Homework List (updated April 26, 2018)
Our homework basically follows \textit{Kac 2016} (not really anymore though)

Homework 11, due Fri May 4 @ 2PM

- Sect. 31: 2, 4, 11, 19, 23, 24, 31
- Sect. 33: 2, 6, 8, 10
- Sect. 48: 4, 15, 18, 29

Homework 10, due Fri Apr 27 @ 2PM

- Sect. 45: 21, 25, 26, 31, 32
- Bonus: We encountered this ascending chain condition on ideals in class. A ring $R$ with this ascending chain condition on ideals is called \textit{Noetherian}. To recall, a ring $R$ is called Noetherian if any (possibly infinite) sequence of ideals $I_0 \subset I_1 \subset I_2 \subset \ldots$ stabilizes, $\cup_{i=0}^{\infty} I_i = I_N$ for $N > 0$. This condition is equivalent to the condition that any ideal in $R$ is finitely generated, and turns out to be quite important.

Hilbert’s basis theorem proposes the following: If $R$ is commutative Noetherian ring, then the polynomial ring $R[x]$ is also Noetherian. Find a proof of Hilbert basis theorem that you understand, and reproduce the proof. Include enough detail so that it is clear that you have a precise understanding of the proof.

1) Use Hilbert’s basis theorem to prove that any finitely generated commutative algebra over a field is Noetherian.

Homework 9, due Fri Apr 20 @ 2PM!!!:

- Sect. 23: 6, 7, 8, 14, 16, 17.

1) Let $A$ be a commutative algebra over a field $F$ which is an integral domain. Suppose additionally that $A$ is finite dimensional over $F$. Prove that $A$ must be a field!!! Here is an outline for your mathtex pleasure:

(a) Consider for any nonzero element $a \in A$ the set map $\lambda_a : A \rightarrow A$, $\lambda_a(b) = ab$.
(b) Observe/show that $\lambda_a$ is a linear homomorphism of $A$ (i.e. $F$-vector space homomorphism).
(c) Decide that $\lambda_a$ is injective.
(d) Make a dimension argument to see that $\lambda_a$ is surjective as well.
(e) etc.

2) Let $F$ be a field. Recall that a polynomial $p(x)$ in $F[x]$ is called irreducible if $p(x)$ is non-zero and any other polynomial $q(x)$ which divides $p(x)$ is such that $p(x) = cq(x)$ for a unit $c \in F$. It is a fact which we will discuss later that a polynomial $p$ is irreducible if and only if $p$ is prime, meaning that if $p|ab$ for polynomials $a$ and $b$, then either $p|a$ or $p|b$. Prove the following:

(a) For any arbitrary non-zero polynomial $p$ the quotient $F[x]/(p)$ is finite dimensional over $F$.
(b) The quotient $F[x]/(p)$ by non-zero $p$ is a domain if and only if $p$ is an irreducible polynomial. (Hint: Use the fact here.)
(c) Conclude that $F[x]/(p)$ is a field whenever $p$ is an irreducible polynomial.

3) Prove that the evaluation map $f_i : \mathbb{R}[x] \rightarrow \mathbb{C}$, $f_i(p) = p(i)$, induces an isomorphism of rings $\mathbb{C} \cong \mathbb{R}[x]/(x^2 + 1)$.

4) What is the kernel of the evaluation map $f_\pi : \mathbb{Q}[x] \rightarrow \mathbb{R}$, $f_\pi(p) = p(\pi)$, where $\pi$ is our transcendental friend $\pi = 3.14159\ldots$?

5) Let $P$ be a prime number (not polynomial). Let $F : \mathbb{Q}[x] \rightarrow \mathbb{C}$ be the evaluation map at $e^{2\pi i/P}$.

- Bonus: Let $F$ be a field. Show that the ideal $(x,y) \subset F[x,y]$ cannot be generated by a single polynomial.

6) Bonus: Let $S$ be a subset in $\mathbb{C}^n$ specified by the vanishing of polynomials $f_1(X_1, \ldots, X_n) = 0, \ldots, f_l(X_1, \ldots, X_n) = 0$. Take

\[ \mathcal{O}(S) := \mathbb{C}[X_1, \ldots, X_n]/(f_1, \ldots, f_l) \]
Show that there is a natural bijection of sets
\[ S \leftrightarrow \{ \text{C-algebra maps } \phi : \mathcal{O}(S) \to \mathbb{C} \}. \]

[Hint: For a map \( \phi : \mathcal{O}(S) \to \mathbb{C} \), think of the values of \( z_i = \phi(X_i) \). These \( z_i \) specify a corresponding map \( \tilde{\phi} : \mathbb{C}[X_1, \ldots, X_n] \to \mathbb{C} \), \( \tilde{\phi}(p) = p(z_1, \ldots, z_n) \), and the \( z_i \) need to have some properties so that we get the factorization \( \phi \) of \( \tilde{\phi} \) through the quotient \( \mathcal{O}(S) \).]

Homework 8, due Fri Apr 13 @ 2PM!!:
- Sect. 26: 3, 10, 12–14, 17, 18, 20, 22, 28, 30, 32, 37
- Sect. 24: 4, 5, 6, 7, 9, 19
- Sect. 22: 1, 4, 8, 10, 14, 24, 25
- Bonus: Let \( P \) be the curve of solutions to the equation \( z_2 = z_1^2 \) in 2-dim’l complex space \( \mathbb{C}^2 = \{(z_1, z_2) : z_i \in \mathbb{C}\} \). Consider the ring homomorphism
  \[ \text{res} : \mathbb{C}[X, Y] \to \text{Fun}(P, \mathbb{C}), \]
  where \( \text{res}(p) \) is the function \( \text{res}(p)(z_1, z_2) = p(z_1, z_2) \). Show that the image of \( \mathbb{C}[X, Y] \) in \( \text{Fun}(P, \mathbb{C}) \) is isomorphic to \( \mathbb{C}[X, Y]/(Y - X^2) \). (So, “the ring of polynomial functions on the complex parabola \( P \)” is the quotient \( \mathbb{C}[X, Y]/(Y - X^2) \).)
  Recall that \( \text{Fun}(P, \mathbb{C}) \) is the ring of set maps \( P \to \mathbb{C} \) where we add and multiply pointwise.

Homework 7, due Fri Apr 6 @ 2PM!!:
- Sect. 18: 1–6, 7, 8, 12, 18, 20, 22, 24, 33, 40 (only \( \mathbb{R} \cong \mathbb{C} \)), 41, 46, 55
- Sect. 19: 1, 3, 4, 6, 8, 10, 12, 17, 23, 25, 27

Homework 6, due Thur Mar 22:
- Write a proof for the Second Sylow Theorem
- Sect. 36: 10, 15, 17, 18, 19, 20
- Sect. 37: 4, 6, 7

Homework 5, due Wed Mar 14:
- Sect. 11: 10, 18, 20, 24, 36, 44, 50, 52
- Sect. 16: 1–3, 8, 9, 11, 13, 14, 15
- Sect. 17: 1–8

Note: The book uses the notation \( S_X \) for the automorphism group \( \text{Aut}_{\text{Set}}(X) \) of a set \( X \). Homework 4, due

Wed Mar 7:
- Sect. 14 (pg 142): 3,7,16, 17–20, 21, 30, 34, 40, 26
- Sect. 15 (pg 151): 13, 34, 37, 38
- Sect. 11 (pg 110): 1, 8, 14, 15
- Find all normal subgroups of \( D_4 \).
- Provide a group isomorphism between \( S^1 \) and \( S^1/\mu_N \) for arbitrary positive \( N \). Here \( S^1 \) is the subgroup of modulus 1 elements in \( \mathbb{C}^\times \), and \( \mu_N \) is the normal subgroup of \( N \)-th roots of 1 in \( S^1 \) (or in \( \mathbb{C}^\times \) if you like).

Homework 3, due Wed Feb 28:
- Sect. 10 (pg 101): 4, 16, 30–32, 39, 45, 46, 47
- Sect. 13 (pg 133): 1–10, 14, 22, 29, 47, 52
- Bonus: Let \( Q_8 \) be the quaternion group (wikiwand.com/en/Quaternion_group). This is an order 8 non-abelian group. Show that every subgroup in \( Q_8 \) is normal. Conclude that there exists no group isomorphism between \( D_4 \) and \( Q_8 \).
- Bonus: Show that \( A_4 \) contains no subgroup of order 6.

Homework 2, due Wed Feb 21:
Homework 1, due Wed Feb 14:

- Sect. 2 (pg 25): 1–4, 7–10
- Bonus: Given a set $S$, the group $(\text{Aut}(S), \circ)$ is abelian if and only if $|S| \leq 2$.
- Sect. 4 (pg 45): 2–8
- Sect. 5 (pg 55): 22, 24, 31, 33, 51
- Sect. 6 (pg 66): 4, 17, 19, 21, 28, 32–34
- Prove the following: Suppose $G$ is a cyclic group. Then either $|G| = \infty$ and $G \cong \mathbb{Z}$, or $|G| = n < \infty$ and $G \cong \mathbb{Z}/n\mathbb{Z}$.

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