Homework List (updated May 9, 2018)
Our homework basically follows Kac 2016 (not really anymore though)

Homework 12, due Fri May 11 @ 2PM
(1) Observe that \(\mathfrak{sl}_n(\mathbb{C})\) is the kernel of the trace operator \(\text{Tr} : \mathfrak{gl}_n(\mathbb{C}) \to \mathbb{C}\). Show that the trace is linear and surjective.
(2) Use (1) to calculate the dimension \(\dim(\mathfrak{sl}_n(\mathbb{C}))\).
(3) Let \(E_{ij} \in \mathfrak{gl}_n(\mathbb{C})\) be the standard elementary matrix, i.e. the matrix \(E_{ij} = [a_{kl}]\) with a single nonzero entry which is of value 1, \(a_{kl} = \delta_{ki}\delta_{lj}\). Define \(h_i = E_{ii} - E_{i+1,i+1} \in \mathfrak{sl}_n(\mathbb{C})\) and consider the set of elements \(B = \{h_1, \ldots, h_{n-1}\} \cup \{E_{ij} : i \neq j\}\). Show that \(B\) is a basis of \(\mathfrak{sl}_n(\mathbb{C})\).
(4) Give a formula for the brackets \([h_i, E_{kl}]\) and \([E_{ij}, E_{kl}]\) in \(\mathfrak{sl}_n(\mathbb{C})\) in terms of the basis \(B\) of (3). For \([h_i, E_{kl}]\), consider the cases \(k = l + 1, k + 1 = l\), and \(\pm(k - l) > 1\).
(5) Take \(e_i = E_{i,i+1}\) and \(f_i = E_{i+1,i}\). Recall from (4) the formulas for \([h_i, e_j]\), \([h_i, f_j]\). Show that \([e_i, f_i] = h_i\).
(Bonus) Show that \(\mathfrak{sl}_n(\mathbb{C})\) is generated by the set of elements Gens = \(\{e_i, f_i : 1 \leq i \leq n - 1\}\), meaning in this case that all the basis elements are obtained by taking successive brackets of elements in Gens.
(6) Show that the linear inclusions \(j_i : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{sl}_n(\mathbb{C})\), \(j_i(e) = e_i, j_i(f) = f_i, j_i(h) = h_i\), are Lie algebra maps for each \(i = 1, \ldots, n - 1\).
(7) Consider finite dimensional vector spaces \(V\) and \(W\) with bases \(\{v_i\}_{i=1}^n\) and \(\{w_j\}_{j=1}^m\) respectively, we define \(V \otimes W\) as the \(N \cdot M\)-dimensional vector space with basis consisting of formal products \(\{v_i \otimes w_j : 1 \leq i \leq n, 1 \leq j \leq m\}\). For arbitrary elements \(v = \sum_i c_i v_i\) and \(w = \sum_j c_j w_j\) we define \(v \otimes w = \sum_{i,j} c_i c_j v_i \otimes w_j\).
For a Lie algebra \(L\), and \(L\)-representations \(V\) and \(W\), show that \(V \otimes W\) is a \(L\)-representation under the action \(x \cdot (v \otimes w) = (x \cdot v) \otimes w + v \otimes (x \cdot w)\).
(8) [Note the change in notation] Consider the representation \(V(1) = \mathbb{C}\{v_1, v_{-1}\}\) over \(\mathfrak{sl}_2(\mathbb{C})\). Show that the linear inclusion \(i_0 : V(0) = \mathbb{C} \to V(1) \otimes V(1), i_0(c) = c(v_1 \otimes v_{-1} - v_{-1} \otimes v_1)\)
is a map of \(\mathfrak{sl}_2(\mathbb{C})\) representations.
(9) Let \(\{w_2, w_0, w_{-2}\}\) be the standard basis for the 3-dimensional \(\mathfrak{sl}_2\)-representation \(V(2)\). Show that the linear inclusion \(i_2 : V(2) \to V(1) \otimes V(1), i_2(w_2) = v_1 \otimes v_1, i_2(w_0) = v_1 \otimes v_{-1} + v_{-1} \otimes v_1, i_2(w_{-2}) = 2v_{-1} \otimes v_{-1}\)
is a maps of \(\mathfrak{sl}_2(\mathbb{C})\)-representations.
(10) Show, from the previous two problems, that the linear map \(I : V(0) \oplus V(2) \to V(1) \otimes V(1), I(v, w) = i_0(v) + i_2(w)\)
is an isomorphism of \(\mathfrak{sl}_2\)-representations.

Homework 11, due Fri May 4 @ 2PM
- Sect. 31: 2, 4, 11, 19, 23, 24, 31
- Sect. 33: 2, 6, 8, 10
- Sect. 36:
  - **Bonus**: Prove that the algebraic closure \(\bar{\mathbb{Q}}\) is not finite over \(\mathbb{Q}\). (Hint: There are irreducible polynomials over \(\mathbb{Q}\) of arbitrarily large degree...)

Homework 10, due Fri Apr 27 @ 2PM
- Sect. 45: 21, 25, 26, 31, 32
• Bonus: We encountered this ascending chain condition on ideals in class. A ring $R$ with this ascending chain condition on ideals is called Noetherian. To recall, a ring $R$ is called Noetherian if any (possibly infinite) sequence of ideals $I_0 \subset I_1 \subset I_2 \subset \ldots$ stabilizes, $\bigcup_{i=0}^{\infty} I_i = I_N$ for $N >> 0$. This condition is equivalent to the condition that any ideal in $R$ is finitely generated, and turns out to be quite important.

Hilbert’s basis theorem proposes the following: If $R$ is commutative Noetherian ring, then the polynomial ring $R[x]$ is also Noetherian. Find a proof of Hilbert basis theorem that you understand, and reproduce the proof. Include enough detail so that it is clear that you have a precise understanding of the proof.

(1) Use Hilbert’s basis theorem to prove that any finitely generated commutative algebra over a field is Noetherian.

Homework 9, due Fri Apr 20 @ 2PM!!:

• Sect. 23: 6, 7, 8, 14, 16, 17.

(1) Let $A$ be a commutative algebra over a field $\mathbb{F}$ which is an integral domain. Suppose additionally that $A$ is finite dimensional over $\mathbb{F}$. Prove that $A$ must be a field!! Here is an outline for your mathing pleasure:

(a) Consider for any nonzero element $a \in A$ the set map $\lambda_a : A \to A$, $\lambda_a(b) = ab$.
(b) Observe/show that $\lambda_a$ is a linear homomorphism of $A$ (i.e. $F$-vector space homomorphism).
(c) Decide that $\lambda_a$ is injective.
(d) Make a dimension argument to see that $\lambda_a$ is surjective as well.
(e) etc.

(2) Let $\mathbb{F}$ be a field. Recall that a polynomial $p(x)$ in $\mathbb{F}[x]$ is called irreducible if $p(x)$ is non-zero and any other polynomial $q(x)$ which divides $p(x)$ is such that $p(x) = cq(x)$ for a unit $c \in \mathbb{F}$. It is a fact which we will discuss later that a polynomial $p$ is irreducible if and only if $p$ is prime, meaning that if $pab$ for polynomials $a$ and $b$, then either $p|a$ or $p|b$. Prove the following:

(a) For any arbitrary non-zero polynomial $p$ the quotient $\mathbb{F}[x]/(p)$ is finite dimensional over $\mathbb{F}$.
(b) The quotient $\mathbb{F}[x]/(p)$ by non-zero $p$ is a domain if and only if $p$ is an irreducible polynomial.

(Hint: Use the fact here.)

(c) Conclude that $\mathbb{F}[x]/(p)$ is a field whenever $p$ is an irreducible polynomial.

(3) Prove that the evaluation map $f_i : \mathbb{R}[x] \to \mathbb{C}$, $f_i(p) = p(i)$, induces an isomorphism of rings $\mathbb{C} \cong \mathbb{R}[x]/(x^2 + 1)$.

(4) What is the kernel of the evaluation map $f_\pi : \mathbb{Q}[x] \to \mathbb{R}$, $f_\pi(p) = p(\pi)$, where $\pi$ is our transcendental friend $\pi = 3.14159...$?

(5) Let $P$ be a prime number (not polynomial). Let $F : \mathbb{Q}[x] \to \mathbb{C}$ be the evaluation map at $e^{2\pi i}/P$.

(Hint: Use the fact here.)

(6) Bonus: Let $\mathbb{F}$ be a field. Show that the ideal $(x, y) \subset \mathbb{F}[x, y]$ cannot be generated by a single polynomial.

(7) Bonus: Let $S$ be a subset in $\mathbb{C}^n$ specified by the vanishing of polynomials $f_1(X_1, \ldots, X_n) = 0, \ldots, f_i(X_1, \ldots, X_n) = 0$. Take

$$\mathcal{O}(S) := \mathbb{C}[X_1, \ldots, X_n]/(f_1, \ldots, f_i)$$

Show that there is a natural bijection of sets

$$S \leftrightarrow \{ \text{C-algebra maps } \phi : \mathcal{O}(S) \to \mathbb{C} \}.$$ 

(Hint: For a map $\phi : \mathcal{O}(S) \to \mathbb{C}$ think of the values of $z_i = \phi(X_i)$. These $z_i$ specify a corresponding map $\tilde{\phi} : \mathbb{C}[X_1, \ldots, X_n] \to \mathbb{C}$, $\tilde{\phi}(p) = p(z_1, \ldots, z_n)$, and the $z_i$ need to have some properties so that we get the factorization $\phi$ of $\tilde{\phi}$ through the quotient $\mathcal{O}(S)$.)

Homework 8, due Fri Apr 13 @ 2PM!!!:

• Sect. 26: 3, 10, 12–14, 17, 18, 20, 22, 28, 30, 32, 37
• Sect. 24: 4, 5, 6, 7, 9, 19
• Sect. 22: 1, 4, 8, 10, 14, 16, 24, 25
Bonus: Let $P$ be the curve of solutions to the equation $z_2 = z_1^2$ in 2-dim'l complex space $\mathbb{C}^2 = \{(z_1, z_2) : z_i \in \mathbb{C}\}$. Consider the ring homomorphism

$$\text{res} : \mathbb{C}[X, Y] \to \text{Fun}(P, \mathbb{C}),$$

where $\text{res}(p)$ is the function $\text{res}(p)(z_1, z_2) = p(z_1, z_2)$. Show that the image of $\mathbb{C}[X, Y]$ in $\text{Fun}(P, \mathbb{C})$ is isomorphic to $\mathbb{C}[X, Y]/(Y - X^2)$. (So, “the ring of polynomial functions on the complex parabola $P$” is the quotient $\mathbb{C}[X, Y]/(Y - X^2)$.)

Recall that $\text{Fun}(P, \mathbb{C})$ is the ring of set maps $P \to \mathbb{C}$ where we add and multiply pointwise.

Homework 7, due Fri Apr 6 @ 2PM!!!:
- Sect. 18: 1–6, 7, 8, 12, 18, 20, 22, 24, 33, 40 (only $\mathbb{R} \cong \mathbb{C}$), 41, 46, 55
- Sect. 19: 1, 3, 4, 6, 8, 10, 12, 17, 23, 25, 27

Homework 6, due Thur Mar 22:
- Write a proof for the Second Sylow Theorem
- Sect. 36: 10, 15, 17, 18, 19, 20
- Sect. 37: 4, 6, 7

Homework 5, due Wed Mar 14:
- Sect. 11: 10, 18, 20, 24, 36, 44, 50, 52
- Sect. 16: 1–3, 8, 9, 11, 13, 14, 15
- Sect. 17: 1–8

Note: The book uses the notation $S_X$ for the automorphism group $\text{Aut}_{\text{Set}}(X)$ of a set $X$. Homework 4, due

Wed Mar 7:
- Sect. 14 (pg 142): 3, 7, 16, 17–20, 21, 30, 34, 40, 26
- Sect. 15 (pg 151): 13, 34, 37, 38
- Sect. 11 (pg 110): 1, 8, 14, 15
- Find all normal subgroups of $D_4$.
- Provide a group isomorphism between $S^1$ and $S^1/\mu_N$ for arbitrary positive $N$. Here $S^1$ is the subgroup of modulus 1 elements in $\mathbb{C}^\times$, and $\mu_N$ is the normal subgroup of $N$-th roots of 1 in $S^1$ (or in $\mathbb{C}^\times$ if you like).

Homework 3, due Wed Feb 28:
- Sect. 10 (pg 101): 4, 16, 30–32, 39, 45, 46, 47
- Sect. 13 (pg 133): 1–10, 14, 22, 29, 47, 52
- Bonus: Let $Q_8$ be the quaternion group ($\text{wikiwand.com/en/Quaternion_group}$). This is an order 8 non-abelian group. Show that every subgroup in $Q_8$ is normal. Conclude that there exists no group isomorphism between $D_4$ and $Q_8$.
- Bonus: Show that $A_4$ contains no subgroup of order 6.

Homework 2, due Wed Feb 21:
- Sect. 7 (pg 72): 1, 3, 6
- Sect. 8 (pg 83): 2, 8, 10, 17, 35 only d g h, 44, 45, 47
- Sect. 9 (pg 94): 1, 7, 10, 11, 13 only a b c, 14, 15, 23, 27 only a c
- Bonus: Show that the elementary braids $s_1$ and $s_2$ in $B_3$ satisfy $s_1s_2s_1 = s_2s_1s_2$. Show that the analogous transpositions $\tau_1$ and $\tau_2$ in $S_3$ also satisfy $\tau_1\tau_2\tau_1 = \tau_2\tau_1\tau_2$.

Homework 1, due Wed Feb 14:
- Sect. 2 (pg 25): 1–4, 7–10
- Bonus: Given a set $S$, the group $(\text{Aut}(S), \circ)$ is abelian if and only if $|S| \leq 2$.
- Sect. 4 (pg 45): 2–8
• Sect. 5 (pg 55): 22, 24, 31, 33, 51
• Sect. 6 (pg 66): 4, 17, 19, 21, 28, 32–34
• Prove the following: Suppose $G$ is a cyclic group. Then either $|G| = \infty$ and $G \cong \mathbb{Z}$, or $|G| = n < \infty$ and $G \cong \mathbb{Z}/n\mathbb{Z}$.

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