module categories, and dynamical quantum group constructions from [61, 23]. We consider a finite abelian group \( \Lambda \) and suppose \( \text{char}(k) \nmid |\Lambda| \).

Suppose \( A \) is a Hopf algebra and \( \Lambda \) is an abelian subgroup in the group of grouplikes \( G(A) \). Let \( J : \Lambda^\vee \to A \otimes A \) be a dynamical twist. In particular, \( J \) is a map into the \( \Lambda \)-invariants \((A \otimes A)^\Lambda \) under the (diagonal) adjoint action which solves a parameter dependent dual cocycle condition.

The forgetful functor \( \text{rep}(A) \to \text{rep}(\Lambda) \) induces an exact module category structure on \( \text{rep}(\Lambda) \) and we use \( J \) to perturb the associativity and hence produce a new module category structure \( M(J) \) on \( \text{rep}(\Lambda) \). Directly, the associativity is given by

\[
\text{assoc}_{M(J)} : X \otimes (Y \otimes V) \to (X \otimes Y) \otimes V, \quad x \otimes y \otimes v \mapsto \sum_{\chi \in \Lambda^\vee} (J(\chi) \otimes P_\chi)(x \otimes y \otimes v),
\]

where \( P_\chi \) is the usual idempotent

\[
|\Lambda|^{-1} \sum_{\lambda \in \Lambda} \chi(\lambda) \lambda^{-1}.
\]

From the action of \( \text{rep}(A) \) on \( M(J) \) we get a faithful monoidal embedding

\[
F(J) : \text{rep}(A) \to \text{bimod}(k[\Lambda]).
\]

We then follow Szlachanyi [60, Theorem 1.8] to produce a Hopf algebroid \( A^J \) over \( k[\Lambda] \) equipped with an equivalence to \( \text{rep}(A) \) over \( \text{bimod}(k[\Lambda]) \),

\[
\text{rep}(A) \xrightarrow{\sim} \text{bimod}(k[\Lambda])
\]

As an algebra \( A^J = A \otimes \text{End}_k(k[\Lambda]) \).

More directly, \( A^J \) is given in [60] as the endomorphism algebra \( \text{End}_A(A \otimes O(\Lambda))^{\text{op}} \) which is canonically identified with \( A \otimes \text{End}_k(k[\Lambda]) \) via the isomorphism

\[
A \otimes \text{End}_k(k[\Lambda]) \to \text{End}_A(A \otimes O(\Lambda))^{\text{op}}, \quad a \otimes f \mapsto (- \cdot a) \otimes f^\ast.
\]

Since the base \( k[\Lambda] \) is separable, \( A^J \) is identified with a weak Hopf algebra by splitting the multiplication for \( k[\Lambda] \), as described in [52, Proof of Theorem 4.1].

**Lemma A.2.** For a dynamical twist \( J : \Lambda^\vee \to A \otimes A \), the weak Hopf algebra \( A^J \) constructed from the corresponding exact module category \( \mathcal{M}(J) \) is equal to the Xu style twisted weak Hopf algebra, as constructed in [23, Proposition 4.2.4].

**Sketch proof.** The constructions of [60] and [23] are both explicit, and both of the proposed twisted weak Hopf algebras are equal to \( A \otimes \text{End}_k(k[\Lambda]) \) as algebras. One simply writes down the coproduct for \( A^J \) and the Xu style twisted algebra and observed directly that they are equal. Specifically, both comultiplications are given by the formula

\[
\Delta^J(a \otimes E_{\mu,\nu}) = \sum_{\tau,\sigma} (1 \otimes P_\sigma) J^{-1}(\mu) \Delta(a) J(\nu) (E_{\sigma,\nu} \otimes P_\tau E_{\mu,\nu}),
\]

where \( E_{\mu,\nu} \in \text{End}_k(k[\Lambda]) \) is the elementary matrix which maps idempotents as \( E_{\mu,\nu}(P_\tau) = \delta_{\nu,\tau} P_\mu \), and the sum is over all \( \tau,\sigma \in \Lambda^\vee \). □
Recall that the vector space dual of a weak Hopf algebra is another weak Hopf algebra. For the twisted weak Hopf algebra $A^J$ we write $A^J_*$ for the vector space dual. We view $A^J_*$ as a “dynamical cocycle twist” of $A^*$. The dual $A^J_*$ is opposite to the Etingof-Varchenko style dynamical twisted algebra for the pair $(A, J)$ [23, Theorem 4.3.1] (cf. [27]).

**Lemma A.3.** For $J$ and $A$ as in Lemma A.2, there is an equivalence of tensor categories $Sz^*(J) : \text{rep}(A^J_*)^{\text{op}} \xrightarrow{\sim} \text{rep}(A)^*_{\text{R}(J)}$.

**Proof.** This is an immediate consequence of the diagram (16) and [52, Theorem 4.2].

**A.3. Constant dynamical twists.** In the case of a constant twist $J : \Lambda^V \rightarrow A$, with constant value $J^c$, the module category $\mathcal{M}(J)$ is $\text{rep}(\Lambda)$ with the constant associator given by multiplying by $J^c$

$$\text{assoc}(J) : X \otimes (Y \otimes V) \rightarrow (X \otimes Y) \otimes V, \quad x \otimes y \otimes v \mapsto J^c_{12}(x \otimes y \otimes v).$$

We note that for such a constant twist the value $J^c \in A \otimes A$ lies in the $\Lambda$-invariants $(A \otimes A)^{\Lambda}$, under the adjoint action.

We have the left and right translation actions of $\Lambda$ on the dual $A^*$. Specifically, for $\lambda \in \Lambda$ we act by the algebra automorphisms $\lambda \cdot f = \lambda^{-1}(f_1)f_2$ and $f \cdot \lambda = f_1 \lambda^{-1}(f_2)$. Restricting along these automorphisms gives an action of $\Lambda^e = \Lambda \times \Lambda^{\text{op}}$ on $\text{rep}(A^*)$, and we may take the (k-linear) equivariantization $\text{rep}(A^*)^{\Lambda^e}$. Similarly, for the $\Lambda$-invariant twist $J^c$ we still have $k[\Lambda] \subset A^{J^c}$ and may take the equivariantization $\text{rep}(A^*_{J^c})^{\Lambda^e}$. Algebraically, the equivariantization is the category of representations over the smash product $A^*_{J^c} \rtimes (\Lambda^e)$.

We note that $\text{rep}(A^*_{J^c})^{\Lambda \times \Lambda^{\text{op}}}$ is not a tensor category under the usual product $\otimes_k$, as the translation actions of $\Lambda$ on $A^*_J$, are not actions by Hopf automorphisms. Via the algebra projection $A^*_J \rightarrow \mathcal{O}(\Lambda)$, and regular left action of $\mathcal{O}(\Lambda)$ on $k[\Lambda]$, we see that $k[\Lambda]$ is an object in $\text{rep}(A^*_{J^c})$. We have the canonical equivariant structure on $k[\Lambda]$ given by the regular left and right actions of $\Lambda$.

**Proposition A.4.** Suppose that $J : \Lambda^V \rightarrow A$ is a constant dynamical twist for a Hopf algebra $A$, with constant value $J^c \in (A \otimes A)^{\Lambda}$. Then there is a k-linear equivalence

$$F : \text{rep}(A)^*_{\mathcal{M}(J)} \xrightarrow{\sim} \text{rep}(A^*_{J^c})^{\Lambda \times \Lambda^{\text{op}}}$$

under which the unit is sent to the $A^*_J$-representation $k[\Lambda]$, with the above equivariant structure.

One can deduce from the proof that there is a canonical Hopf algebroid structure on the smash product $A^*_{J^c} \rtimes (\Lambda^e)$ so that the above equivalence is an equivalence of tensor categories.

**Sketch proof.** Take $R = k[\Lambda]$. One sees, by restricting the module structure on $\mathcal{M}(J)$ along the equivalence $\text{rep}(A^{J^c}) \xrightarrow{\sim} \text{rep}(A)$, that it suffices to consider the case $J = 1 \otimes 1$. Also, via the coalgebra isomorphism $A^* \rtimes (\Lambda^e) \cong (A^*)^{\text{op}} \rtimes (\Lambda^e)$ given by the antipode and inversion on the $\Lambda$ factors we may prove the result with $A^*$ replaced by $(A^*)^{\text{op}}$.

For $J = 1$, we want to show that $\text{rep}(A)^*_{\text{rep}(\Lambda)}$ is equivalent to the category of $(A^*)^{\text{op}}$-representations equipped with a compatible $\Lambda$-bimodule structure. Such a
representation is exactly a left $A$-comodule $M$ with left and right $\Lambda$-actions satisfying
\[(\lambda m)_{-1} \otimes (\lambda m)_0 = (\lambda m_{-1}) \otimes (\lambda m_0)\] (17)
and
\[(m\lambda)_{-1} \otimes (m\lambda)_0 = (m_{-1}\lambda) \otimes (m_0\lambda),\] (18)
where $\rho(m) = m_{-1} \otimes m_0$ denotes the coaction, $m \in M$ and $\lambda \in \Lambda$.

For any such $M$ we have family of isomorphisms
\[\gamma^V_{M,X} : M \otimes X \otimes V \to X \otimes M \otimes V, \quad m \otimes x \otimes v \mapsto m_{-1} x \otimes m_0 \otimes v,\]
for $X$ in $\text{rep}(A)$ and $V$ in $\text{rep}(\Lambda)$. After composing with the projection to $X \otimes M \otimes_R V$, one see from the compatibility (18) that $\gamma^V_{M,X}$ induces a map
\[\gamma^V_{M,X} : M \otimes_R (X \otimes V) \to X \otimes (M \otimes_R V).\]

The compatibility (17) implies that $\gamma^V_{M,X}$ is a map of $\Lambda$-representations. One checks that $\gamma^V_{M,X}$ is an isomorphism by constructing the inverse (directly via the comultiplication on $M$ and the antipode). These isomorphisms $\gamma^V_{M,X}$ are natural in $X$ and $V$, and hence give the endofunctor $M \otimes_R - : \text{rep}(\Lambda) \to \text{rep}(\Lambda)$ a $\text{rep}(A)$-linear structure $(M \otimes_R - , \gamma^V_{M,X})$. Whence we have a functor
\[\text{rep}(A^*)^{\Lambda \times \Lambda^{op}} \to \text{rep}(A)^*_{\text{rep}(\Lambda)}, \quad M \mapsto (M, \gamma_{M,-}).\]

For the inverse map, consider any $\text{rep}(A)$-linear endofunctor $F$ of $\text{rep}(\Lambda)$. We may write $F$ as $M_F \otimes_R -$, for some $\Lambda$-bimodule $M_F$, and the $\text{rep}(\Lambda)$-linearity provides an isomorphism
\[\gamma^R_{F,A} : M_F \otimes_R (A \otimes R) \to A \otimes (M_F \otimes_R R) \cong A \otimes M_F\]
which is natural in $A$ and $R$. Such an isomorphism is determined by its value on $M_F$, which we embed in $M_F \otimes_R (A \otimes R)$ as $m \mapsto m \otimes 1 \otimes 1$. We thus get a left $A$-coaction $\rho_F$ on $M_F$ by taking $\rho_F := \gamma^R_{F,A}|_{M_F}$. The fact that $\gamma^R_{F,A}$ is a map of $\Lambda$-representations implies the relation (18), and the fact that $\gamma^R_{F,A}$ is well-defined over $\otimes_R$ implies (17). Whence we see that $(M_F, \rho_F)$ is an object in the equivariantization $\text{rep}(A^*)^{\Lambda \times \Lambda^{op}}$. This gives the desired inverse
\[\text{rep}(A)^*_{\text{rep}(\Lambda)} \to \text{rep}(A^*)^{\Lambda \times \Lambda^{op}}, \quad F \mapsto (M_F, \rho_F).\]

One calculates directly that the identity maps to $k[\Lambda]$ with the proposed equivariant structure under this equivalence. \hfill \Box

**Appendix B. Simisimplicity of the Müger Center and Pointed Hopf Algebras**

We prove Proposition 11.12. We suppose $k$ is algebraically closed and of characteristic 0. For a pointed Hopf algebra $A$ we let $\text{Prim}^g(A)$ denote the space of $(g,1)$-primitives in $A$, and $\text{Prim}^g(A)'$ denote the sum of the eigenspaces for the adjoint action of $\text{Ad}_g$ with eigenvalues $\neq 1$. We take
\[\text{Prim}(A)' = \sum_{g \in G(A)} \text{Prim}^g(A)'.\]

Consider the Hopf subalgebra $T(g,v)$ generated by $g$ and $v$ for any $(g,1)$-skew primitive $v$ which is an eigenvector for the adjoint action of $g$. If $v$ has eigenvalue 1 then $v \in k(1-g)$. Otherwise $v$, would provide a primitive in the (finite dimensional)
Hopf algebra $T(g,v)/(g-1)$. So if $v$ is not in the grouplikes then $v$ has eigenvalue $q \in k^* - \{1\}$. Now, from the Taft-Wilson theorem, the first portion of the coradical filtration appears as
\[ F_1 A = k[G] \oplus (k[G] \cdot \text{Prim}(A')). \]
We note that multiplication provides an isomorphism $k[G] \otimes \text{Prim}(A') \cong k[G] \cdot \text{Prim}(A')$, as each $h \cdot \text{Prim}(A')$ is the space of $(hg,h)$-skew primitives.

B.1. **The spectrum: duality and twists.** Recall that the spectrum $\sigma(A)$ of a pointed Hopf algebra $A$ is the union of the spectra of the adjoint operators $\text{Ad}_g$ on the respective spaces of $(g,1)$-skew primitives $\text{Prim}^g(A)/k(1-g)$.

**Lemma B.1.** Let $A$ be a pointed with abelian group of grouplikes. Then
\begin{enumerate}[(i)]  
  \item $\sigma(A) = \sigma(\text{gr}_FA)$, where $\text{gr}_FA$ is the associated graded Hopf algebra with respect to the coradical filtration.  
  \item If additionally $A$ has the Chevalley property, then $\sigma(A) = \sigma(\text{gr}_J A)$, where $\text{gr}_J A$ is the associated graded Hopf algebra with respect to the Jac(A)-adic filtration.  
\end{enumerate}

**Proof.** Claim (i) is clear, since the $(g,1)$-skew primitives of degree 1 in $\text{gr}_FA$ are exactly the space $\text{Prim}^g(A)/k(1-g)$, and the degree 0 skew primitives are just $k(1-g)$. For (ii), suppose that $A$ has the Chevalley property. Take $G = G(A)$, $K = A/\text{Jac}(A)$, and $\pi : A \to K$ the reduction modulo the Jacobson radical. Since $k[G]$ is semisimple, and thus contains no nilpotent ideals, we see that the composite $k[G] \to K$ is injective. Dually, we find that $K^* \to k[G]^*$ is injective. So the $k[G] \to K$ is an isomorphism, and we may identify $K$ with $k[G]$.

Since $A$ is pointed with abelian grouplikes, we know that $A$ is generated by its grouplikes and skew primitives [3, Theorem 4.15]. Furthermore, since the adjoint action of any $g \in G$ on $k[G]$ is trivial we must have $\text{Prim}(A') \subset \text{Jac}(A)$. So we conclude that $\text{Jac}(A)$ is generated by the space $\text{Prim}(A')$, and we have a surjective map $k[G] \cdot \text{Prim}(A') \to \text{Jac}(A)/\text{Jac}(A)^2$.

Recall that $(\text{gr}_J A)^* = \text{gr}_FA^*$. Since $\text{gr}_FA^*$ is pointed with abelian grouplikes, and hence generated by grouplikes and skew primitives, the ideal of positive elements is generated in degree 1. Rather, multiplication restricts of surjections $\text{gr}_FA^*(2^n) \to \text{gr}_FA^*(n)$ for each positive $n$. Dually, we find that the filtration on $\text{gr}_J A$ given by the grading agrees with the coradical filtration, and hence that all skew primitives lay in degree 1. It follows that the map $\text{Prim}(A') \to \text{Jac}(A)/\text{Jac}(A)^2$ is injective, and subsequently that the surjection from $k[G] \cdot \text{Prim}(A')$ is an isomorphism. Whence the spectra for $A$ and $\text{gr}_J A$ agree. \hfill \Box

**Lemma B.2.** Suppose $A$ is a pointed with abelian group of grouplikes, and also that $A$ has the Chevalley property. Then $A^*$ is also pointed and the spectra of $A$ and $A^*$ are conjugate, $\sigma(A^*) = \sigma(A)$.

**Proof.** Take $G = G(A)$. By considering the sequence $k[G] \to A \to A/\text{Jac}(A)$ and its dual $F_0 A^* \to A^* \to k[G]^*$ we see that the map $k[G] \to A/\text{Jac}(A)$ is a Hopf isomorphism. Whence we identify $A/\text{Jac}(A)$ with $k[G]$. (One argues precisely as in the proof of Lemma B.1 (ii).)

By Lemma B.1 we may assume $A$ is coradically graded. Since $G$ is abelian, the entire group acts on each $(g,1)$-skew primitive space $\text{Prim}^g(A')$, and hence on all of $\text{Prim}(A')$. Take $V = \text{Prim}(A')$ and $V^g = \text{Prim}^g(A')$. We fix a non-degenerate
symmetric form \( b \) on the group \( G = G(A) \) to identify \( G \) with its character group \( G^c = G(A^*) \).

Decompose each primitive space into character spaces \( V^g = \oplus_{h \in G} V^g_h \), where for \( v \in V^g_h \) we have \( \lambda \nu \lambda^{-1} = b(h, \lambda) v \) for each \( \lambda \in G \). So
\[
\sigma(A) = \{ b(g, h) : g, h \in G, \ V^g_h \neq 0 \}.
\]

We embed \( (V^g_h)^* \) in \( A^* \) as the space of functions which vanish on all degrees other that 1, and on \( A_1 \) are defined by \( f(gv) = f(v) \), for \( v \in V \) and \( g \in G \). Since multiplication gives an isomorphism \( k[G] \otimes V \cong A_1 \) we can in fact make such a definition.

A direct calculation show that \( (\lambda f \lambda^{-1})(hv) = b(g, \lambda)f(v) \) for \( v \in V^g_h \), and \( \lambda f \lambda^{-1} \) vanishes on all other summands \( k[G]V^g_h \). Also
\[
\Delta(f)(\lambda \otimes v) = f(v), \quad \Delta(f)(v \otimes \lambda) = b(h, \lambda^{-1})f(v),
\]
so that \( \Delta(f) = 1 \otimes f + h^{-1} \otimes f \). Whence \( (V^g_h)^* \subset \text{Prim}^{h^{-1}}(A^*)_g \). By ranging over all \( g \) and \( h \), and counting dimensions, we see that \( (V^g_h)^* \subset \text{Prim}^{h^{-1}}(A^*)_g \) for each \( g, h \in G \). Whence we have
\[
\sigma(A^*) = \{ b(g, h^{-1}) : V^g_h \neq 0 \} = \{ b(g, h)^{-1} : V_h^g \neq 0 \} = \sigma(A).
\]

\[\square\]

**Lemma B.3.** Suppose that \( A \) is pointed with abelian group of grouplikes. Then for any cocycle twist \( \tau : A \otimes A \to k \) we have \( \sigma(A) = \sigma(A_\tau) \).

**Proof.** Since \( \sigma(A) = \sigma(\text{gr}_F A) \) we may assume that \( A \) is coradically graded and that \( \tau \) is restricted from a 2-cocycle for the grouplikes. We have \( G(A) = G(A_\tau) \) and \( \text{Prim}^g(A) = \text{Prim}^g(A_\tau) \). Take any \( (g, 1) \)-skew primitive \( v \) in \( A \) which is an eigenvector for the adjoint action of \( g \), with associated eigenvalue \( q \). Then one simply applies the cocycle condition to find
\[
(g \cdot v) \cdot g^{-1} = \tau(g, g) \tau^{-1}(g, 1) \tau(g^2, g^{-1}) \tau^{-1}(g, g^{-1}) gvg^{-1} = q \tau(g, g) \tau^{-1}(g^2, g) \tau(g, g) v = q v.
\]
So we see that the \( \text{Prim}^g(A) = \text{Prim}^g(A_\tau) \) as a \( (g) \)-representation, for each \( g \in G(A) \). Hence the spectra agree. \[\square\]

**B.2. Proof of Proposition 11.12.**

**Proof of Proposition 11.12.** Any symmetric category \( E \) is equivalent to the representation category of a quasitriangular Hopf algebra of the form \( \text{Wedge}(V) \rtimes \Pi \), where \( \Pi \) is a finite group equipped with an order 2 element \( z \in \Pi \) such that
\[
\Delta(v) = v \otimes 1 + z \otimes v \quad \text{and} \quad \text{Ad}_z(v) = -v \quad \text{for each} \quad v \in V.
\]
This follows by Deligne's work [15, 14] (cf. [2]). Thus an embedding \( E \to \text{rep}(A) \) is given by restricting along a surjective Hopf map \( A \to (\text{Wedge}(V) \rtimes \Pi)^J \), where \( J \) is some Drinfeld twist. Suppose we have such an embedding, and corresponding Hopf projection. We claim that \( V \) must be 0.

Take \( B = (\text{Wedge}(V) \rtimes \Pi)^J \). As \( B \) is the surjective image of a pointed Hopf algebra, it is itself pointed with group of grouplikes given by the image of \( G = G(A) \). In particular, the group of grouplikes are abelian. Furthermore, since Drinfeld twists preserve the Chevalley property, \( B \) still has the Chevalley property. As we saw in the proof of Lemma B.1, we have an isomorphism \( k[G(B)] \cong B/\text{Jac}(B) = k[\Pi]^J \),
where $\bar{J}$ is the reduction of $J$ along the projection from $\text{Wedge}(V) \times \Pi$. In particular, $k[\Pi]^{\bar{J}}$ is commutative, and $\Pi$ is abelian.

Since $A$ is generated by grouplikes and skew primitives $\text{Prim}(A)'$, $B$ is generated by its grouplikes and the image of $\text{Prim}(A)'$. One argues as in the proof of Lemma B.1 (ii) to find that $\text{Jac}(B)/\text{Jac}(B)^2$ is generated by the image $\text{Prim}(A)'$, as a module over the grouplikes. Whence the image of $\text{Prim}(A)'$ provides all skew primitives in the associated graded algebra $\text{gr} \, \text{Jac} B$, and we have an inclusion $\sigma(\text{gr} \, \text{Jac} B) \subset \sigma(\text{A})$. By the equality $\sigma(\text{B}) = \sigma(\text{gr} \, \text{Jac} B)$ of Lemma B.1 we get $\sigma(\text{B}) \subset \sigma(\text{A})$.

Now, we take the dual to find
\[
\sigma(\text{B}^\ast) = \sigma((\text{Wedge}(V^\ast) \times \Pi^\ast),)
= \sigma(\text{Wedge}(V^\ast) \times \Pi^\ast)
= \sigma(\text{Wedge}(V) \times \Pi) = \begin{cases} 0 & \text{if } V = 0 \\ \{-1\} & \text{otherwise.} \end{cases}
\]

The inclusion $\sigma(\text{B}) \subset \sigma(\text{A})$, and the assumption $-1 \notin \sigma(\text{A})$, therefore implies $V = 0$. Thus $\text{Wedge}(V) \times \Pi = k[\Pi]$, and $\mathcal{E} \cong \text{rep}(\Pi)$. Hence $\mathcal{E}$ is semisimple.

Supposing additionally that $|G|$ is odd, the Hopf projection $k[G] \to k[\Pi]^{\bar{J}}$ implies that $|\Pi|$ is odd as well. Therefore $\Pi$ contains no degree 2 elements, the element $z$ introduced at the beginning of this proof is 1, and the braiding on $\mathcal{E} \cong \text{rep}(\Pi)$ is the trivial one. Thus $\mathcal{E}$ is Tannakian. □

**References**


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E-mail address: negronc@mit.edu

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MA USA

E-mail address: julia@math.tamu.edu

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX USA