Generalized Ehrhart Polynomials

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Outline

- Ehrhart Theorem
- Generalized Ehrhart polynomials by examples
- Main theorem in three equivalent versions
- Proof: “writing in base n” trick
- Further questions
Ehrhart Theorem

Let $P \subset \mathbb{R}^d$ be a polytope with rational vertices.

$$(0, 1) \quad (0, 0) \quad \frac{1}{2}, 0$$

$$(0, n) \quad (\frac{1}{2}n, 0)$$

$$\text{#}(nP \cap \mathbb{Z}^2) = \begin{cases} \frac{1}{4}n^2 + n + 1 & n \text{ even} \\ \frac{1}{4}n^2 + n + \frac{3}{4} & n \text{ odd} \end{cases}$$

Definition

We call $f(n)$ a quasi-polynomial, if $f(n) = f_i(n)$ ($n \equiv i \mod T$), for some $T \in \mathbb{N}$ and polynomials $f_i(n)$'s.

Theorem (Ehrhart)

$i(P, n) = \text{#}(nP \cap \mathbb{Z}^d)$ is a quasi-polynomial. In particular, if $P$ has integral vertices, $i(P, n)$ is a polynomial, called the **Ehrhart polynomial**.
Example: non homogenous “dilation”

The number of integer points on this diagonal is

\[ f(n) = \# \{(x, y) \in \mathbb{Z}_{\geq 0} \mid 2x + y = n + 1 \} \]
Example: non homogenous “dilation”

\[(0, n+1) \rightarrow (0, 0) \rightarrow \left(\frac{n}{2} + \frac{1}{2}, 0\right)\]

\[
\begin{align*}
2x + y &\leq n + 1 \\
x &\geq 0 \\
y &\geq 0
\end{align*}
\]

\[
\begin{align*}
2x + y &= n + 1 \\
x &\geq 0 \\
y &\geq 0
\end{align*}
\] + \[
\begin{align*}
2x + y &\leq n \\
x &\geq 0 \\
y &\geq 0
\end{align*}
\]

The number of integer points on this diagonal is

\[f(n) = \#\{(x, y) \in \mathbb{Z}_{\geq 0}^2 \mid 2x + y = n + 1\}\]
Example: non homogenous “dilation”

\[
\begin{align*}
\text{(0, } n+1) & \quad \text{(0, 0)} \quad \text{(n/2 + 1/2, 0)} \\
(0, n+1) & \quad \text{\text{The number of integer points on this diagonal is}} \\
(n/2 + 1/2, 0) & \\
\end{align*}
\]

The number of integer points on this diagonal is

\[
f(n) = \#\{(x, y) \in \mathbb{Z}_{\geq 0}^2 \mid 2x + y = n + 1\}
\]
Theorem (Popoviciu’s Formula)

The number of nonnegative integer solutions \((x, y)\) to \(ax + by = m\), where \(a, b\) are coprime integers, is given by the formula

\[
\frac{m}{ab} - \left\{ \frac{ma'}{b} \right\} - \left\{ \frac{mb'}{a} \right\} + 1,
\]

where \(\{r\} = r - \lfloor r \rfloor\) and \(a'\) and \(b'\) are any integers satisfying \(aa' + bb' = 1\).

Example

For \(2x + y = n + 1\), we have

\[
a = 2, \ b = 1, \ a' = 1, \ b' = -1
\]

Then

\[
\frac{n + 1}{2} - \left\{ \frac{n + 1}{1} \right\} - \left\{ \frac{-(n + 1)}{2} \right\} + 1 = \begin{cases} 
\frac{n+3}{2} & \text{if } n \text{ odd} \\
\frac{n}{2} + 1 & \text{if } n \text{ even}
\end{cases}
\]
Example: nonlinear “dilation”

\[(0, n^2 + 2n) \rightarrow (2n + 4, 0)\]

The number of integer points on the diagonal is

\[f(n) = \#\{(x, y) \in \mathbb{Z}^2_{\geq 0} \mid (n^2 + 2n)x + (2n + 4)y = (2n + 4)(n^2 + 2n)\}\]

We can not use Popoviciu’s Formula directly, so we need the following generalizations:

- Replace \(GCD(a, b) = 1, a, b \in \mathbb{Z}\) by \(GCD(a(n), b(n)) = 1\) for all \(n \in \mathbb{Z}\), where \(a(n), b(n) \in \mathbb{Z}[n]\)
- Replace \(\left\{\frac{n-1}{2}\right\}\) by function \(\left\{\frac{a(n)}{b(n)}\right\}\), \(a(n), b(n) \in \mathbb{Z}[n]\).
Example: nonlinear “dilation”

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Example

Compute \( \left\{ \frac{n^2 + 2n}{2n+4} \right\} \) and \( \text{GCD}(n^2 + 2n, 2n + 4) \).

- \( n = 2m, \ n^2 + 2n = 4m^2 + 4m = m(4m + 4). \)
  \( \left\{ \frac{n^2 + 2n}{2n+4} \right\} = 0 \) and \( \text{GCD}(n^2 + 2n, 2n + 4) = 2n + 4. \)

- \( n = 2m + 1, \ n^2 + 2n = 4m^2 + 8m + 3 = m(4m + 6) + (2m + 3). \)
  \( \left\{ \frac{n^2 + 2n}{2n+4} \right\} = \left\{ m + \frac{2m+3}{4m+6} \right\} = \frac{2m+3}{4m+6} = \frac{1}{2}. \)

For \( \text{GCD} \), we apply Euclidean algorithm and get \( 4m + 6 = 2(2m + 3) \), So \( \text{GCD}(n^2 + 2n, 2n + 4) = 2m + 3 = n + 2 \)

Therefore,

\[
\left\{ \frac{n^2 + 2n}{2n+4} \right\} = \begin{cases} 
0 & \text{n even} \\
\frac{1}{2} & \text{n odd}
\end{cases}
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Therefore,

\[
\left\{ \frac{n^2 + 2n}{2n + 4} \right\} = \begin{cases} 0 & n \text{ even} \\ \frac{1}{2} & n \text{ odd} \end{cases} \quad \text{and} \quad GCD(n^2 + 2n, 2n + 4) = \begin{cases} 2n + 4 & n \text{ even} \\ n + 2 & n \text{ odd} \end{cases}
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For \( \text{GCD} \), we apply Euclidean algorithm and get
\[ 4m + 6 = 2(2m + 3), \] So \( \text{GCD}(n^2 + 2n, 2n + 4) = 2m + 3 = n + 2 \)
Therefore,
\[ \left\{ \frac{n^2 + 2n}{2n + 4} \right\} = \begin{cases} 
0 & n \text{ even} \\
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\end{cases} \] and \( \text{GCD}(n^2 + 2n, 2n + 4) = \begin{cases} 
2n + 4 & n \text{ even} \\
n + 2 & n \text{ odd} 
\end{cases} \)
Definition
Let \( f(n), g(n) \) be polynomial functions \( \mathbb{Z} \to \mathbb{Z} \). Define functions \( q \), the quotient, \( r \), the remainder and \( ggcd \), the generalized GCD of \( f \) and \( g \) as follows: for each \( n \in \mathbb{Z} \),

\[
q(n) = \left\lfloor \frac{f(n)}{g(n)} \right\rfloor, \quad r(n) = \left\{ \frac{f(n)}{g(n)} \right\} g(n), \quad \text{and} \quad ggcd(n) = \text{GCD}(f(n), g(n)).
\]

Theorem (Chen, L., Sam)
Functions \( q, r, ggcd \) are quasi-polynomials for \( n \) sufficiently large.
Back to example: nonlinear dilation

\( P(n) = \) 

\( (0, 0) \rightarrow (0, n^2 + 2n) \rightarrow (2n + 4, 0) \)

\[
\#\{ P(n) \cap \mathbb{Z}^2 \} = \frac{1}{2}((n^2 + 2n + 1)(2n + 5) + f(n)), \text{ where}
\]

\[
f(n) = \#\{ (x, y) \in \mathbb{Z}_{\geq 0}^2 \mid (n^2 + 2n)x + (2n + 4)y = (2n + 4)(n^2 + 2n) \}.
\]

Now by generalized division and GCD, we can apply Popoviciu's Formula and get

\[
f(n) = \begin{cases} 
2n + 5 & \text{n even} \\
 n + 3 & \text{n odd}
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\[ f(n) = \begin{cases} 2n + 5 & \text{n even} \\ n + 3 & \text{n odd} \end{cases} \]
Generalized Ehrhart (quasi) polynomials

**Theorem**
Let $P(n)$ be a polytope in $\mathbb{R}^d$ and the coordinates of its vertices are given by polynomial functions of $n$ (rational functions of $n$). Then $i(P, n) = #(P(n) \cap \mathbb{Z}^d)$ is a quasi-polynomial of $n$ for $n$ sufficiently large.

**Theorem**
Define a rational polytope $P(n) = \{x \in \mathbb{R}^d | V(n)x \geq c(n)\}$, where $V(n)$ is an $r \times d$ matrix, and $c(n)$ is an $r \times 1$ column vector, both with entries in $\mathbb{Z}[n]$. Then $#(P(n) \cap \mathbb{Z}^d)$ is a quasi-polynomial of $n$ for $n$ sufficiently large.

**Theorem**
Let $A(n)$ be an $m \times k$ matrix and $b(n)$ be a column vector of length $m$, both with entries in $\mathbb{Z}[n]$. If $f(n)$ denotes the number of nonnegative integer vectors $x$ satisfying $A(n)x = b(n)$ (assuming that these values are finite), then $f(n)$ is a quasi-polynomial of $n$ for $n$ sufficiently large.

The above three theorems are equivalent.
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The above three theorems are equivalent.
Lemma
\[ f(n) = \# \{x \in (\mathbb{Z}_{\geq 0})^k \mid A(n)x = b(n)\} \] is a quasi-polynomial for \( n \) sufficiently large under the following assumptions:

- \( A(n) \) is a constant matrix \( A \),
- terms in vector \( b(n) \) are linear functions \( cn + d \).

Proof.
The base case is true by Ehrhart Theorem, when there are no nonhomogeneous (in)equalities. By induction on the number of variables and the number of nonhomogeneous (in)equalities. For example,
\[ f(n) = \{(x, y) \in \mathbb{Z}_\geq 0^2 \mid 2x + y \leq n + 1\} \]
\[ = \{(x, y) \in \mathbb{Z}_\geq 0^2 \mid 2x + y \leq n\} \cup \{y = n + 1 - 2x\}. \]
Special case

Lemma

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By induction on the number of variables and the number of nonhomogeneous (in)equalities.
For example, \( f(n) = \{(x, y) \in \mathbb{Z}^2_{\geq 0} \mid 2x + y \leq n + 1\} \)
\[ = \{(x, y) \in \mathbb{Z}^2_{\geq 0} \mid 2x + y \leq n\} \cup \{y = n + 1 - 2x\}. \]
Fix a natural number $n$. Then for any integer $x$, there is a unique expression of $x$ in base $n$:

$$x = x_d n^d + x_{d-1} n^{d-1} + \cdots + x_1 n + x_0$$

for some natural number $d$ and $0 \leq x_i < n$.

Now let $n$ grow, and consider $x$ to be polynomial functions of $n$.

- $f(n) = 2n^2 + 3n + 5$. 
  $2n^2 + 3n + 5$ is the expression of $f(n)$ in base $n$ for any $n > 5$

- $g(n) = 2n^2 - n + 3$
  $n^2 + (n - 1)n + 3$ is the expression of $g(n)$ in base $n$ for any $n > 3$. 

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Reduce $A(n)x = b(n)$ to $Ax = an + b$

For example,

$$2x_1 + (n + 1)x_2 + n^2x_3 = 4n^2 + 3n - 5$$

**Step one:** Writing in base $n$:

- $4n^2 + 3n - 5 = 4n^2 + 2n + (n - 5)$
- $x_1 = x_{12}n^2 + x_{11}n + x_{10}, \ x_2 = x_{21}n + x_{20}$ and $x_3 = x_{30}$, with $0 \leq x_{ij} < n$.

The original equation gives a upper boundary for the degree of $n$ in the expression of the $x_i$, so there are only finitely many new variables $x_{ij}$.

**Step two:** Expand the equation

$$(2x_{12} + x_{21} + x_{30})n^2 + (2x_{11} + x_{21} + x_{20})n + (2x_{10} + x_{20}) = 4n^2 + 2n + (n - 5).$$
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Reduce \( A(n)x = b(n) \) to \( Ax = an + b \) (2)

**Step three:** Compare coefficients of \( n \) on both sides

\[
(2x_{12} + x_{21} + x_{30})n^2 + (2x_{11} + x_{21} + x_{20})n + (2x_{10} + x_{20}) = 4n^2 + 2n + (n - 5).
\]

Note \( 0 \leq x_{ij} < n \).

- **\( n^0 \):** \( (2x_{10} + x_{20}) \equiv (n - 5) \) (mod \( n \)) \( \in \{ n - 5, 2n - 5, 3n - 5 \} \), for \( 0 \leq x_{ij} < n \). Let \( A_i^0 = \{ 2x_{10} + x_{20} = n - 5 + in \}, i = 0, 1, 2 \)

- **\( n^1 \):** Subtract the \( n^0 \) term from both sides. If \( x \) satisfies \( A_i^0 \), we are left with

\[
(2x_{12} + x_{21} + x_{30})n^2 + (2x_{11} + x_{21} + x_{20} + i)n = 4n^2 + 2n.
\]

So let \( A_{ij}^1 = \{ 2x_{11} + x_{21} + x_{20} + i = 2 + jn \} \), where \( j = 0, 1, 2, 3, 4 \).

- **\( n^2 \):** Subtract the \( n^0 \) and \( n^1 \) terms. If \( x \) satisfies \( A_{ij}^1 \), we are left with

\[
(2x_{12} + x_{21} + x_{30} + j)n^2 = 4n^2.
\]

So let \( A_j^2 = \{ 2x_{12} + x_{21} + x_{30} + j = 4 \} \).

Notice that all these equations \( A_i^0, A_{ij}^1, \) and \( A_j^2 \) are of the form \( Ax = an + b \).
Reduce $A(n)x = b(n)$ to $Ax = an + b$ (2)

Step three: Compare coefficients of $n$ on both sides

\[(2x_{12} + x_{21} + x_{30})n^2 + (2x_{11} + x_{21} + x_{20})n + (2x_{10} + x_{20}) = 4n^2 + 2n + (n - 5).\]

Note $0 \leq x_{ij} < n$.

- $n^0$: $(2x_{10} + x_{20}) \equiv (n - 5) \pmod{n} \in \{n - 5, 2n - 5, 3n - 5\}$, for $0 \leq x_{ij} < n$. Let $A^0_i = \{2x_{10} + x_{20} = n - 5 + in\}$, $i = 0, 1, 2$
- $n^1$: Subtract the $n^0$ term from both sides. If $x$ satisfies $A^0_i$, we are left with
  \[(2x_{12} + x_{21} + x_{30})n^2 + (2x_{11} + x_{21} + x_{20} + i)n = 4n^2 + 2n.\]
  So let $A^1_{ij} = \{2x_{11} + x_{21} + x_{20} + i = 2 + jn\}$, where $j = 0, 1, 2, 3, 4$.
- $n^2$: Subtract the $n^0$ and $n^1$ terms. If $x$ satisfies $A^1_{ij}$, we are left with
  \[(2x_{12} + x_{21} + x_{30} + j)n^2 = 4n^2.\]
  So let $A^2_j = \{2x_{12} + x_{21} + x_{30} + j = 4\}$.

Notice that all these equations $A^0_i$, $A^1_{ij}$, and $A^2_j$ are of the form $Ax = an + b$. 
Reduce $A(n)x = b(n)$ to $Ax = an + b$ (2)

Step three: Compare coefficients of $n$ on both sides

$(2x_{12} + x_{21} + x_{30})n^2 + (2x_{11} + x_{21} + x_{20})n + (2x_{10} + x_{20}) = 4n^2 + 2n + (n - 5)$.

Note $0 \leq x_{ij} < n$.

- $n^0$: $(2x_{10} + x_{20}) \equiv (n - 5) \pmod{n} \in \{n - 5, 2n - 5, 3n - 5\}$, for $0 \leq x_{ij} < n$. Let $A^0_i = \{2x_{10} + x_{20} = n - 5 + in\}$, $i = 0, 1, 2$
- $n^1$: Subtract the $n^0$ term from both sides. If $x$ satisfies $A^0_i$, we are left with

$$(2x_{12} + x_{21} + x_{30})n^2 + (2x_{11} + x_{21} + x_{20} + i)n = 4n^2 + 2n.$$ 

So let $A^1_{ij} = \{2x_{11} + x_{21} + x_{20} + i = 2 + jn\}$, where $j = 0, 1, 2, 3, 4$.

- $n^2$: Subtract the $n^0$ and $n^1$ terms. If $x$ satisfies $A^1_{ij}$, we are left with

$$(2x_{12} + x_{21} + x_{30} + j)n^2 = 4n^2.$$ 

So let $A^2_j = \{2x_{12} + x_{21} + x_{30} + j = 4\}$.

Notice that all these equations $A^0_i$, $A^1_{ij}$, and $A^2_j$ are of the form $Ax = an + b$. 
Reduce $A(n)x = b(n)$ to $Ax = an + b$ (2)

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- $n^2$: Subtract the $n^0$ and $n^1$ terms. If $x$ satisfies $A^1_{ij}$, we are left with

$$(2x_{12} + x_{21} + x_{30} + j)n^2 = 4n^2.$$  

So let $A^2_j = \{2x_{12} + x_{21} + x_{30} + j = 4\}$.

Notice that all these equations $A^0_i$, $A^1_{ij}$, and $A^2_j$ are of the form $Ax = an + b$.  

Reduce $A(n)x = b(n)$ to $Ax = an + b$ (3)

When $n$ is sufficiently large,

$$\{(x_1, x_2, x_3) \in \mathbb{Z}_3^{\geq 0} \mid 2x_1 + (n + 1)x_2 + n^2x_3 = 4n^2 + 3n - 5\}$$

is in bijection with the set

$$\{x = (x_{12}, x_{11}, x_{10}, x_{21}, x_{20}, x_{30}) \in \mathbb{Z}_6^{\geq 0}, 0 \leq x_{ij} < n\},$$

such that $x$ satisfies

$$\begin{pmatrix}
A_0^0 & A_0^1 & A_0^2 \\
A_1^0 & A_1^1 & A_1^2 \\
A_2^0 & A_2^1 & A_2^2 \\
\end{pmatrix}
\begin{pmatrix}
A_{00}^1 & A_{01}^1 & \cdots & A_{04}^1 \\
A_{10}^1 & A_{11}^1 & \cdots & A_{14}^1 \\
A_{20}^1 & A_{21}^1 & \cdots & A_{24}^1 \\
\end{pmatrix}
\begin{pmatrix}
A_0^2 \\
A_1^2 \\
\vdots \\
A_4^2 \\
\end{pmatrix}.$$ 

Where $AB = A \cap B$, $A + B = A \cup B$ (disjoint union) with all equations $A_i^j$ of the form $Ax = an + b$
Open questions

Theorem (Dahmen-Micchelli, 1988)
For an integral matrix $A$ and a vector $m = (m_1, \ldots, m_k)^T$, denote
\[ t(A|m) = \# \{ x \in (\mathbb{Z}_{\geq 0})^k \mid Ax = m \}. \]

Then $t(A|m)$ is a piece-wise quasi-polynomial function in $m_1, \ldots, m_k$.

Question (Multivariable Case)
If we let terms in $A$ and $m$ be polynomials in variables $m_1, \ldots, m_k$, is it possible that $t(A|m)$ is still a piece-wise quasi-polynomial function in $m_1, \ldots, m_k$?

Conjecture (Ehrhart’s Conjecture)
The statement is true if all polynomial functions are linear.