Homework List (updated April 13, 2018)
Our homework basically follows Kac 2016 (not really anymore though)

Homework 9, due Fri Apr 20 @ 2PM!!!:

- Sect. 23: 6, 7, 8, 14, 16, 17.
- Sect. 22: 1, 4, 8, 10, 14.
- Sect. 24: 4, 5, 6, 7, 9, 19
- Sect. 26: 3, 10, 12–14, 17, 18, 20, 22, 28, 30, 32, 37

1. Let $A$ be a commutative algebra over a field $F$ which is an integral domain. Suppose additionally that $A$ is finite dimensional over $F$. Prove that $A$ must be a field!!! Here is an outline for your mathing pleasure:
   - (a) Consider for any nonzero element $a \in A$ the set map $\lambda_a : A \to A$, $\lambda_a(b) = ab$.
   - (b) Observe/show that $\lambda_a$ is a linear homomorphism of $A$ (i.e. $F$-vector space homomorphism).
   - (c) Decide that $\lambda_a$ is injective.
   - (d) Make a dimension argument to see that $\lambda_a$ is surjective as well.
   - (e) etc.

2. Let $\mathbb{F}$ be a field. Recall that a polynomial $p(x)$ in $\mathbb{F}[x]$ is called irreducible if $p(x)$ is non-zero and any other polynomial $q(x)$ which divides $p(x)$ is such that $p(x) = cq(x)$ for a unit $c \in \mathbb{F}$. It is a fact which we will discuss later that a polynomial $p$ is irreducible if and only if $p$ is prime, meaning that if $p|ab$ for polynomials $a$ and $b$, then either $p|a$ or $p|b$. Prove the following:
   - (a) For any arbitrary non-zero polynomial $p$ the quotient $\mathbb{F}[x]/(p)$ is finite dimensional over $\mathbb{F}$.
   - (b) The quotient $\mathbb{F}[x]/(p)$ by non-zero $p$ is a domain if and only if $p$ is an irreducible polynomial.

(Hint: Use the fact here.)

3. Prove that the evaluation map $f_i : \mathbb{R}[x] \to \mathbb{C}$, $f_i(p) = p(i)$, induces an isomorphism of rings $\mathbb{C} \cong \mathbb{R}[x]/(x^2 + 1)$.

4. What is the kernel of the evaluation map $f_\pi : \mathbb{Q}[x] \to \mathbb{R}$, $f_\pi(p) = p(\pi)$, where $\pi$ is our transcendental friend $\pi = 3.14159...$?

5. Let $P$ be a prime number (not polynomial). Let $F : \mathbb{Q}[x] \to \mathbb{C}$ be the evaluation map at $e^{2\pi i}/P$.
   - Prove that $\ker(F) = (\Phi_P)$, where $\Phi_P$ is the $P$-th cyclotomic polynomial.
   - Bonus: Let $\mathbb{F}$ be a field. Show that the ideal $(x, y) \subset \mathbb{F}[x,y]$ cannot be generated by a single polynomial.

6. Bonus: Let $S$ be a subset in $\mathbb{C}^n$ specified by the vanishing of polynomials $f_1(X_1, \ldots, X_n) = 0, \ldots, f_l(X_1, \ldots, X_n) = 0$. Take

$$O(S) := \mathbb{C}[X_1, \ldots, X_n]/(f_1, \ldots, f_l)$$

Show that there is a natural bijection of sets

$$S \leftrightarrow \{\mathbb{C}\text{-algebra maps } \phi : O(S) \to \mathbb{C} \}.$$

(Hint: For a map $\phi : O(S) \to \mathbb{C}$ think of the values of $z_i = \phi(X_i)$. These $z_i$ specify a corresponding map $\bar{\phi} : \mathbb{C}[X_1, \ldots, X_n] \to \mathbb{C}$, $\bar{\phi}(p) = p(z_1, \ldots, z_n)$, and the $z_i$ need to have some properties so that we get the factorization $\phi$ of $\bar{\phi}$ through the quotient $O(S)$.)

Homework 8, due Fri Apr 13 @ 2PM!!!:

- Sect. 26: 3, 10, 12–14, 17, 18, 20, 22, 28, 30, 32, 37
- Sect. 24: 4, 5, 6, 7, 9, 19
- Sect. 22: 1, 4, 8, 10, 14, 24, 25
- Bonus: Let $P$ be the curve of solutions to the equation $z_2 = z_1^2$ in 2-dim’l complex space $\mathbb{C}^2 = \{(z_1, z_2) : z_i \in \mathbb{C} \}$. Consider the ring homomorphism

$$\text{res} : \mathbb{C}[X,Y] \to \text{Fun}(P, \mathbb{C}),$$

where $\text{res}(p)$ is the function $\text{res}(p)(z_1, z_2) = p(z_1, z_2)$. Show that the image of $\mathbb{C}[X,Y]$ in $\text{Fun}(P, \mathbb{C})$ is isomorphic to $\mathbb{C}[X,Y]/(Y - X^2)$. (So, “the ring of polynomial functions on the complex parabola $P^2$ is the quotient $\mathbb{C}[X,Y]/(Y - X^2)$.)

Recall that $\text{Fun}(P, \mathbb{C})$ is the ring of set maps $P \to \mathbb{C}$ where we add and multiply pointwise.
Homework 7, due Fri Apr 6 @ 2PM!!!:
• Sect. 18: 1–6, 7, 8, 12, 18, 22, 24, 33, 40 (only \( \mathbb{R} \cong \mathbb{C} \)), 41, 46, 55
• Sect. 19: 1, 3, 4, 6, 8, 10, 12, 17, 23, 25, 27

Homework 6, due Thur Mar 22:
• Write a proof for the Second Sylow Theorem
• Sect. 36: 10, 15, 17, 19, 20
• Sect. 37: 4, 6, 7

Homework 5, due Wed Mar 14:
• Sect. 11: 10, 18, 20, 24, 36, 44, 50, 52
• Sect. 16: 1–3, 8, 9, 11, 13, 14, 15
• Sect. 17: 1–8

Note: The book uses the notation \( S_X \) for the automorphism group \( \text{Aut}_{\text{Set}}(X) \). Homework 4, due

Wed Mar 7:
• Sect. 14 (pg 142): 3, 7, 16, 17–20, 21, 30, 34, 40, 26
• Sect. 15 (pg 151): 13, 34, 37, 38
• Sect. 11 (pg 110): 1, 8, 14, 15
• Find all normal subgroups of \( D_4 \).
• Provide a group isomorphism between \( S^1 \) and \( S^1/\mu_N \) for arbitrary positive \( N \). Here \( S^1 \) is the subgroup of modulus 1 elements in \( \mathbb{C}^\times \), and \( \mu_N \) is the normal subgroup of \( N \)-th roots of 1 in \( S^1 \) (or in \( \mathbb{C}^\times \) if you like).

Homework 3, due Wed Feb 28:
• Sect. 10 (pg 101): 4, 16, 30–32, 39, 45, 46, 47
• Sect. 13 (pg 133): 1–10, 14, 22, 29, 47, 52
• Bonus: Let \( Q_8 \) be the quaternion group (wikiwand.com/en/Quaternion_group). This is an order 8 non-abelian group. Show that every subgroup in \( Q_8 \) is normal. Conclude that there exists no group isomorphism between \( D_4 \) and \( Q_8 \).
• Bonus: Show that \( A_4 \) contains no subgroup of order 6.

Homework 2, due Wed Feb 21:
• Sect. 7 (pg 72): 1, 3, 6
• Sect. 8 (pg 83): 2, 8, 10, 17, 35 only d g h, 44, 45, 47
• Sect. 9 (pg 94): 1, 7, 10, 11, 13 only a b c, 14, 15, 23, 27 only a c
• Bonus: Show that the elementary braids \( s_1 \) and \( s_2 \) in \( B_3 \) satisfy \( s_1 s_2 s_1 = s_2 s_1 s_2 \). Show that the analogous transpositions \( \tau_1 \) and \( \tau_2 \) in \( S_3 \) also satisfy \( \tau_1 \tau_2 \tau_1 = \tau_2 \tau_1 \tau_2 \).

Homework 1, due Wed Feb 14:
• Sect. 2 (pg 25): 1–4, 7–10
• Bonus: Given a set \( S \), the group \( \langle \text{Aut}(S), \circ \rangle \) is abelian if and only if \( |S| \leq 2 \).
• Sect. 4 (pg 45): 2–8
• Sect. 5 (pg 55): 22, 24, 31, 33, 51
• Sect. 6 (pg 66): 4, 17, 19, 21, 28, 32–34
• Prove the following: Suppose \( G \) is a cyclic group. Then either \( |G| = \infty \) and \( G \cong \mathbb{Z} \), or \( |G| = n < \infty \) and \( G \cong \mathbb{Z}/n\mathbb{Z} \).

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