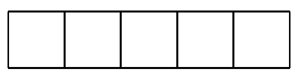
7 Enumeration under group action.

In Sections 5 and 6 we considered the quotient poset B_n/G, where G is a subgroup of the symmetric group $\mathfrak{S}_n$. If p_i is the number of elements of rank i of this poset, then the sequence $p_0, p_1, \ldots, p_n$ is rank-symmetric and rank-unimodal. Thus it is natural to ask whether there is some nice formula for the numbers p_i. For instance, in Theorem 5.10 p_i is the number of nonisomorphic graphs with m vertices (where $n = \binom{m}{2}$) and i edges; is there some nice formula for this number? For the group $G_{mn} = \mathfrak{S}_n \wr \mathfrak{S}_m$ of Theorem 6.6 we obtained a simple generating function for p_i (i.e., a formula for the polynomial $\sum_i p_i q^i$), but this was a very special situation. In this section we will present a general theory for enumerating inequivalent objects subject to a group of symmetries, which will include a formula for the generating function $\sum_i p_i q^i$ as a special case, where p_i is the number of elements of rank i of B_n/G. The chief architect of this theory is G. Pólya (1887–1985) (though much of it was anticipated by J. H. Redfield) and hence is often called *Pólya's theory of enumeration* or just *Pólya theory.*

Pólya theory is most easily understood in terms of "colorings" of some geometric or combinatorial object. For instance, consider a row of five squares:



In how many ways can we color the squares using n colors? Each square can be colored any of the n colors, so there are n^5 ways in all. These colorings can by indicated as

A	B	C	D	E

where A, B, C, D, E are the five colors. Now assume that we are allowed to rotate the five squares 180°, and that two colorings are considered the same if one can be obtained from the other by such a rotation. (We may think that we have cut the row of five squares out of paper and colored them on one side.) We say that two colorings are *equivalent* if they are the same or can be transformed into one another by a 180° rotation. The first naive assumption is that every coloring is equivalent to exactly one other (besides itself), so the number of inequivalent colorings is $n^5/2$. Clearly this reasoning cannot be correct since $n^5/2$ is not always an integer! The problem, of course, is that some colorings stay the same when we rotate 180°. In fact, these are exactly the colorings

A	B	C	B	A

where A, B, C are any three colors. There are n^3 such colorings, so the total number of inequivalent colorings is given by

$$\frac{1}{2}(\text{number of colorings which don't equal their } 180^\circ \text{ rotation})$$

$$+(\text{number of colorings which equal their } 180^\circ \text{ rotation}$$

$$\begin{aligned} &= \frac{1}{2}(n^5 - n^3) + n^3 \\ &= \frac{1}{2}(n^5 + n^3). \end{aligned}$$

Pólya theory gives a systematic method for obtaining formulas of this sort for any underlying symmetry group.

The general setup is the following. Let X be a finite set, and G a subgroup of the symmetric group $\mathfrak{S}_X$. Think of G as a group of symmetries of X. Let C be another set (which may be infinite), which we think of as a set of "colors." A *coloring* of X is a function $f : X \to C$. For instance, X could be the set of four squares of a 2×2 chessboard, labelled as follows:

1	2
3	4

Let $C = \{r, b, y\}$ (the colors red, blue, and yellow). A typical coloring of X would then look like

r	b
y	r

The above diagram thus indicates the function $f : X \to C$ given by $f(1) = r, f(2) = b, f(3) = y, f(4) = r$.

We define two colorings f and g to be *equivalent* (or G-*equivalent*, when it is necessary to specify the group), denoted $f \sim g$ or $f \overset{G}{\sim} g$, if there exists an element $\pi \in G$ such that

$$g(\pi(x)) = f(x) \text{ for all } x \in X.$$

We may write this condition more succinctly as $g\pi = f$, where $g\pi$ denotes the composition of functions (from right to left). It is easy to check, using the fact that G is a group, that $\sim$ is an equivalence relation. One should think that equivalent functions are the same "up to symmetry."

7.1 Example. Let X be the 2×2 chessboard and $C = \{r, b, y\}$ as above. There are many possible choices of a symmetry group G, and this will affect when two colorings are equivalent. For instance, consider the following groups:

- G_1 consists of only the identity permutation $(1)(2)(3)(4)$.
- G_2 is the group generated by a vertical reflection. It consists of the two elements $(1)(2)(3)(4)$ (the identity element) and $(1,2)(3,4)$ (the vertical reflection).
- G_3 is the group generated by a reflection in the main diagonal. It consists of the two elements $(1)(2)(3)(4)$ (the identity element) and $(1)(4)(2,3)$ (the diagonal reflection).
- G_4 is the group of all rotations of X. It is a cyclic group of order four with elements $(1)(2)(3)(4)$, $(1,2,4,3)$, $(1,4)(2,3)$, and $(1,3,4,2)$.
- G_5 is the dihedral group of all rotations and reflections of X. It has eight elements, namely, the four elements of G_4 and the four reflections $(1,2)(3,4)$, $(1,3)(2,4)$, $(1)(4)(2,3)$, and $(2)(3)(1,4)$.
- G_6 is the symmetric group of *all* 24 permutations of X. Although this is a perfectly valid group of symmetries, it no longer has any connection with the geometric representation of X as the squares of a 2×2 chessboard.

Consider the inequivalent colorings of X with two red squares, one blue square, and one yellow square, in each of the six cases above.

(G_1) There are twelve colorings in all with two red squares, one blue square, and one yellow square, and all are inequivalent under the trivial group (the group with one element). In general, whenever G is the trivial group then two colorings are equivalent if and only if they are the same [why?].

(G_2) There are now six inequivalent colorings, represented by

r	r		r	b		r	y		b	y		r	b		r	y
b	y		r	y		r	b		r	r		y	r		b	r

Each equivalence class contains two elements.

(G_3) Now there are seven classes, represented by

r	r		r	r		b	y		y	b		r	b		b	r		y	r
b	y		y	b		r	r		r	r		y	r		r	y		r	b

The first five classes contain two elements each and the last two classes only one element. Although G_2 and G_3 are isomorphic as abstract groups, as permutation groups they have a different structure. Specifically, the generator $(1,2)(3,4)$ of G_2 has two cycles of length two, while the generator $(1)(4)(2,3)$ has two cycles of length one and one of length two. As we will see below, it is the lengths of the cycles of the elements of G that determine the sizes of the equivalence classes. This explains why the number of classes for G_2 and G_3 are different.

(G_4) There are three classes, each with four elements. The size of each class is equal to the order of the group because none of the colorings have any symmetry with respect to the group, i.e., for any coloring f, the only group element π that fixes f (so $f\pi = f$) is the identity ($\pi = (1)(2)(3)(4)$).

r	r		r	r		r	b
y	b		b	y		y	r

(G_5) Under the full dihedral group there are now two classes.

r	r
b	y

r	b
y	r

The first class has eight elements and the second four elements. In general, the size of a class is the index in G of the subgroup fixing some fixed coloring in that class [why?]. For instance, the subgroup fixing the second coloring above is $\{(1)(2)(3)(4), (1,4)(2)(3)\}$, which has index four in the dihedral group of order eight.

(G_6) Under the group $\mathfrak{S}_4$ of all permutations of the squares there is clearly only one class, with all twelve colorings. In general, for any set X if the group is the symmetric group $\mathfrak{S}_X$ then two colorings are equivalent if and only if each color appears the same number of times [why?].

Our object in general is to count the number of equivalence classes of colorings which use each color a specified number of times. We will put the information into a *generating function* — a polynomial whose coefficients are the numbers we seek. Consider for example the set X, the group $G = G_5$ (the dihedral group), and the set $C = \{r, b, y\}$ of colors in Example 7.1 above. Let $\kappa(i,j,k)$ be the number of inequivalent colorings using red i times, blue j times, and yellow k times. Think of the colors r, b, y as *variables*, and form the polynomial

$$F_G(r,b,y) = \sum_{i+j+k=4} \kappa(i,j,k) r^i b^j y^k.$$

Note that we sum only over i, j, k satisfying $i + j + k = 4$ since a total of four colors will be used to color the four-element set X. The reader should check that

$$\begin{aligned} F_G(r,b,y) &= (r^4 + b^4 + y^4) + (r^3 b + r b^3 + r^3 y + r y^3 + b^3 y + b y^3) \\ &\quad +2(r^2 b^2 + r^2 y^2 + b^2 y^2) + 2(r^2 b y + r b^2 y + r b y^2). \end{aligned}$$

For instance, the coefficient of $r^2 by$ is two because, as we have seen above, there are two inequivalent colorings using the colors r, r, b, y. Note that $F_G(r, b, y)$ is a *symmetric function* of the variables r, b, y (i.e., it stays the same if we permute the variables in any way), because insofar as counting inequivalent colorings goes, it makes no difference what *names* we give the colors. As a special case we may ask for the *total* number of inequivalent colorings with four colors. This obtained by setting $r = b = y = 1$ in $F_G(r, b, y)$ [why?], yielding $F_G(1, 1, 1) = 3 + 6 + 2 \cdot 3 + 2 \cdot 3 = 21$.

What happens to the generating function F_G in the above example when we use the n colors $r_1, r_2, \ldots, r_n$ (which can be thought of as different shades of red)? Clearly all that matters are the *multiplicities* of the colors, without regard for their order. In other words, there are five cases: (a) all four colors the same, (b) one color used three times and another used once, (c) two colors used twice each, (d) one color used twice and two others once each, and (e) four colors used once each. These five cases correspond to the five partitions of 4, i.e., the five ways of writing 4 as a sum of positive integers without regard to order: 4, 3+1, 2+2, 2+1+1, 1+1+1+1. Our generating function becomes

$$F_G(r_1, r_2, \ldots, r_n) = \sum_i r_i^4 + \sum_{i \neq j} r_i^3 r_j + 2 \sum_{i<j} r_i^2 r_j^2 + 2 \sum_{\substack{i \neq j \\ i \neq k \\ j<k}} r_i^2 r_j r_k + 3 \sum_{i<j<k<l} r_i r_j r_k r_l,$$

where the indices in each sum lie between 1 and n. If we set all variables equal to one (obtaining the total number of colorings with n colors), then simple combinatorial reasoning yields

$$\begin{aligned} F_G(1, 1, \ldots, 1) &= n + n(n-1) + 2\binom{n}{2} + 2n\binom{n-1}{2} + 3\binom{n}{4} \\ &= \frac{1}{8}(n^4 + 2n^3 + 3n^2 + 2n). \end{aligned} \tag{38}$$

Note that the polynomial (38) has the following description: The denominator 8 is the order of the group G_5, and the coefficient of n^i in the numerator is just the number of permutations in G_5 with i cycles! For instance, the coefficient of n^2 is 3, and G_5 has the three elements $(1, 2)(3, 4)$, $(1, 3)(2, 4)$, and $(1, 4)(2, 3)$ with two cycles. We want to prove a general result of this nature.

The basic tool which we will use is a simple result from the theory of permutation groups known as *Burnside's lemma*. It was actually first proved by Cauchy when G is transitive (i.e., $|Y/G| = 1$) and by Frobenius in the general case, and is sometimes called the *Cauchy-Frobenius lemma*.

7.2 Lemma. (Burnside's lemma) *Let Y be a finite set and G a subgroup of $\mathfrak{S}_Y$. For each $\pi \in G$, let*

$$\mathrm{Fix}(\pi) = \{y \in Y : \pi(y) = y\},$$

so $|\mathrm{Fix}(y)|$ is the number of cycles of length one in the permutation π. Let Y/G be the set of orbits of G. Then

$$|Y/G| = \frac{1}{|G|} \sum_{\pi \in G} |\mathrm{Fix}(\pi)|.$$

An equivalent form of Burnside's lemma is that statement that the average number of elements of Y fixed by an element of G is equal to the number of orbits. Before proceeding to the proof, let us consider an example.

7.3 Example. Let $Y = \{a, b, c, d\}$, $G = \{(a)(b)(c)(d), (a, b)(c, d), (a, c), (b, d), (a, d)(b, c)\}$, and $G' = \{(a)(b)(c)(d), (a, b)(c)(d), (a)(b)(c, d), (a, b)(c, d)\}$. Both groups are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ (compare Example 5.1(c) and (d)). By Burnside's lemma the number of orbits of G is $\frac{1}{4}(4 + 0 + 0 + 0) = 1$. Indeed, given any two elements $i, j \in Y$, it is clear by inspection that there is a $\pi \in G$ (which happens to be unique) such that $\pi(i) = j$. On the other hand, the number of orbits of G' is $\frac{1}{4}(4 + 2 + 2 + 0) = 2$. Indeed, the two orbits are $\{a, b\}$ and $\{c, d\}$.

Proof of Burnside's lemma. For $y \in Y$ let $G_y = \{\pi \in G : \pi \cdot y = y\}$ (the set of permutations fixing y). Then

$$\begin{aligned} \frac{1}{|G|} \sum_{\pi \in G} |\mathrm{Fix}(\pi)| &= \frac{1}{|G|} \sum_{\pi \in G} \sum_{\substack{y \in Y \\ \pi \cdot y = y}} 1 \\ &= \frac{1}{|G|} \sum_{y \in Y} \sum_{\substack{\pi \in G \\ \pi \cdot y = y}} 1 \\ &= \frac{1}{|G|} \sum_{y \in Y} |G_y|. \end{aligned}$$

Now (as in the proof of Lemma 5.7) the multiset of elements $\pi \cdot y$, $\pi \in G$, contains every element in the orbit Gy the same number of times, namely

$|G|/|G_y|$ times. Thus y occurs $|G|/|G_y|$ times among the $\pi \cdot y$, so

$$\frac{|G|}{|Gy|} = |G_y|.$$

Thus

$$\begin{aligned} \frac{1}{|G|}\sum_{\pi \in G} |\mathrm{Fix}(\pi)| &= \frac{1}{|G|}\sum_{y \in Y} \frac{|G|}{|Gy|} \\ &= \sum_{y \in Y} \frac{1}{|Gy|}. \end{aligned}$$

How many times does a term $1/|\mathcal{O}|$ appear in the above sum, where $\mathcal{O}$ is a fixed orbit? We are asking for the number of y such that $Gy = \mathcal{O}$. But $Gy = \mathcal{O}$ if and only if $y \in \mathcal{O}$, so $1/|\mathcal{O}|$ appears $|\mathcal{O}|$ times. Thus each orbit gets counted exactly once, so the above sum is equal to the number of orbits. □

7.4 Example. How many inequivalent colorings of the vertices of a regular hexagon H are there using n colors, under cyclic symmetry? Let $\mathcal{C}_n$ be the set of all n-colorings of H. Let G be the group of all permutations of $\mathcal{C}_n$ which permute the colors cyclically, so $G \cong \mathbb{Z}_6$. We are asking for the number of orbits of G [why?]. We want to apply Burnside's lemma, so for each of the six elements σ of G we need to compute the number of colorings fixed by that element. Let π be a generator of G.

- $\sigma = 1$ (the identity): All n^6 colorings are fixed by σ.
- $\sigma = \pi, \pi^{-1}$: Only the n colorings with all colors equal are fixed.
- $\sigma = \pi^2, \pi^4$: Any coloring of the form *ababab* is fixed (writing the colors linearly in the order they appear around the hexagon, starting at any fixed vertex). There are n choices for a and n for b, so n^2 colorings in all.
- $\sigma = \pi^3$: The fixed colorings are of the form *abcabc*, so n^3 in all.

Hence by Burnside's lemma, we have

$$\text{number of orbits } = \frac{1}{6}(n^6 + n^3 + 2n^2 + 2n).$$

The reader who has followed the preceding example will have no trouble understanding the following result.

7.5 Theorem. *Let G be a group of permutations of a finite set X. Then the number of inequivalent (with respect to G) n-colorings of X is equal to*

$$\frac{1}{|G|}\sum_{\pi\in G} n^{c(\pi)},$$

where $c(\pi)$ denotes the number of cycles of π.

Proof. Let π_n denote the action of $\pi \in G$ on the set $\mathcal{C}_n$ of n-colorings of X. We want to determine the set $\mathrm{Fix}(\pi_n)$, so that we can apply Burnside's lemma. Let C be the set of n colors. If $f : X \to C$ is a coloring fixed by π, then for all $x \in X$ we have

$$f(x) = \pi_n \cdot f(x) = f(\pi_n \cdot x).$$

Thus $f \in \mathrm{Fix}(\pi_n)$ if and only if $f(x) = f(y)$ whenever $\pi(x) = y$. In other words, we must have $f(x) = f(\pi(x))$. Hence $f(x) = f(\pi^k(x))$ for any $k \geq 1$ [why?]. The elements y of X of the form $\pi^k(x)$ for $k \geq 1$ are just the elements of the cycle of π containing x. Thus to obtain $f \in \mathrm{Fix}(\pi_n)$, we should take the cycles $\sigma_1, \ldots, \sigma_{c(\pi)}$ of π and color each element of σ_i the same color. There are n choices for each σ_i, so $n^{c(\pi)}$ colorings in all fixed by π. In other words, $|\mathrm{Fix}(\pi_n)| = n^{c(\pi)}$, and the proof follows by Burnside's lemma. $\square$

We would now like not just to count the *total* number of inequivalent colorings with n-colors, but more strongly to specify the number of occurences of each color. We will need to use not just the number $c(\pi)$ of cycles of each $\pi \in G$, but rather the lengths of each of the cycles of π. Thus given a permutation π of an n-element set X, define the *type* of π to be

$$\mathrm{type}(\pi) = (c_1, c_2, \ldots, c_n),$$

where π has c_i i-cycles. For instance, if $\pi = 4, 7, 3, 8, 2, 10, 11, 1, 6, 9, 5$, then

$$\begin{aligned} \mathrm{type}(\pi) &= \mathrm{type}\ (1,4,8)(2,7,11,5)(3)(6,10,9) \\ &= (1,0,2,1,0,0,0,0,0,0,0). \end{aligned}$$

Note that we always have $\sum_i ic_i = n$ [why?]. Define the *cycle indicator* of π to be the monomial

$$Z_\pi = z_1^{c_1} z_2^{c_2} \cdots z_n^{c_n}.$$

(Many other notations are used for the cycle indicator. The use of Z_π comes from the German work *Zyklus* for cycle. The original paper of Pólya was written in German.) Thus for the example above, we have $Z_\pi = z_1 z_3^2 z_4$.

Now given a subgroup G of $\mathfrak{S}_X$, the *cycle indicator* (or *cycle index polynomial*) of G is defined by

$$Z_G = Z_G(z_1, \ldots, z_n) = \frac{1}{|G|} \sum_{\pi \in G} Z_\pi.$$

Thus Z_G (also denoted P_G, $\mathrm{Cyc}(G)$, etc.) is a polynomial in the variables $z_1, \ldots, z_n$.

7.6 Example. If X consists of the vertices of a square and G is the group of rotations of X (a cyclic group of order 4), then

$$Z_G = \frac{1}{4}(z_1^4 + z_2^2 + 2z_4).$$

If reflections are also allowed (so G is the dihedral group of order 8), then

$$Z_G = \frac{1}{8}(z_1^4 + 3z_2^2 + 2z_1^2 z_2 + 2z_4).$$

We are now ready to state the main result of this section.

7.7 Theorem. (Pólya's theorem, 1937) *Let G be a group of permutations of the n-element set X. Let $C = \{r_1, r_2, \ldots\}$ be a set of colors. Let $\kappa(i_1, i_2, \ldots)$ be the number of inequivalent (under the action of G) colorings $f : X \to C$ such that color r_j is used i_j times. Define*

$$F_G(r_1, r_2, \ldots) = \sum_{i_1, i_2, \ldots} \kappa(i_1, i_2, \ldots) r_1^{i_1} r_2^{i_2} \cdots.$$

(Thus F_G is a polynomial or a power series in the variables $r_1, r_2, \ldots$, depending on whether or not C is finite or infinite.) Then

$$F_G(r_1, r_2, \ldots) = Z_G(r_1 + r_2 + r_3 + \cdots, r_1^2 + r_2^2 + r_3^2 + \cdots, \ldots, r_1^j + r_2^j + r_3^j + \cdots).$$

(In other words, substitute $\sum_i r_i^j$ for z_j in Z_G.)

Before giving the proof let us consider an example.

7.8 Example. Suppose that in Example 7.6 our set of colors is $C = \{a, b, c, d\}$, and that we take G to be the group of cyclic symmetries. Then

$$\begin{aligned} F_G(a,b,c,d) &= \frac{1}{4}\left((a+b+c+d)^4 + (a^2+b^2+c^2+d^2)^2 + 2(a^4+b^4+c^4+d^4)\right) \\ &= (a^4+\cdots) + (a^3b+\cdots) + 2(a^2b^2+\cdots) + 3(a^2bc+\cdots) + 6abcd. \end{aligned}$$

An expression such as $(a^2b^2+\cdots)$ stands for the sum of all monomials in the variables a, b, c, d with exponents $2, 2, 0, 0$ (in some order). The coefficient of all such monomials is 2, indicating two inequivalent colorings using one color twice and another color twice. If instead G were the full dihedral group, we would get

$$\begin{aligned} F_G(a,b,c,d) &= \frac{1}{8}\big((a+b+c+d)^4 + 3(a^2+b^2+c^2+d^2)^2 \\ &\quad + 2(a+b+c+d)^2(a^2+b^2+c^2+d^2) + 2(a^4+b^4+c^4+d^4)\big) \\ &= (a^4+\cdots) + (a^3b+\cdots) + 2(a^2b^2+\cdots) + 2(a^2bc+\cdots) + 3abcd. \end{aligned}$$

Proof of Pólya's theorem. Let $|X| = t$ and $i_1 + i_2 + \cdots = t$, where each $i_j \geq 0$. Let $\mathbf{i} = (i_1, i_2, ...)$, and let $\mathcal{C}_\mathbf{i}$ denote the set of all colorings of X with color r_j used i_j times. The group G acts on $\mathcal{C}_\mathbf{i}$, since if $f \in \mathcal{C}_\mathbf{i}$ and $\pi \in G$, then $\pi \cdot f \in \mathcal{C}_\mathbf{i}$. ("Rotating" a colored object does not change how many times each color appears.) Let $\pi_\mathbf{i}$ denote the action of π on $\mathcal{C}_\mathbf{i}$. We want to apply Burnside's lemma to compute the number of orbits, so we need to find $|\mathrm{Fix}(\pi_\mathbf{i})|$.

In order for $f \in \mathrm{Fix}(\pi_\mathbf{i})$, we must color X so that (a) in any cycle of π, all the elements get the same color, and (b) the color r_j appears i_j times. Consider the product

$$H_\pi = \prod_j (r_1^j + r_2^j + \cdots)^{c_j(\pi)},$$

where $c_j(\pi)$ is the number of j-cycles (cycles of length j) of π. When we expand this product as a sum of monomials $r_1^{j_1} r_2^{j_2} \cdots$, we get one of these monomials by choosing a term r_k^j from each factor of H_π and multiplying these terms together. Choosing r_k^j corresponds to coloring all the elements of some j-cycle with r_k. Since a factor $r_1^j + r_2^j + \cdots$ occurs precisely $c_j(\pi)$ times in H_π, choosing a term r_k^j from every factor corresponds to coloring X so that every cycle is monochromatic (i.e., all the elements of that cycle get the same color). The product of these terms r_k^j will be the monomial $r_1^{j_1} r_2^{j_2} \cdots$, where we have used color r_k a total of j_k times. It follows that the coefficient of $r_i^{i_1} r_2^{i_2} \cdots$ in H_π is equal to $|\mathrm{Fix}(\pi_{\mathbf{i}})|$. Thus

$$H_\pi = \sum_{\mathbf{i}} |\mathrm{Fix}(\pi_{\mathbf{i}})| r_1^{i_1} r_2^{i_2} \cdots. \tag{39}$$

Now sum both sides of (39) over all $\pi \in G$ and divide by $|G|$. The left-hand side becomes

$$\frac{1}{|G|} \sum_{\pi \in G} \prod_j (r_1^j + r_2^j + \cdots)^{c_j(\pi)} = Z_G(r_1 + r_2 + \cdots, r_1^2 + r_2^2 + \cdots, \ldots).$$

On the other hand, the right-hand side becomes

$$\sum_{\mathbf{i}} \left[\frac{1}{|G|} \sum_{\pi \in G} |\mathrm{Fix}(\pi_{\mathbf{i}})| \right] r_1^{i_1} r_2^{i_2} \cdots.$$

By Burnside's lemma, the expression in brackets is just the number of orbits of $\pi_{\mathbf{i}}$ acting on $\mathcal{C}_{\mathbf{i}}$, i.e., the number of inequivalent colorings using color r_j a total of i_j times, as was to be proved. □

7.9 Example. (Necklaces) A *necklace* of length ℓ is a circular arrangement of ℓ (colored) beads. Two necklaces are considered the same if they are cyclic rotations of one another. Let X be a set of ℓ (uncolored) beads, say $X = \{1, 2, \ldots, \ell\}$. Regarding the beads as being placed equidistantly on a circle in the order $1, 2 \ldots, \ell$, let G be the cyclic group group of rotations of X. Thus if π is the cycle $(1, 2, \ldots, \ell)$, then $G = \{1, \pi, \pi^2, \ldots, \pi^{\ell-1}\}$. For example, if $\ell = 6$ then the elements of G are

$$\begin{aligned}
\pi^0 &= (1)(2)(3)(4)(5)(6) \\
\pi &= (1,2,3,4,5,6) \\
\pi^2 &= (1,3,5)(2,4,6) \\
\pi^3 &= (1,4)(2,5)(3,6) \\
\pi^4 &= (1,5,3)(2,6,4) \\
\pi^5 &= (1,6,5,4,3,2).
\end{aligned}$$

In general, if d is the greatest common divisor of m and ℓ (denoted $d = \gcd(m,\ell)$), then π^m has d cycles of length ℓ/d. An integer m satisfies $1 \leq m \leq \ell$ and $\gcd(m,\ell) = d$ if and only if $1 \leq m/d \leq \ell/d$ and $\gcd(m/d, \ell/d) = 1$. Hence the number of such integers m is given by the Euler phi-function (or totient function) $\phi(\ell/d)$, which by definition is equal to the number of integers $1 \leq i \leq \ell/d$ such that $\gcd(i, \ell/d) = 1$. Recall that $\phi(k)$ can be computed by the formula

$$\phi(k) = k \prod_{\substack{p|k \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right).$$

For instance, $\phi(1000) = 1000(1 - \frac{1}{2})(1 - \frac{1}{5}) = 400$. Putting all this together gives the following formula for the cycle enumerator $Z_G(z_1, \ldots, z_\ell)$:

$$Z_G(z_1, \ldots, z_\ell) = \frac{1}{\ell} \sum_{d|\ell} \phi(\ell/d) z_{\ell/d}^d,$$

or (substituting ℓ/d for d),

$$Z_G(z_1, \ldots, z_\ell) = \frac{1}{\ell} \sum_{d|\ell} \phi(d) z_d^{\ell/d}.$$

There follows from Pólya's theorem the following result (originally proved by P. A. MacMahon (1854–1929) before Pólya discovered his general result).

7.10 Theorem.

(a) *The number $N_\ell(n)$ of n-colored necklaces of length ℓ is given by*

$$N_\ell(n) = \frac{1}{\ell} \sum_{d|\ell} \phi(\ell/d) n^d. \qquad (40)$$

(b) *We have*

$$F_G(r_1, r_2, \ldots) = \frac{1}{\ell} \sum_{d|\ell} \phi(d)(r_1^d + r_2^d + \cdots)^{\ell/d}.$$

NOTE: (b) reduces to (a) if $r_1 = r_2 = \cdots = 1$. Moreover, since clearly $N_\ell(1) = 1$, putting $n = 1$ in (40) yields the famous identity

$$\sum_{d|\ell} \phi(\ell/d) = \ell.$$

What if we are allowed to flip necklaces over, not just rotate them? Now the group becomes the dihedral group of order 2ℓ, and the corresponding inequivalent colorings are called *dihedral necklaces*. We leave to the reader to work out the cycle enumerators

$$\frac{1}{2\ell}\left(\sum_{d|\ell}\phi(d)z_d^{\ell/d}+mz_1^2z_2^{m-1}+mz_2^m\right),\quad \text{if } \ell=2m$$

$$\frac{1}{2\ell}\left(\sum_{d|\ell}\phi(d)z_d^{\ell/d}+\ell z_1z_2^m\right),\quad \text{if } \ell=2m+1.$$

7.11 Example. Let $G=\mathfrak{S}_\ell$, the group of all permutations of $\{1,2,\ldots,\ell\}=X$. Thus for instance

$$\begin{aligned} Z_{\mathfrak{S}_3}(z_1,z_2,z_3) &= \frac{1}{6}(z_1^3+3z_1z_2+2z_3) \\ Z_{\mathfrak{S}_4}(z_1,z_2,z_3,z_4) &= \frac{1}{24}(z_1^4+6z_1^2z_2+3z_2^2+8z_1z_3+6z_4). \end{aligned}$$

It is easy to count the number of inequivalent colorings in $\mathcal{C}_{\mathbf{i}}$. If two colorings of X use each color the same number of times, then clearly there is *some* permutation of X which sends one of the colorings to the other. Hence $\mathcal{C}_{\mathbf{i}}$ consists of a single orbit. Thus

$$F_{\mathfrak{S}_\ell}(r_1,r_2,\ldots)=\sum_{i_1+i_2+\cdots=\ell} r_1^{i_1}r_2^{i_2}\cdots,$$

the sum of all monomials of degree ℓ.

To count the total number of inequivalent n-colorings, note that

$$\sum_{\ell\geq 0}F_{\mathfrak{S}_\ell}(r_1,r_2,\ldots)x^\ell=\frac{1}{(1-r_1x)(1-r_2x)\cdots}. \tag{41}$$

since if we expand each factor on the right-hand side into the series $\sum_{j\geq 0} r_i^j x^j$ and multiply, the coefficient of x^ℓ will just be the sum of all monomials of degree ℓ. For fixed n, let $f_n(\ell)$ denote the number of inequivalent n-colorings of X. Since $f_n(\ell) = F_{\mathfrak{S}_\ell}(1,1,\ldots,1)$ (n 1's in all), there follows from (41) that

$$\sum_{\ell \geq 0} f_n(\ell) x^\ell = \frac{1}{(1-x)^n}.$$

The right-hand side can be expanded (e.g. by Taylor's theorem) as

$$\frac{1}{(1-x)^n} = \sum_{\ell \geq 0} \binom{n+\ell-1}{\ell} x^\ell.$$

Hence

$$f_n(\ell) = \binom{n+\ell-1}{\ell}.$$

It is natural to ask whether there might be a more direct proof of such a simple result. This is actually a standard result in elementary enumerative combinatorics. For fixed ℓ and n we want the number of solutions to $i_1 + i_2 + \cdots + i_n = \ell$ in nonnegative integers. Setting $k_j = i_j + 1$, this is the same as the number of solutions to $k_1 + k_2 + \cdots + k_n = \ell + n$ in *positive* integers. Place $\ell + n$ dots in a horizontal line. There are $\ell + n - 1$ spaces between the dots. Choose $n - 1$ of these spaces and draw a vertical bar in them in $\binom{n+\ell-1}{n-1} = \binom{n+\ell-1}{\ell}$ ways. For example, if $n = 5$ and $\ell = 6$, then one way of drawing the bars is

$$\bullet \quad \bullet \;\Big|\; \bullet \quad \bullet \quad \bullet \;\Big|\; \bullet \quad \bullet \;\Big|\; \bullet \;\Big|\; \bullet \quad \bullet \quad \bullet$$

The number of dots in each "compartment," read from left to right, gives the numbers $k_1, \ldots, k_n$. For the above example we get $2+3+2+1+3 = 11$, corresponding to the original solution $1+2+1+0+2 = 6$ (i.e., one element of X colored r_1, two elements colored r_2, one colored r_3, and two colored r_5). Since this correspondence between solutions to $i_1 + i_2 + \cdots + i_n = \ell$ and sets of bars is clearly a bijection, we get $\binom{n+\ell-1}{\ell}$ solutions as claimed.

Recall (Theorem 7.5) that the number of inequivalent n-colorings of X (with respect to any group G of permutations of X) is given by

$$\frac{1}{|G|} \sum_{\pi \in G} n^{c(\pi)},$$

where $c(\pi)$ denotes the number of cycles of π. Hence for $G = \mathfrak{S}_\ell$ we get the identity

$$\begin{aligned}\frac{1}{\ell!}\sum_{\pi\in\mathfrak{S}_\ell} n^{c(\pi)} &= \binom{n+\ell-1}{\ell} \\ &= \frac{1}{\ell!}n(n+1)(n+2)\cdots(n+\ell-1).\end{aligned}$$

Multiplying by $\ell!$ yields

$$\sum_{\pi\in\mathfrak{S}_\ell} n^{c(\pi)} = n(n+1)(n+2)\cdots(n+\ell-1). \qquad (42)$$

Equivalently [why?], if we define $c(\ell,k)$ to be the number of permutations in $\mathfrak{S}_\ell$ with k cycles (called a *signless Stirling number of the first kind*), then

$$\sum_{k=1}^{\ell} c(\ell,k)x^k = x(x+1)(x+2)\cdots(x+\ell-1).$$

For instance, $x(x+1)(x+2)(x+3) = x^4+6x^3+11x^2+6x$, so (taking the coefficient of x^2) eleven permutations in $\mathfrak{S}_4$ have two cycles, namely, (123)(4), (132)(4), (124)(3), (142)(3), (134)(2), (143)(2), (234)(1), (243)(1), (12)(34), (13)(24), (14)(23).

Although it was easy to compute the generating function $F_{\mathfrak{S}_\ell}(r_1,r_2,\ldots)$ directly without the necessity of computing the cycle indicator $Z_{\mathfrak{S}_\ell}(z_1,\ldots,z_\ell)$, we can still ask whether there is a formula of some kind for this polynomial. First we determine explicitly its coefficients.

7.12 Theorem. *Let $\sum ic_i = \ell$. The number of permutations $\pi \in \mathfrak{S}_\ell$ with c_i cycles of length i (or equivalently, the coefficient of $z_1^{c_1}z_2^{c_2}\cdots$ in $\ell! Z_{\mathfrak{S}_\ell}(z_1,\ldots,z_\ell)$) is equal to $\ell!/1^{c_1}c_1!2^{c_2}c_2!\cdots$.*

Example. The number of permutations in $\mathfrak{S}_{15}$ with three 1-cycles, two 2-cycles, and two 4-cycles is $15!/1^3\cdot 3!\cdot 2^2\cdot 2!\cdot 4^2\cdot 2! = 851,350,500$.

Proof of Theorem 7.12. Fix $\mathbf{c} = (c_1,c_2,\ldots)$ and let $X_\mathbf{c}$ be the set of all permutations $\pi \in \mathfrak{S}_\ell$ with c_i cycles of length i. Given a permutation $\sigma = a_1a_2\cdots a_\ell$ in $\mathfrak{S}_\ell$, construct a permutation $f(\sigma) \in X_\mathbf{c}$ as follows. Let the 1-cycles of $f(\sigma)$ be $(a_1), (a_2), \ldots, (a_{c_1})$. Then let the 2-cycles of $f(\sigma)$ be $(a_{c_1+1}, a_{c_1+2}), (a_{c_1+3}, a_{c_1+4}), \ldots$, $(a_{c_1+2c_2-1}, a_{c_1+2c_2})$. Then let the 3-cycles of $f(\sigma)$ be $(a_{c_1+2c_2+1}, a_{c_1+2c_2+2}, a_{c_1+2c_2+3})$, $(a_{c_1+2c_2+4}, a_{c_1+2c_2+5}, a_{c_1+2c_2+6})$, $\ldots$, $(a_{c_1+2c_2+3c_3-2}, a_{c_1+2c_2+3c_3-1}, a_{c_1+2c_2+3c_3})$, etc., continuing until we reach a_ℓ and have produced a permutation in $X_{\mathbf{c}}$. For instance, if $\ell = 11, c_1 = 3, c_2 = 2, c_4 = 1$, and $\sigma = 4, 9, 6, 11, 7, 1, 3, 8, 10, 2, 5$, then

$$f(\sigma) = (4)(9)(6)(11, 7)(1, 3)(8, 10, 2, 5).$$

We have a defined a function $f : \mathfrak{S}_\ell \to X_{\mathbf{c}}$. Given $\pi \in X_{\mathbf{c}}$, what is $\#f^{-1}(\pi)$, the number of permutations sent to π by f? A cycle of length i can be written in i ways, namely,

$$(b_1, b_2, \ldots, b_i) = (b_2, b_3, \ldots, b_i, b_1) = \cdots = (b_i, b_1, b_2, \ldots, b_{i-1}).$$

Moreover, there are $c_i!$ ways to order the c_i cycles of length i. Hence

$$\#f^{-1}(\pi) = c_1! c_2! c_3! \cdots 1^{c_1} 2^{c_2} 3^{c_3} \cdots,$$

the same number for any $\pi \in X_{\mathbf{c}}$. It follows that

$$\begin{aligned} \#X_{\mathbf{c}} &= \frac{\#\mathfrak{S}_\ell}{c_1! c_2! \cdots 1^{c_1} 2^{c_2} \cdots} \\ &= \frac{\ell!}{c_1! c_2! \cdots 1^{c_1} 2^{c_2} \cdots}, \end{aligned}$$

as was to be proved. □

As for the polynomial $Z_{\mathfrak{S}_\ell}$ itself, we have the following result.

7.13 Theorem. *We have*

$$\sum_{\ell \geq 0} Z_{\mathfrak{S}_\ell}(z_1, z_2, \ldots) x^\ell = e^{z_1 x + z_2 \frac{x^2}{2} + z_3 \frac{x^3}{3} + \cdots}.$$

Proof. There are some sophisticated ways to prove this theorem which "explain" why the exponential function appears, but we will be content here with a "naive" proof. Write

$$\begin{aligned} e^{z_1 x + z_2\frac{x^2}{2} + z_3\frac{x^3}{3}+\cdots} &= e^{zx}\cdot e^{z_2\frac{x^2}{2}}\cdot e^{z_3\frac{x^3}{3}}\cdots \\ &= \left(\sum_{n\geq 0}\frac{z_1^n x^n}{n!}\right)\left(\sum_{n\geq 0}\frac{z_2^n x^{2n}}{2^n n!}\right)\left(\sum_{n\geq 0}\frac{z_3^n x^{3n}}{3^n n!}\right)\cdots. \end{aligned}$$

When we multiply this product out, the coefficient of $z_1^{c_1} z_2^{c_2}\cdots x^\ell$, where $\ell = c_1 + 2c_2 + \cdots$, is given by

$$\frac{1}{1^{c_1}c_1!2^{c_2}c_2!\cdots} = \frac{1}{\ell!}\left(\frac{\ell!}{1^{c_1}c_1!2^{c_2}c_2!\cdots}\right).$$

By Theorem 7.12 this is just the coefficient of $z_1^{c_1} z_2^{c_2}\cdots$ in $Z_{\mathfrak{S}_\ell}(z_1, z_2, \ldots)$, as was to be proved. □

As a check of Theorem 7.13, set each $z_i = n$ to obtain

$$\begin{aligned} \sum_{\ell\geq 0} Z_{\mathfrak{S}_\ell}(n,n,\ldots)x^\ell &= e^{nx+n\frac{x^2}{2}+n\frac{x^3}{3}+\cdots} \\ &= e^{n(x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots)} \\ &= e^{n\log(1-x)^{-1}} \\ &= \frac{1}{(1-x)^n} \\ &= \sum_{\ell\geq 0}\binom{-n}{\ell}(-x)^\ell \\ &= \sum_{\ell\geq 0}\binom{n+\ell-1}{\ell}x^\ell, \end{aligned}$$

the last step following from the easily checked equality $\binom{-n}{\ell} = (-1)^\ell\binom{n+\ell-1}{\ell}$. Equating coefficients of x^ℓ in the first and last terms of the above string of equalities gives

$$\begin{aligned} Z_{\mathfrak{S}_\ell}(n,n,\ldots) &= \binom{n+\ell-1}{\ell} \\ &= \frac{n(n+1)\cdots(n+\ell-1)}{\ell!}, \end{aligned}$$

agreeing with Theorem 7.5 and equation (42).

Quotients of boolean algebra. We will show how to apply Pólya theory to the problem of counting the number of elements of given rank in a quotient poset B_X/G. Here X is a finite set, B_X is the boolean algebra of all subsets of X, and G is a group of permutations of X (with an induced action on B_X). What do colorings of X have to do with subsets? The answer is very simple: A 2-coloring $f : X \rightarrow \{0, 1\}$ corresponds to a subset S_f of X by the usual rule

$$s \in S_f \Longleftrightarrow f(s) = 1.$$

Note that two 2-colorings f and g are G-equivalent if and only if S_f and S_g are in the same orbit of G (acting on B_X). Thus the number of inequivalent 2-colorings f of X with i values equal to 1 is just $\#(B_X/G)_i$, the number of elements of B_X/G of rank i. As an immediate application of Pólya's theorem (Theorem 7.7) we obtain the following result.

7.14 Corollary. *We have*

$$\sum_i \#(B_X/G)_i q^i = Z_G(1+q, 1+q^2, 1+q^3, \ldots).$$

Proof. If $c(i, j)$ denotes the number of inequivalent 2-colorings of X with the colors 0 and 1 such that 0 is used j times and 1 is used i times (so $i + j = \#X$), then by Pólya's theorem we have

$$\sum_{i,j} c(i, j) x^i y^j = Z_G(x+y, x^2+y^2, x^3+y^3, \ldots).$$

Setting $x = q$ and $y = 1$ yields the desired result [why?]. □

Combining Corollary 7.14 with the rank-unimodality of B_X/G (Theorem 5.9) yields the following corollary.

7.15 Corollary. *For any finite group G of permutations of a finite set X, the polynomial $Z_G(1+q, 1+q^2, 1+q^3, \ldots)$ has symmetric, unimodal, integer coefficients.*

7.16 Example. (a) For the poset P of Example 5.3(a) we have $G =$

$\{(1)(2)(3),(1,2)(3)\}$, so $Z_G(z_1,z_2,z_3)=\frac{1}{2}(z_1^3+z_1z_2)$. Hence

$$\begin{aligned}\sum_{i=0}^{3}(\#P_i)q^i &= \frac{1}{2}\left((1+q)^3+(1+q)(1+q^2)\right)\\ &= 1+2q+2q^2+q^3.\end{aligned}$$

(b) For the poset P of Example 5.3(b) we have $G=\{(1)(2)(3)(4)(5),$ $(1,2,3,4,5),(1,3,5,2,4),(1,4,2,5,3),(1,5,4,3,2)\}$, so $Z_G(z_1,z_2,z_3,z_4,z_5)=\frac{1}{5}(z_1^5+4z_5)$. Hence

$$\begin{aligned}\sum_{i=0}^{5}(\#P_i)q^i &= \frac{1}{5}\left((1+q)^5+4(1+q^5)\right)\\ &= 1+q+2q^2+2q^3+q^4+q^5.\end{aligned}$$

(c) Let X be the squares of a 2×2 chessboard, labelled as follows:

1	2
3	4

Let G be the wreath product $\mathfrak{S}_2 \wr \mathfrak{S}_2$, as defined in Section 6. Then

$$\begin{aligned}G=\{&(1)(2)(3)(4),(1,2)(3)(4),(1)(2)(3,4),(1,2)(3,4),\\ &(1,3)(2,4),(1,4)(2,3),(1,3,2,4),(1,4,2,3)\},\end{aligned}$$

so

$$Z_G(z_1,z_2,z_3,z_4)=\frac{1}{8}(z_1^4+2z_1^2z_2+3z_2^2+2z_4).$$

Hence

$$\begin{aligned}\sum_{i=0}^{4}(\#P_i)q^i &= \frac{1}{4}\left((1+q)^4+2(1+q)^2(1+q^2)+3(1+q^2)^2+2(1+q^4)\right)\\ &= 1+q+2q^2+q^3+q^4\\ &= \begin{bmatrix}4\\2\end{bmatrix},\end{aligned}$$

agreeing with Theorem 6.6.

Using more sophisticated methods (such as the representation theory of the symmetric group), the following generalization of Corollary 7.15 can be proved: Let $P(q)$ be any polynomial with symmetric, unimodal, nonnegative, integer coefficients, such as $1+q+3q^2+3q^3+8q^4+3q^5+3q^6+q^7+q^8$ or q^5+q^6 $(= 0 + 0x + \cdots + 0x^4 + x^5 + x^6 + 0x^7 + \cdots + 0x^{11})$. Then the polynomial $Z_G(P(q), P(q^2), P(q^3), \ldots)$ has symmetric, unimodal, nonnegative, integer coefficients.

Graphs. A standard application of Pólya theory is to the enumeration of nonisomorphic graphs. We saw at the end of Section 5 that if M is an m-element vertex set, $X = \binom{M}{2}$, and $\mathfrak{S}_m^{(2)}$ is the group of permutations of X induced by permutations of M, then an orbit of i-element subsets of X may be regarded as an isomorphism class of graphs on the vertex set M with i-edges. Thus $\#(B_X/\mathfrak{S}_m^{(2)})_i$ is the number of nonisomorphic graphs (without loops or multiple edges) on the vertex set M with i edges. It follows from Corollary 7.14 that if $g_i(m)$ denotes the number of nonisomorphic graphs with m vertices and i edges, then

$$\sum_{i=0}^{\binom{m}{2}} g_i(m)q^i = Z_{\mathfrak{S}_m^{(2)}}(1+q, 1+q^2, 1+q^3, \ldots).$$

Thus we would like to compute the cycle enumerator $Z_{\mathfrak{S}_m^{(2)}}(z_1, z_2, \ldots)$. If two permutations π and σ of M have the same cycle type (number of cycles of each length), then their actions on X also have the same cycle type [why?]. Thus for each possible cycle type of a permutation of M (i.e., for each partition of m) we need to compute the induced cycle type on X. We also know from Theorem 7.12 the number of permutations of M of each type. For small values of m we can pick some permutation π of each type and compute directly its action on X in order to determine the induced cycle type. For $m = 4$ we have:

CYCLE LENGTHS OF π	NUMBER	π	INDUCED PERMUTATION π'	CYCLE LENGTHS OF π'
$1,1,1,1$	1	$(1)(2)(3)(4)$	$(12)(13)(14)(23)24)(34)$	$1,1,1,1,1,1$
$2,1,1$	6	$(1,2)(3)(4)$	$(12)(12,23)(14,24)(34)$	$2,2,1,1$
$3,1$	8	$(1,2,3)(4)$	$(12,23,13)(14,24,34)$	$3,3$
$2,2$	3	$(1,2)(3,4)$	$(12)(13,24)(14,23)(34)$	$2,2,1,1$
4	6	$(1,2,3,4)$	$(12,23,34,14)(13,24)$	$4,2$

It follows that

$$Z_{\mathfrak{S}_4^{(2)}}(z_1,z_2,z_3,z_4,z_5,z_6) = \frac{1}{24}(z_1^6 + 9z_1^2z_2^2 + 8z_3^2 + 6z_2z_4).$$

If we set $z_i = 1+q^i$ and simplify, we obtain the polynomial

$$\sum_{i=0}^{6} g_i(4)q^i = 1 + q + 2q^2 + 3q^3 + 2q^4 + q^5 + q^6.$$

Suppose that we instead wanted to count the number $h_i(4)$ of nonisomorphic graphs with four vertices and i edges, where now we allow at most *two* edges between any two vertices. We can take M, X, and $G = \mathfrak{S}_4^{(2)}$ as before, but now we have three colors: red for no edges, blue for one edge, and yellow for two edges. A monomial $r^ib^jy^k$ corresponds to a coloring with i pairs of vertices having no edges between them, j pairs having one edge, and k pairs having two edges. The total number e of edges is $j+2k$. Hence if we let $r=1, b=q, y=q^2$, then the monomial $r^ib^jy^k$ becomes $q^{j+2k}=q^e$. It follows that

$$\begin{aligned}\sum_{i=0}^{i(i-1)} h_i(4)q^i &= Z_{\mathfrak{S}_4^{(2)}}(1+q+q^2, 1+q^2+q^4, 1+q^3+q^6, \ldots)\\ &= \frac{1}{24}\big((1+q+q^2)^6 + 9(1+q+q^2)^2(1+q^2+q^4)^2\\ &\quad +8(1+q^3+q^6)^2 + 6(1+q^2+q^4)(1+q^4+q^8)\big)\\ &= 1 + q + 3q^2 + 5q^3 + 8q^4 + 9q^5 + 12q^6 + 9q^7 + 8q^8 + 5q^9\\ &\quad +3q^{10} + q^{11} + q^{12}.\end{aligned}$$

The total number of nonisomorphic graphs on four vertices with edge multiplicities at most two is $\sum_i h_i(4) = 66$.

It should now be clear that if we restrict the edge multiplicity to be r, then the corresponding generating function is $Z_{\mathfrak{S}_4^{(2)}}(1+q+q^2+\cdots+q^{r-1}, 1+q^2+q^4+\cdots+q^{2r-2},\ldots)$. In particular, to obtain the *total* number $N(r,4)$ of nonisomorphic graphs on four vertices with edge multiplicity at most r, we simply set each $z_i = r$, obtaining

$$\begin{aligned} N(r,4) &= Z_{\mathfrak{S}_4^{(2)}}(r,r,r,r,r,r) \\ &= \frac{1}{24}(r^6+9r^4+14r^2). \end{aligned}$$

This is the same as number of inequivalent r-colorings of the set $X = \binom{M}{2}$ (where $\#M = 4$) [why?].

Of course the same sort of reasoning can be applied to any number of vertices. For five vertices our table becomes the following (using such notation as 1^5 to denote a sequence of five 1's).

CYCLE LENGTHS OF π	NO.	π	INDUCED PERMUTATION π'	CYCLE LENGTHS OF π'
1^5	1	$(1)(2)(3)(4)(5)$	$(12)(13)\cdots(45)$	1^{10}
$2,1^3$	10	$(1,2)(3)(4)(5)$	$(12)(13,23)(14,25)(15,25)(34)(35)(45)$	$2^3,1^4$
$3,1^2$	20	$(1,2,3)(4)(5)$	$(12,23,13)(14,24,34)(15,25,35)(45)$	$3^3,1$
$2^2,1$	15	$(1,2)(3,4)(5)$	$(12)(13,24)(14,23)(15,25)(34)(35,45)$	$2^4,1^2$
$4,1$	30	$(1,2,3,4)(5)$	$(12,23,34,14)(13,24)(15,25,35,45)$	$4^2,2$
$3,2$	20	$(1,2,3)(4,5)$	$(12,23,13)(14,25,34,15,24,35)(45)$	$6,3,1$
5	24	$(1,2,3,4,5)$	$(12,23,34,45,15)(13,24,35,14,25)$	5^2

Thus

$$Z_{\mathfrak{S}_5^{(2)}}(z_1,\ldots,z_{10}) = \frac{1}{120}(z_1^{10}+10z_1^4z_2^3+20z_1z_3^3+15z_1^2z_2^4+30z_2z_4^2+20z_1z_3z_6+24z_5^2),$$

from which we compute

$$\begin{aligned} \sum_{i=0}^{10} g_i(5)q^i &= Z_{\mathfrak{S}_5^{(2)}}(1+q, 1+q^2,\ldots,1+q^{10}) \\ &= 1+q+2q^2+4q^3+6q^4+6q^5+6q^6+4q^7+2q^8+q^9+q^{10}. \end{aligned}$$

For an arbitrary number $m = \#M$ of vertices there exist explicit formulas for the cycle indicator of the induced action of $\pi \in \mathfrak{S}_M$ on $\binom{M}{2}$, thereby obviating the need to compute π' explicitly as we did in the above tables, but the overall expression for $Z_{\mathfrak{S}_m^{(2)}}$ cannot be simplified significantly or put into a simple generating function as we did in Theorem 7.13. For reference we record

$$\begin{aligned} Z_{\mathfrak{S}_6^{(2)}} &= \frac{1}{6!}(z_1^{15} + 15z_1^7 z_2^4 + 40z_1^3 z_3^4 + 45z_1^3 z_2^6 + 90z_1 z_2 z_4^3 + 120z_1 z_2 z_3^2 z_6 \\ &\quad +144z_5^3 + 15z_1^3 z_2^6 + 90z_1 z_2 z_4^3 + 40z_3^5 + 120z_3 z_6^2) \end{aligned}$$

$$(g_0(6), g_1(6), \ldots, g_{15}(6)) = (1, 1, 2, 5, 9, 15, 21, 24, 24, 21, 15, 9, 5, 2, 1, 1).$$

Moreover if $u(n)$ denotes the number of nonisomorphic simple graphs with n vertices, then

$$\begin{aligned} &(u(0), u(1), \ldots, u(11)) \\ &= (1, 1, 2, 4, 11, 34, 156, 1044, 12346, 274668, 12005168, 1018997864). \end{aligned}$$

A table of $u(n)$ for $n \leq 75$ is given at

`http://www.research.att.com/∼njas/sequences/b000088.txt`

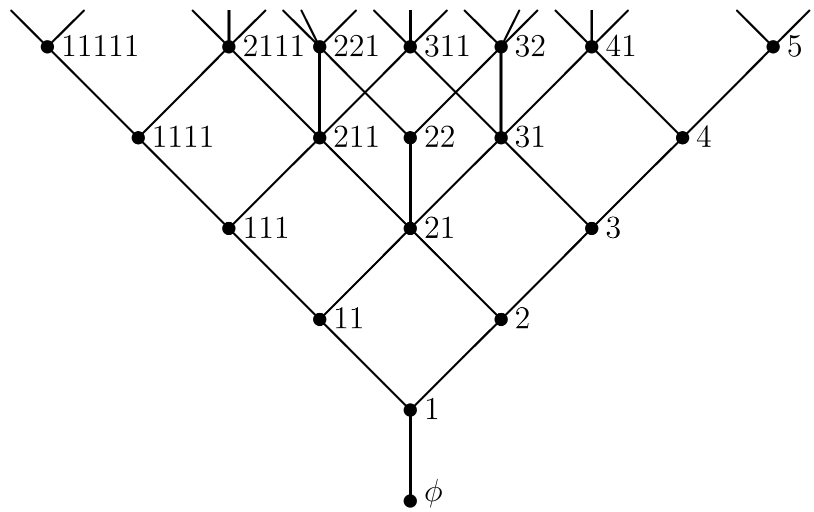
In particular,

$$\begin{aligned} u(75) = \; & 91965776790545918117055311393231179873443957239 \\ & 0555232344598910500368551136102062542965342147 \\ & 8723210428876893185920222186100317580740213865 \\ & 7140377683043095632048495393006440764501648363 \\ & 4760490012493552274952950606265577383468983364 \\ & 6883724923654397496226869104105041619919159586 \\ & 8518775275216748149124234654756641508154401414 \\ & 8480274454866344981385848105320672784068407907 \\ & 1134767688676890584660201791139593590722767979 \\ & 8617445756819562952590259920801220117529208077 \\ & 0705444809177422214784902579514964768094933848 \\ & 3173060596932480677345855848701061537676603425 \\ & 1254842843718829212212327337499413913712750831 \end{aligned}$$

0550986833980707875560051306072520155744624852
0263616216031346723897074759199703968653839368
77636080643275926566803872596099072.

8 A glimpse of Young tableaux.

We defined in Section 6 Young's lattice Y, the poset of all partitions of all nonnegative integers, ordered by containment of their Young diagrams.



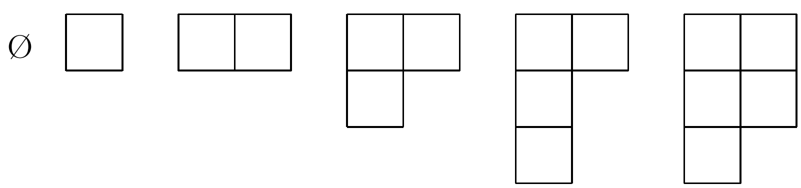
Young's lattice

Here we will be concerned with the counting of certain walks in the Hasse diagram (considered as a graph) of Y. Note that since Y is infinite, we cannot talk about its eigenvalues and eigenvectors. We need different techniques for counting walks. (It will be convenient to denote the length of a walk by n, rather than by ℓ as in previous sections.)

Note that Y is a graded poset (of infinite rank), with Y_i consisting of all partitions of i. In other words, we have $Y = Y_0 \cup Y_1 \cup \cdots$ (disjoint union), where every maximal chain intersects each level Y_i exactly once. We call Y_i the ith *level* of Y.

Since the Hasse diagram of Y is a simple graph (no loops or multiple edges), a walk of length n is specified by a sequence $\lambda^0, \lambda^1, \ldots, \lambda^n$ of vertices of Y. We will call a walk in the Hasse diagram of a poset a *Hasse walk*. Each λ^i is a partition of some integer, and we have either (a) $\lambda^i < \lambda^{i+1}$ and $|\lambda^i| = |\lambda^{i+1}| - 1$, or (b) $\lambda^i > \lambda^{i+1}$ and $|\lambda^i| = |\lambda^{i+1}| + 1$. A step of type (a) is denoted by U (for "up," since we move up in the Hasse diagram), while a step of type (b) is denoted by D (for "down"). If the walk W has steps of types $A_1, A_2, \ldots, A_n$, respectively, where each A_i is either U or D, then we say that W is of *type* $A_n A_{n-1} \cdots A_2 A_1$. Note that the type of a walk is written in the *opposite* order to that of the walk. This is because we will soon regard U and D as linear transformations, and we multiply linear transformations *right-to-left* (opposite to the usual left-to-right reading order). For instance (abbreviating a partition $(\lambda_1, \ldots, \lambda_m)$ as $\lambda_1 \cdots \lambda_m$), the walk $\emptyset, 1, 2, 1, 11, 111, 211, 221, 22, 21, 31, 41$ is of type $UUDDUUUUDUU = U^2D^2U^4DU^2$.

There is a nice combinatorial interpretation of walks of type U^n which begin at $\emptyset$. Such walks are of course just saturated chains $\emptyset = \lambda^0 < \lambda^1 < \cdots < \lambda^n$. In other words, they may be regarded as sequences of Young diagrams, beginning with the empty diagram and adding one new square at each step. An example of a walk of type U^5 is given by



We can specify this walk by taking the final diagram and inserting an i into square s if s was added at the ith step. Thus the above walk is encoded by the "tableau"

1	2
3	5
4	

Such an object τ is called a *standard Young tableaux* (or SYT). It consists of the Young diagram D of some partition λ of an integer n, together with the numbers $1, 2, \ldots, n$ inserted into the squares of D, so that each number appears exactly once, and every row and column is *increasing*. We call λ the *shape* of the SYT τ, denoted $\lambda = \mathrm{sh}(\tau)$. For instance, there are five SYT of shape $(2, 2, 1)$, given by

$$
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}
\qquad
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}
\qquad
\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}
\qquad
\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}
\qquad
\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}
$$

Let f^λ denote the number of SYT of shape λ, so for instance $f^{(2,2,1)} = 5$. The numbers f^λ have many interesting properties; for instance, there is a famous explicit formula for them known as the Frame-Robinson-Thrall hook formula. For the sake of completeness we state this formula without proof, though it is not needed in what follows.

Let u be a square of the Young diagram of the partition λ. Define the *hook* $H(u)$ of u (or at u) to be the set of all squares directly to the right of u or directly below u, including u itself. The size (number of squares) of $H(u)$ is called the *hook length* of u (or at u), denoted $h(u)$. In the diagram of the partition $(4, 2, 2)$ below, we have inserted the hook length $h(u)$ inside each square u.

$$
\begin{array}{|c|c|c|c|}\hline 6 & 5 & 2 & 1 \\ \hline 3 & 2 \\ \cline{1-2} 2 & 1 \\ \cline{1-2}\end{array}
$$

8.1 Theorem. (hook formula) *Let $\lambda \vdash n$. Then*

$$f^\lambda = \frac{n!}{\prod_{u \in \lambda} h(u)}.$$

Here the notation $u \in \lambda$ means that u ranges over all squares of the Young diagram of λ.

For instance, the diagram of the hook lengths of $\lambda = (4, 2, 2)$ above gives

$$f^\lambda = \frac{8!}{6 \cdot 5 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 2 \cdot 1} = 56.$$

In this section we will be concerned with the connection between SYT and counting walks in Young's lattice. If $w = A_n A_{n-1} \cdots A_1$ is some word in U and D and $\lambda \vdash n$, then let us write $\alpha(w, \lambda)$ for the number of Hasse walks in Y of type w which start at the empty partition $\emptyset$ and end at λ. For instance, $\alpha(UDUU, 11) = 2$, the corresponding walks being $\emptyset, 1, 2, 1, 11$ and $\emptyset, 1, 11, 1, 11$. Thus in particular $\alpha(U^n, \lambda) = f^\lambda$ [why?]. In a similar fashion, since the number of Hasse walks of type $D^n U^n$ which begin at $\emptyset$, go up to a partition $\lambda \vdash n$, and then back down to $\emptyset$ is given by $(f^\lambda)^2$, we have

$$\alpha(D^n U^n, \emptyset) = \sum_{\lambda \vdash n} (f^\lambda)^2. \tag{43}$$

Our object is to find an explicit formula for $\alpha(w, \lambda)$ of the form $f^\lambda c_w$, where c_w does not depend on λ. (It is by no means *a priori* obvious that such a formula should exist.) In particular, since $f^\emptyset = 1$, we will obtain by setting $\lambda = \emptyset$ a simple formula for the number of (closed) Hasse walks of type w from $\emptyset$ to $\emptyset$ (thus including a simple formula for (43)).

There is an easy condition for the existence of *any* Hasse walks of type w from $\emptyset$ to λ, given by the next lemma.

8.2 Lemma. *Suppose $w = D^{s_k} U^{r_k} \cdots D^{s_2} U^{r_2} D^{s_1} U^{r_1}$, where $r_i \geq 0$ and $s_i \geq 0$. Let $\lambda \vdash n$. Then there exists a Hasse walk of type w from $\emptyset$ to λ if and only if:*

$$\sum_{i=1}^{k} (r_i - s_i) = n$$

$$\sum_{i=1}^{j} (r_i - s_i) \geq 0 \text{ for } 1 \leq j \leq k.$$

Proof. Since each U moves up one level and each D moves down one level, we see that $\sum_{i=1}^k (r_i - s_i)$ is the level at which a walk of type w beginning at $\emptyset$ ends. Hence $\sum_{i=1}^k (r_i - s_i) = |\lambda| = n$.

After $\sum_{i=1}^j (r_i + s_i)$ steps we will be at level $\sum_{i=1}^j (r_i - s_i)$. Since the lowest level is level 0, we must have $\sum_{i=1}^{j}(r_i - s_i) \geq 0$ for $1 \leq j \leq k$.

The easy proof that the two conditions of the lemma are *sufficient* for the existence of a Hasse walk of type w from $\emptyset$ to λ is left to the reader. $\square$

If w is a word in U and D satisfying the conditions of Lemma 8.2, then we say that w is a *valid λ-word*. (Note that the condition of being a valid λ-word depends only on $|\lambda|$.)

The proof of our formula for $\alpha(w, \lambda)$ will be based on linear transformations analogous to those defined by (18) and (19). As in Section 4 let $\mathbb{R}Y_j$ be the real vector space with basis Y_j. Define two linear transformations $U_i : \mathbb{R}Y_i \to \mathbb{R}Y_{i+1}$ and $D_i : \mathbb{R}Y_i \to \mathbb{R}Y_{i-1}$ by

$$U_i(\lambda) = \sum_{\substack{\mu \vdash i+1 \\ \lambda < \mu}} \mu$$

$$D_i(\lambda) = \sum_{\substack{\nu \vdash i-1 \\ \nu < \lambda}} \nu,$$

for all $\lambda \vdash i$. For instance (using abbreviated notation for partitions)

$$U_{21}(54422211) = 64422211 + 55422211 + 54432211 + 54422221 + 544222111$$

$$D_{21}(54422211) = 44422211 + 54322211 + 54422111 + 5442221.$$

It is clear [why?] that if r is the number of *distinct* (i.e., unequal) parts of λ, then $U_i(\lambda)$ is a sum of $r + 1$ terms and $D_i(\lambda)$ is a sum of r terms. The next lemma is an analogue for Y of the corresponding result for B_n (Lemma 4.6).

8.3 Lemma. *For any $i \geq 0$ we have*

$$D_{i+1}U_i - U_{i-1}D_i = I_i, \tag{44}$$

the identity linear transformation on $\mathbb{R}Y_i$.

Proof. Apply the left-hand side of (44) to a partition λ of i, expand in terms of the basis Y_i, and consider the coefficient of a partition μ. If $\mu \neq \lambda$ and μ can be obtained from λ by adding one square s to (the Young diagram of) λ and then removing a (necessarily different) square t, then there is exactly one choice of s and t. Hence the coefficient of μ in $D_{i+1}U_i(\lambda)$ is equal to 1. But then there is exactly one way to remove a square from λ and then add a square to get μ, namely, remove t and add s. Hence the coefficient of μ in $U_{i-1}D_i(\lambda)$ is also 1, so the coefficient of μ when the left-hand side of (44) is applied to λ is 0.

If now $\mu \neq \lambda$ and we cannot obtain μ by adding a square and then deleting a square from λ (i.e., μ and λ differ in more than two rows), then clearly when we apply the left-hand side of (44) to λ, the coefficient of μ will be 0.

Finally consider the case $\lambda = \mu$. Let r be the number of distinct (unequal) parts of λ. Then the coefficient of λ in $D_{i+1}U_i(\lambda)$ is $r+1$, while the coefficient of λ in $U_{i-1}D_i(\lambda)$ is r, since there are $r+1$ ways to add a square to λ and then remove it, while there are r ways to remove a square and then add it back in. Hence when we apply the left-hand side of (44) to λ, the coefficient of λ is equal to 1.

Combining the conclusions of the three cases just considered shows that the left-hand side of (44) is just I_i, as was to be proved. □

We come to one of the main results of this section.

8.4 Theorem. *Let λ be a partition and $w = A_n A_{n-1} \cdots A_1$ a valid λ-word. Let $S_w = \{i : A_i = D\}$. For each $i \in S_w$, let a_i be the number of D's in w to the right of A_i, and let b_i be the number of U's in w to the right of A_i. Then*

$$\alpha(w, \lambda) = f^\lambda \prod_{i \in S_w} (b_i - a_i).$$

Before proving Theorem 8.4, let us give an example. Suppose $w = U^3 D^2 U^2 D U^3 = UUUDDUUDUUU$ and $\lambda = (2, 2, 1)$. Then $S_w = \{4, 7, 8\}$ and $a_4 = 0$, $b_4 = 3$, $a_7 = 1$, $b_7 = 5$, $a_8 = 2$, $b_8 = 5$. We have also seen earlier that $f^{221} = 5$. Thus

$$\alpha(w, \lambda) = 5(3 - 0)(5 - 1)(5 - 2) = 180.$$

Proof of Theorem 8.4. [to be replaced by a simpler proof] For notational simplicity we will omit the subscripts from the linear transformations U_i and D_i. This should cause no confusion since the subscripts will be uniquely determined by the elements on which U and D act. For instance, the expression $UDUU(\lambda)$ where $\lambda \vdash i$ must mean $U_{i+1}D_{i+2}U_{i+1}U_i(\lambda)$; otherwise it would be undefined since U_j and D_j can only act on elements of $\mathbb{R}Y_j$, and moreover U_j raises the level by one while D_j lowers it by one.

By (44) we can replace DU in any word y in the letters U and D by $UD + I$. This replaces y by a sum of two words, one with one fewer D and the other with one D moved one space to the right. For instance, replacing the first DU in $UUDUDDU$ by $UD + I$ yields $UUUDDDU + UUDDU$. If we begin with the word w and iterate this procedure, replacing a DU in any word with $UD+I$, eventually there will be no U's to the right of any D's and the procedure will come to an end. At this point we will have expressed w as a linear combination (with integer coefficients) of words of the form U^iD^j. Since the operation of replacing DU with $UD + I$ preserves the difference between the number of U's and D's in each word, all the words U^iD^j which appear will have $i - j$ equal to some constant n (namely, the number of U's minus the number of D's in w). Specifically, say we have

$$w = \sum_{i-j=n} r_{ij}(w) U^i D^j, \tag{45}$$

where each $r_{ij}(w) \in \mathbb{Z}$. (We also define $r_{ij}(w) = 0$ if $i < 0$ or $j < 0$.) We claim that the $r_{ij}(w)$'s are uniquely determined by w. Equivalently [why?], if we have

$$\sum_{i-j=n} d_{ij} U^i D^j = 0 \tag{46}$$

(as an identity of linear transformations acting on the space $\mathbb{R}Y_k$ for *any* k), where each $d_{ij} \in \mathbb{Z}$ (or $d_{ij} \in \mathbb{R}$, if you prefer), then each $d_{ij} = 0$. Let j' be the least integer for which $d_{j'+n,j'} \neq 0$. Let $\mu \vdash j'$, and apply both sides of (46) to μ. The left-hand side has exactly one nonzero term, namely, the term with $j = j'$ [why?]. The right-hand side, on the other hand[1], is 0, a contradiction. Thus the $r_{ij}(w)$'s are unique.

[1]The phrase "the right-hand side, on the other hand" does not mean the left-hand side!

Now apply U on the left to (45). We get

$$Uw = \sum_{i,j} r_{ij}(w) U^{i+1} D^j.$$

Hence (using uniqueness of the r_{ij}'s) there follows [why?]

$$r_{ij}(Uw) = r_{i-1,j}(w). \tag{47}$$

We next want to apply D on the left to (45). It is easily proved by induction on i (left as an exercise) that

$$DU^i = U^i D + iU^{i-1}. \tag{48}$$

(We interpret U^{-1} as being 0, so that (48) is true for $i = 0$.) Hence

$$\begin{aligned} Dw &= \sum_{i,j} r_{ij}(w) DU^i D^j \\ &= \sum_{i,j} r_{ij}(w)(U^i D + iU^{i-1}) D^j, \end{aligned}$$

from which it follows [why?] that

$$r_{ij}(Dw) = r_{i,j-1}(w) + (i+1) r_{i+1,j}(w). \tag{49}$$

Setting $j = 0$ in (47) and (49) yields

$$r_{i0}(Uw) = r_{i-1,0}(w) \tag{50}$$

$$r_{i0}(Dw) = (i+1) r_{i+1,0}(w). \tag{51}$$

Now let (45) operate on $\emptyset$. Since $D^j(\emptyset) = 0$ for all $j > 0$, we get $w(\emptyset) = r_{n0}(w) U^n(\emptyset)$. Thus the coefficient of λ in $w(\emptyset)$ is given by

$$\alpha(w, \lambda) = r_{n0}(w) \alpha(U^n, \lambda) = r_{n0} f^\lambda,$$

where as usual $\lambda \vdash n$. It is easy to see from (50) and (51) that

$$r_{n0}(w) = \prod_{j \in S_w} (b_j - a_j),$$

and the proof follows. □

An interesting special case of the previous theorem allows us to evaluate equation (43).

8.5 Corollary. *We have*

$$\alpha(D^nU^n, \varnothing) = \sum_{\lambda\vdash n} (f^\lambda)^2 = n!$$

Proof. When $w = D^nU^n$ in Theorem 8.4 we have $S_w = \{n+1, n+2, \ldots, 2n\}$, $a_i = n - i$, and $b_i = n$, from which the proof is immediate. □

Note (for those familiar with the representation theory of finite groups). It can be shown that the numbers f^λ, for $\lambda \vdash n$, are the degrees of the irreducible representations of the symmetric group $\mathcal{S}_n$. Given this, Corollary 8.5 is a special case of the result that the sum of the squares of the degrees of the irreducible representations of a finite group G is equal to the order $|G|$ of G. There are many other intimate connections between the representation theory of $\mathcal{S}_n$, on the one hand, and the combinatorics of Young's lattice and Young tableaux, on the other. There is also an elegant combinatorial proof of Corollary 8.5, based on the *RSK algorithm* (after Gilbert de Beauregard Robinson, Craige Schensted, and Donald Knuth) or *Robinson-Schensted correspondence*, with many fascinating properties and with deep connections with representation theory. In the Appendix to this section we give the definition of the RSK algorithm.

We now consider a variation of Theorem 8.4 in which we are not concerned with the type w of a Hasse walk from $\varnothing$ to w, but only with the number of steps. For instance, there are three Hasse walks of length three from $\varnothing$ to the partition 1, given by $\varnothing, 1, \varnothing, 1$; $\varnothing, 1, 2, 1$; and $\varnothing, 1, 11, 1$. Let $\beta(\ell, \lambda)$ denote the number of Hasse walks of length ℓ from $\varnothing$ to λ. Note the two following easy facts:

(F1) $\beta(\ell, \lambda) = 0$ unless $\ell \equiv |\lambda| \pmod 2$.

(F2) $\beta(\ell, \lambda)$ is the coefficient of λ in the expansion of $(D+U)^\ell(\varnothing)$ as a linear combination of partitions.

Because of (F2) it is important to write $(D+U)^\ell$ as a linear combination of terms U^iD^j, just as in the proof of Theorem 8.4 we wrote a word w in U and D in this form. Thus define integers $b_{ij}(\ell)$ by

$$(D+U)^\ell = \sum_{i,j} b_{ij}(\ell)U^iD^j. \tag{52}$$

Just as in the proof of Theorem 8.4, the numbers $b_{ij}(\ell)$ exist and are well-defined.

8.6 Lemma. *We have $b_{ij}(\ell)=0$ if $\ell-i-j$ is odd. If $\ell-i-j=2m$ then*

$$b_{ij}(\ell) = \frac{\ell!}{2^m\, i!\, j!\, m!}. \tag{53}$$

Proof. The assertion for $\ell-i-j$ odd is equivalent to (F1) above, so assume $\ell-i-j$ is even. The proof is by induction on ℓ. It's easy to check that (53) holds for $\ell=1$. Now assume true for some fixed $\ell\geq 1$. Using (52) we obtain

$$\begin{aligned}
\sum_{i,j} b_{ij}(\ell+1)U^iD^j &= (D+U)^{\ell+1}\\
&= (D+U)\sum_{i,j} b_{ij}(\ell)U^iD^j\\
&= \sum_{i,j} b_{ij}(\ell)(DU^iD^j+U^{i+1}D^j).
\end{aligned}$$

In the proof of Theorem 8.4 we saw that $DU^i=U^iD+iU^{i-1}$ (see equation (48)). Hence we get

$$\sum_{i,j} b_{ij}(\ell+1)U^iD^j = \sum_{i,j} b_{ij}(\ell)(U^iD^{j+1}+iU^{i-1}D^j+U^{i+1}D^j). \tag{54}$$

As mentioned after (52), the expansion of $(D+U)^{\ell+1}$ in terms of U^iD^j is unique. Hence equating coefficients of U^iD^j on both sides of (54) yields the recurrence

$$b_{ij}(\ell+1) = b_{i,j-1}(\ell) + (i+1)b_{i+1,j}(\ell) + b_{i-1,j}(\ell). \tag{55}$$

It is a routine matter to check that the function $\ell!/2^m i!j!m!$ satisfies the same recurrence (55) as $b_{ij}(\ell)$, with the same intial condition $b_{00}(0) = 1$. From this the proof follows by induction. □

From Lemma 8.6 it is easy to prove the following result.

8.7 Theorem. *Let $\ell \geq n$ and $\lambda \vdash n$, with $\ell - n$ even. Then*

$$\beta(\ell, \lambda) = \binom{\ell}{n}(1 \cdot 3 \cdot 5 \cdots (\ell - n - 1))f^\lambda.$$

Proof. Apply both sides of (52) to $\varnothing$. Since $U^i D^j(\varnothing) = 0$ unless $j = 0$, we get

$$\begin{aligned} (D+U)^\ell(\varnothing) &= \sum_i b_{i0}(\ell) U^i(\varnothing) \\ &= \sum_i b_{i0}(\ell) \sum_{\lambda \vdash i} f^\lambda \lambda. \end{aligned}$$

Since by Lemma 8.6 we have $b_{i0}(\ell) = \binom{\ell}{i}(1 \cdot 3 \cdot 5 \cdots (\ell - i - 1))$ when $\ell - i$ is even, the proof follows from (F2). □

NOTE. The proof of Theorem 8.7 only required knowing the value of $b_{i0}(\ell)$. However, in Lemma 8.6 we computed $b_{ij}(\ell)$ for all j. We could have carried out the proof so as only to compute $b_{i0}(\ell)$, but the general value of $b_{ij}(\ell)$ is so simple that we have included it too.

8.8 Corollary. *The total number of Hasse walks in Y of length $2m$ from $\varnothing$ to $\varnothing$ is given by*

$$\beta(2m, \varnothing) = 1 \cdot 3 \cdot 5 \cdots (2m - 1).$$

Proof. Simply substitute $\lambda = \varnothing$ (so $n = 0$) and $\ell = 2m$ in Theorem 8.7. □

The fact that we can count various kinds of Hasse walks in Y suggests that there may be some finite graphs related to Y whose eigenvalues we can also compute. This is indeed the case, and we will discuss the simplest case here. Let $Y_{j-1,j}$ denote the restriction of Young's lattice Y to ranks $j-1$ and j. Identify $Y_{j-1,j}$ with its Hasse diagram, regarded as a (bipartite) graph. Let $p(i) = |Y_i|$, the number of partitions of i. (The function $p(i)$ has been extensively studied, beginning with Euler, though we will not discuss its fascinating properties here.)

8.9 Theorem. *The eigenvalues of $Y_{j-1,j}$ are given as follows: 0 is an eigenvalue of multiplicity $p(j) - p(j-1)$; and for $1 \le s \le j$, the numbers $\pm\sqrt{s}$ are eigenvalues of multiplicity $p(j-s) - p(j-s-1)$.*

Proof. Let $\boldsymbol{A}$ denote the adjacency matrix of $Y_{j-1,j}$. Since $\mathbb{R}Y_{j-1,j} = \mathbb{R}Y_{j-1} \oplus \mathbb{R}Y_j$ (vector space direct sum), any vector $v \in \mathbb{R}Y_{j-1,j}$ can be written uniquely as $v = v_{j-1} + v_j$, where $v_i \in \mathbb{R}Y_i$. The matrix $\boldsymbol{A}$ acts on the vector space $\mathbb{R}Y_{j-1,j}$ as follows [why?]:

$$\boldsymbol{A}(v) = D(v_j) + U(v_{j-1}). \tag{56}$$

Just as Theorem 4.7 followed from Lemma 4.6, we deduce from Lemma 8.3 that for any i we have that $U_i : \mathbb{R}Y_i \to \mathbb{R}Y_{i+1}$ is one-to-one and $D_i : \mathbb{R}Y_i \to \mathbb{R}Y_{i-1}$ is onto. It follows in particular that

$$\begin{aligned} \dim(\ker(D_i)) &= \dim \mathbb{R}Y_i - \dim \mathbb{R}Y_{i-1} \\ &= p(i) - p(i-1), \end{aligned}$$

where ker denotes kernel.

Case 1. Let $v \in \ker(D_j)$, so $v = v_j$. Then $\boldsymbol{A}v = Dv = 0$. Thus $\ker(D_j)$ is an eigenspace of $\boldsymbol{A}$ for the eigenvalue 0, so 0 is an eigenvalue of multiplicity at least $p(j) - p(j-1)$.

Case 2. Let $v \in \ker(D_s)$ for some $0 \le s \le j-1$. Let

$$v^* = \pm\sqrt{j-s}\,U^{j-1-s}(v) + U^{j-s}(v).$$

Note that $v^* \in \mathbb{R}Y_{j-1,j}$, with $v^*_{j-1} = \pm\sqrt{j-s}\,U^{j-1-s}(v)$ and $v^*_j = U^{j-s}(v)$. Using equation (48), we compute

$$\boldsymbol{A}(v^*) = U(v^*_{j-1}) + D(v^*_j)$$

$$
\begin{aligned}
&= \pm\sqrt{j-s}U^{j-s}(v) + DU^{j-s}(v) \\
&= \pm\sqrt{j-s}U^{j-s}(v) + U^{j-s}D(v) + (j-s)U^{j-s-1}(v) \\
&= \pm\sqrt{j-s}U^{j-s}(v) + (j-s)U^{j-s-1}(v) \\
&= \pm\left(\sqrt{j-s}\right) v^*. \qquad (57)
\end{aligned}
$$

It's easy to verify (using the fact that U is one-to-one) that if $v(1), \ldots, v(t)$ is a basis for $\ker(D_s)$, then $v(1)^*, \ldots, v(t)^*$ are linearly independent. Hence by (57) we have that $\pm\sqrt{j-s}$ is an eigenvalue of $\boldsymbol{A}$ of multiplicity at least $t = \dim(\ker(D_s)) = p(s) - p(s-1)$.

We have found a total of

$$
p(j) - p(j-1) + 2\sum_{s=0}^{j-1}(p(s) - p(s-1)) = p(j-1) + p(j)
$$

eigenvalues of $\boldsymbol{A}$. (The factor 2 above arises from the fact that both $+\sqrt{j-s}$ and $-\sqrt{j-s}$ are eigenvalues.) Since the graph $Y_{j-1,j}$ has $p(j-1)+p(j)$ vertices, we have found all its eigenvalues. □

An elegant combinatorial consequence of Theorem 8.9 is the following.

8.10 Corollary. *Fix $j \geq 1$. The number of ways to choose a partition λ of j, then delete a square from λ (keeping it a partition), then insert a square, then delete a square, etc., for a total of m insertions and m deletions, ending back at λ, is given by*

$$
\sum_{s=1}^{j}[p(j-s) - p(j-s-1)]s^m, \; m > 0. \qquad (58)
$$

Proof. Exactly half the closed walks in $Y_{j-1,j}$ of length $2m$ begin at an element of Y_j [why?]. Hence if $Y_{j-1,j}$ has eigenvalues $\theta_1, \ldots, \theta_r$, then by Corollary 1.3 the desired number of walks is given by $\frac{1}{2}(\theta_1^{2m} + \cdots + \theta_r^{2m})$. Using the values of $\theta_1, \ldots, \theta_r$ given by Theorem 8.9 yields (58). □

For instance, when $j = 7$, equation (58) becomes $4 + 2 \cdot 2^m + 2 \cdot 3^m +$

$4^m + 5^m + 7^m$. When $m = 1$ we get 30, the number of edges of the graph $Y_{6,7}$ [why?].

APPENDIX: THE RSK ALGORITHM

We will describe a bijection between permutations $w \in \mathfrak{S}_n$ and pairs (P, Q) of SYT of the same shape $\lambda \vdash n$. Define a *near Young tableau* (NYT) to be the same as an SYT, except that the entries can be any distinct integers, not necessarily the integers $1, 2, \ldots, n$. Let P_{ij} denote the entry in row i and column j of P. The basic operation of the RSK-algorithm consists of the *row insertion* $P \leftarrow k$ of a positive integer k into an NYT $P = (P_{ij})$. The operation $P \leftarrow k$ is defined as follows: Let r be the largest integer such that $P_{1,r-1} < k$. (If $P_{11} > k$ then let $r = 1$.) If P_{1r} doesn't exist (i.e., P has $r-1$ columns), then simply place k at the end of the first row. The insertion process stops, and the resulting NYT is $P \leftarrow k$. If, on the other hand, P has at least r columns so that P_{1r} exists, then replace P_{1r} by k. The element k then "bumps" $P_{1r} := k'$ into the second row, i.e., insert k' into the second row of P by the insertion rule just described. Continue until an element is inserted at the end of a row (possibly as the first element of a new row). The resulting array is $P \leftarrow k$.

8.11 Example. Let

$$P = \begin{array}{llll} 3 & 7 & 9 & 14 \\ 6 & 11 & 12 & \\ 10 & 16 & & \\ 13 & & & \\ 15 & & & \end{array}$$

Then $P \leftarrow 8$ is shown below, with the elements inserted into each row (either by bumping or by the final insertion in the fourth row) in boldface. Thus the 8 bumps the 9, the 9 bumps the 11, the 11 bumps the 16, and the 16 is inserted at the end of a row. Hence

$$(P \leftarrow 8) = \begin{array}{llll} 3 & 7 & \mathbf{8} & 14 \\ 6 & \mathbf{9} & 12 & \\ 10 & \mathbf{11} & & \\ 13 & \mathbf{16} & & \\ 15. & & & \end{array}$$

We omit the proof, which is fairly straightforward, that if P is an NYT, then so is $P \leftarrow k$. We can now describe the RSK algorithm. Let $w =$

$a_1a_2\cdots a_n \in \mathfrak{S}_n$. We will inductively construct a sequence (P_0, Q_0), (P_1, Q_1), $\dots, (P_n, Q_n)$ of pairs (P_i, Q_i) of NYT of the same shape, where P_i and Q_i each have i squares. First, define $(P_0, Q_0) = (\emptyset, \emptyset)$. If (P_{i-1}, Q_{i-1}) have been defined, then set $P_i = P_{i-1} \leftarrow a_i$. In other words, P_i is obtained from P_{i-1} by row inserting a_i. Now define Q_i to be the NYT obtained from Q_{i-1} by inserting i so that Q_i and P_i have the same shape. (The entries of Q_{i-1} don't change; we are simply placing i into a certain new square and not row-inserting it into Q_{i-1}.) Finally let $(P, Q) = (P_n, Q_n)$.

8.12 Example. Let $w = 4273615 \in \mathfrak{S}_7$. The pairs $(P_1, Q_1), \dots,$ $(P_7, Q_7) = (P, Q)$ are as follows:

P_i	Q_i
4	1
2 4	1 2
2 7 4	1 3 2
2 3 4 7	1 3 2 4
2 3 6 4 7	1 3 5 2 4
1 3 6 2 7 4	1 3 5 2 4 6
1 3 5 2 6 4 7	1 3 5 2 4 6 7

8.13 Theorem. The RSK algorithm defines a bijection between the symmetric group $\mathfrak{S}_n$ and the set of all pairs (P, Q) of SYT of the same shape, where the shape λ is a partition of n.

Proof (sketch). The key step is to define the inverse of RSK. In other words, if $w \mapsto (P, Q)$, then how can we recover w uniquely from (P, Q)? Moreover, we need to find w for *any* (P, Q). Observe that the position occupied by n in Q is the last position to be occupied in the insertion process. Suppose that k occupies this position in P. It was bumped into this position by some element j in the row above k that is currently the largest element of its row less than k. Hence we can "inverse bump" k into the position occupied by j, and now inverse bump j into the row above it by the same procedure. Eventually an element will be placed in the first row, inverse bumping another element t out of the tableau altogether. Thus t was the last element of w to be inserted, i.e., if $w = a_1 a_2 \cdots a_n$ then $a_n = t$. Now locate the position occupied by $n-1$ in Q and repeat the procedure, obtaining a_{n-1}. Continuing in this way, we uniquely construct w one element at a time from right-to-left, such that $w \mapsto (P, Q)$. $\square$