11 Cycles, bonds, and electrical networks.

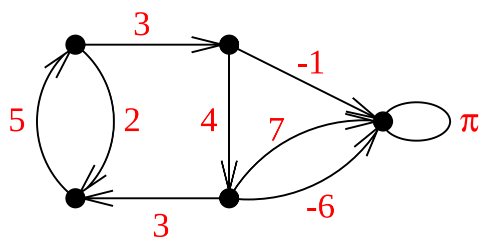
NOTE. This section is in a preliminary form.

11.1 The cycle space and bond space.

In this section we will deal with some interesting linear algebra related to the structure of a directed graph. Let $D = (V, E)$ be a digraph. A function $f : E \to \mathbb{R}$ is called a *circulation* if for every vertex $v \in V$, we have

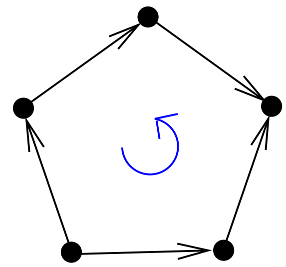
$$\sum_{\substack{e \in E \\ \mathrm{init}(e)=v}} f(e) = \sum_{\substack{e \in E \\ \mathrm{fin}(e)=v}} f(e). \tag{66}$$

Thus if we think of the edges as pipes and f as measuring the flow (quantity per unit of time) of some commodity (such as oil) through the pipe in the specified direction (so that a negative value of $f(e)$ means a flow of $|f(e)|$ in the direction opposite the direction of e), then equation (66) simply says that the amount flowing into each vertex equals the amount flowing out. In other words, the flow is *conservative*. The figure below illustrates a circulation in a digraph D.



Let $\mathcal{C} = \mathcal{C}_D$ denote the set of all circulations on D. Clearly if $f, g \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{R}$ then $\alpha f + \beta g \in \mathcal{C}$. Hence $\mathcal{C}$ is a (real) vector space, called the *cycle space* of D. Thus if $q = |E|$, then $\mathcal{C}_D$ is a subspace of the q-dimensional vector space $\mathbb{R}^E$ of all functions $f : E \to \mathbb{R}$.

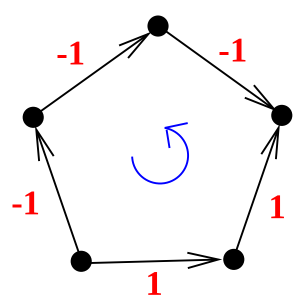
What do circulations have do with something "circulating," and what does the cycle space have to do with actual cycles? To see this, define a *circuit* or *elementary cycle* in D to be a set of edges of a closed walk, *ignoring* *the direction of the arrows*, with no repeated vertices except the first and last. Suppose that C has been assigned an orientation (direction of travel) $\mathfrak{o}$. (Note that this meaning of orientation is not the same as that appearing in Definition 9.5.)



Define a function $f_C : E \to \mathbb{R}$ (which also depends on the orientation $\mathfrak{o}$, though we suppress it from the notation) by

$$f_C(e) = \begin{cases} 1, & \text{if } e \in C \text{ and } e \text{ agrees with } \mathfrak{o} \\ -1, & \text{if } e \in C \text{ and } e \text{ is opposite to } \mathfrak{o} \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that f_C is a circulation. Later we will see that the circulations f_C span the cycle space $\mathcal{C}$, explaining the terminology "circulation" and "cycle space." The figure below shows a circuit C with an orientation $\mathfrak{o}$, and the corresponding circulation f_C.



Given a function $p : V \to \mathbb{R}$, define a new function $\delta p : E \to \mathbb{R}$, called the *coboundary*[3] of p, by

$$\delta p(e) = p(v) - p(u), \quad \text{if } u = \text{init}(e) \text{ and } v = \text{fin}(e).$$

[3] The term "coboundary" arises from algebraic topology, but we will not explain the connection here.

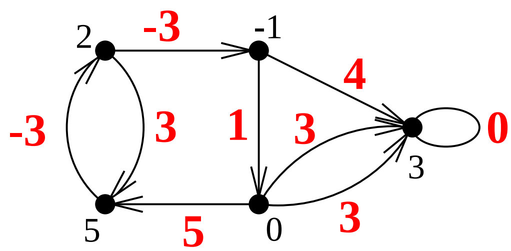


Figure 2: A function and its coboundary

Figure 2 shows a digraph D with the value $p(v)$ of some function $p : V \to \mathbb{R}$ indicated at each vertex v, and the corresponding values $\delta p(e)$ shown at each edge e.

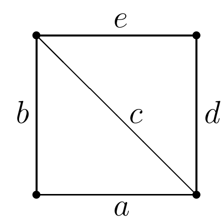
One should regard δ as an operator which takes an element p of the vector space $\mathbb{R}^V$ of all functions $V \to \mathbb{R}$ and produces an element of the vector space $\mathbb{R}^E$ of all functions $E \to \mathbb{R}$. It is immediate from the definition of δ that δ is *linear*, i.e.,

$$\delta(\alpha p + \beta q) = \alpha \cdot \delta p + \beta \cdot \delta q,$$

for all $p, q \in \mathbb{R}^V$ and $\alpha, \beta \in \mathbb{R}$. Thus δ is simply a linear transformation $\delta : \mathbb{R}^V \to \mathbb{R}^E$ between two finite-dimensional vector spaces.

A function $g : E \to \mathbb{R}$ is called a *potential difference* on D if $g = \delta p$ for some $p : V \to \mathbb{R}$. (Later we will see the connection with electrical networks that accounts for the terminology "potential difference.") Let $\mathcal{B} = \mathcal{B}_D$ be the set of all potential differences on D. Thus $\mathcal{B}$ is just the image of the linear transformation δ and is hence a real vector space, called the *bond space* of D.

Let us explain the reason behind the terminology "bond space." A *bond* in a digraph D is a set B of edges such that (a) removing B from D disconnects some (undirected) component of D (that is, removing B creates a digraph which has more connected components, as an undirected graph, than D), and (b) no proper subset of B has this property. A subset of edges satisfying (a) is called a *cutset*, so a bond is just a minimal cutset. Suppose, for example, that D is given as follows (with no arrows drawn since they are irrelevant to the definition of bond):



Then the bonds are the six subsets $ab, de, acd, bce, ace, bcd$.

Let B be a bond. Suppose B disconnects the component (V', E') into two pieces (a bond always disconnects some component into exactly two pieces [why?]) with vertex set S in one piece and $\bar{S}$ in the other. Thus $S \cup \bar{S} = V'$ and $S \cap \bar{S} = \emptyset$. Define

$$[S, \bar{S}] = \{e \in E : \text{exactly one vertex of } e \text{ lies in } S \text{ and one lies in } \bar{S}\}.$$

Clearly $B = [S, \bar{S}]$. It is often convenient to use the notation $[S, \bar{S}]$ for a bond.

Given a bond $B = [S, \bar{S}]$ of D, define a function $g_B : E \to \mathbb{R}$ by

$$g_B(e) = \begin{cases} 1, & \text{if } \operatorname{init}(e) \in \bar{S},\ \operatorname{fin}(e) \in S \\ -1, & \text{if } \operatorname{init}(e) \in S,\ \operatorname{fin}(e) \in \bar{S} \\ 0, & \text{otherwise.} \end{cases}$$

Note that g_B really depends not just on B, but on whether we write B as $[S, \bar{S}]$ or $[\bar{S}, S]$. Writing B in the reverse way simply changes the sign of g_B. Whenever we deal with g_B we will assume that some choice $B = [S, \bar{S}]$ has been made.

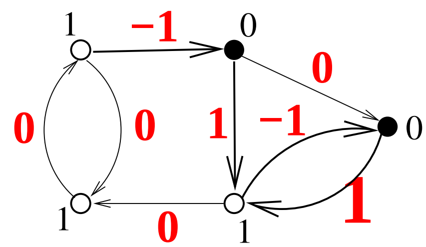
Now note that $g_B = \delta p$, where

$$p(v) = \begin{cases} 1, & \text{if } v \in S \\ 0, & \text{if } v \in \bar{S}. \end{cases}$$

Hence $g_B \in \mathcal{B}$, the bond space of D. We will later see that $\mathcal{B}$ is in fact spanned by the functions g_B, explaining the termininology "bond space."

11.1 Example. In the digraph below, open (white) vertices indicate an element of S and closed (black) vertices an element of $\bar{S}$ for a certain bond

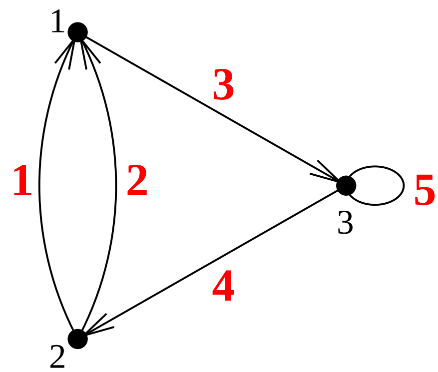
$B = [S, \bar{S}]$. The elements of B are drawn darker than the other edges. The edges are labelled by the values of g_B, and the vertices by the function p for which $g_B = \delta p$.



Recall that in Definition 9.5 we defined the incidence matrix $\boldsymbol{M}(G)$ of a loopless undirected graph G with respect to an orientation $\mathfrak{o}$. We may just as well think of G together with its orientation $\mathfrak{o}$ as a directed graph. We also will allow loops. Thus if $D = (V, E)$ is any (finite) digraph, define the *incidence matrix* $\boldsymbol{M} = \boldsymbol{M}(D)$ to be the $p \times q$ matrix whose rows are indexed by V and columns by E, as follows. The entry in row $v \in V$ and column $e \in E$ is denoted $m_v(e)$ and is given by[4]

$$m_v(e) = \begin{cases} -1, & \text{if } v = \text{init}(e) \text{ and } e \text{ is not a loop} \\ 1, & \text{if } v = \text{fin}(e) \text{ and } e \text{ is not a loop} \\ 0, & \text{otherwise.} \end{cases}$$

For instance, if D is given by



[4]Actually, this definition gives the *negative* of the matrix defined in Definition 9.5, though it makes no difference here. We will fix this inconsistency of notation in a later version of these notes.

then

$$\boldsymbol{M}(D) = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}.$$

11.2 Theorem. *The row space of $\boldsymbol{M}(D)$ is the bond space $\mathcal{B}_D$. Equivalently, the functions $m_v : E \to \mathbb{R}$, where v ranges over all vertices of D, span $\mathcal{B}$.*

Proof. Let $g = \delta p$ be a potential difference on D, so

$$\begin{aligned} g(e) &= p(\mathrm{fin}(e)) - p(\mathrm{init}(e)) \\ &= \sum_{v \in V} p(v) m_v(e). \end{aligned}$$

Thus $g = \sum_{v \in V} p(v) m_v$, so g belongs to the row space of $\boldsymbol{M}$.

Conversely, if $g = \sum_{v \in V} q(v) m_v$ is in the row space of $\boldsymbol{M}$, where $q : V \to \mathbb{R}$, then $g = \delta q \in \mathcal{B}$. $\square$

We now define a scalar product (or inner product) on the space $\mathbb{R}^E$ by

$$\langle f, g \rangle = \sum_{e \in E} f(e) g(e),$$

for any $f, g \in \mathbb{R}^E$. If we think of the numbers $f(e)$ and $g(e)$ as the coordinates of f and g with respect to the basis E, then $\langle f, g \rangle$ is just the usual dot product of f and g. Because we have a scalar product, we have a notion of what it means for f and g to be *orthogonal*, viz., $\langle f, g \rangle = 0$. If $\mathcal{V}$ is any subspace of $\mathbb{R}^E$, then define the *orthogonal complement* $\mathcal{V}^\perp$ of $\mathcal{V}$ by

$$\mathcal{V}^\perp = \{ f \in \mathbb{R}^E : \langle f, g \rangle = 0 \text{ for all } g \in \mathbb{R}^E \}.$$

Recall from linear algebra that

$$\dim \mathcal{V} + \dim \mathcal{V}^\perp = \dim \mathbb{R}^E = \#E. \tag{67}$$

Furthermore, $\left(\mathcal{V}^\perp\right)^\perp = \mathcal{V}$.

Intuitively there is a kind of "duality" between elementary cycles and bonds. Cycles "hold vertices together," while bonds "tear them apart." The precise statement of this duality is given by the next result.

11.3 Theorem. *The cycle and bond spaces of D are related by $\mathcal{C} = \mathcal{B}^{\perp}$. (Equivalently, $\mathcal{B} = \mathcal{C}^{\perp}$.)*

Proof. Let $f : E \to \mathbb{R}$. Then f is a circulation if and only if

$$\sum_{e \in E} m_v(e) f(e) = 0$$

for all $v \in V$ [why?]. But this is exactly the condition that $f \in \mathcal{B}^{\perp}$. $\square$

11.2 Bases for the cycle space and bond space.

We want to examine the incidence matrix $\boldsymbol{M}(D)$ in more detail. In particular, we would like to determine which rows and columns of $\boldsymbol{M}(D)$ are linearly independent, and which span the row and column spaces. As a corollary, we will determine the dimension of the spaces $\mathcal{B}$ and $\mathcal{C}$. We begin by defining the *support* $\|f\|$ of $f : E \to \mathbb{R}$ to be the set of edges $e \in E$ for which $f(e) \neq 0$.

11.4 Lemma. *If $0 \neq f \in \mathcal{C}$, then $\|f\|$ contains an undirected circuit.*

Proof. If not, then $\|f\|$ has a vertex of degree one [why?], which is clearly impossible. $\square$

11.5 Lemma. *If $0 \neq g \in \mathcal{B}$, then $\|g\|$ contains a bond.*

Proof. Let $0 \neq g \in \mathcal{B}$, so $g = \delta p$ for some $p : V \to \mathbb{R}$. Choose a vertex v which is incident to an edge of $\|g\|$, and set

$$U = \{u \in V : p(u) = p(v)\}.$$

Let $\bar{U} = V - U$. Note that $\bar{U} \neq \emptyset$, since otherwise p is constant so $g = 0$. Since $g(e) \neq 0$ for all $e \in [U, \bar{U}]$ [why?], we have that $\|g\|$ contains the cutset $[U, \bar{U}]$. Since a bond is by definition a minimal cutset, it follows that $\|g\|$ contains a bond. $\square$

11.6 Definition. A matrix $\boldsymbol{B}$ is called a *basis matrix* of $\mathcal{B}$ if the rows of $\boldsymbol{B}$ form a basis for $\mathcal{B}$. Similary define a basis matrix $\boldsymbol{C}$ of $\mathcal{C}$.

Recall the notation of Theorem 9.4: Let A be a matrix with at least as many columns as rows, whose columns are indexed by the elements of a set T. If $S \subseteq T$, then $A[S]$ denotes the submatrix of A consisting of the columns indexed by the elements of S. In particular, $A[e]$ (short for $A[\{e\}]$) denotes the column of A indexed by e. We come to our first significant result about bases for the vector spaces $\mathcal{B}$ and $\mathcal{C}$.

11.7 Theorem. *Let $\boldsymbol{B}$ be a basis matrix of $\mathcal{B}$, and $\boldsymbol{C}$ a basis matrix of $\mathcal{C}$. (Thus the columns of $\boldsymbol{B}$ and $\boldsymbol{C}$ are indexed by the edges $e \in E$ of D.) Let $S \subseteq E$, Then:*

(i) *The columns of $\boldsymbol{B}[S]$ are linearly independent if and only if S is acyclic (i.e., contains no circuit as an undirected graph).*

(ii) *The columns of $\boldsymbol{C}[S]$ are linearly independent if and only if S contains no bond.*

Proof. The columns of $\boldsymbol{B}[S]$ are linearly dependent if and only if there exists a function $f : E \to \mathbb{R}$ such that

$$f(e) \neq 0 \text{ for some } e \in S$$

$$f(e) = 0 \text{ for all } e \notin S$$

$$\sum_{e \in E} f(e)\boldsymbol{B}[e] = \mathbf{0} \text{ the column vector of 0's.} \tag{68}$$

The last condition is equivalent to $\langle f, m_v \rangle = 0$ for all $v \in V$, i.e., f is a circulation. Thus the columns of $\boldsymbol{B}[S]$ are linearly dependent if and only if there exists a nonzero circulation f such that $\|f\| \subseteq S$. By Lemma 11.4, $\|f\|$ (and therefore S) contains a circuit. Conversely, if S contains a circuit C then $0 \neq f_C \in \mathcal{C}$ and $\|f_C\| = C \subseteq S$, so f_C defines a linear dependence relation (68) among the columns. Hence the columns of $\boldsymbol{B}[S]$ are linearly independent if and only if S is acyclic, proving (i). (Part (i) can also be deduced from Lemma 9.7.)

The proof of (ii) is similar and is left as an exercise. □

11.8 Corollary. *Let $D = (V, E)$ be a digraph with p vertices, q edges, and k connected components (as an undirected graph). Then*

$$\begin{aligned} \dim \mathcal{B} &= p - k \\ \dim \mathcal{C} &= q - p + k. \end{aligned}$$

Proof. For any matrix X, the rank of X is equal to the maximum number of linearly independent columns. Now let $\boldsymbol{B}$ be a basis matrix of $\mathcal{B}$. By Theorem 11.7(i), the rank of $\boldsymbol{B}$ is then the maximum size (number of elements) of an acyclic subset of E. In each connected component D_i of D, the largest acyclic subsets are the spanning trees, whose number of edges is $p(D_i) - 1$, where $p(D_i)$ is the number of vertices of D_i. Hence

$$\begin{aligned} \operatorname{rank} \boldsymbol{B} &= \sum_{i=1}^{k} (p(D_i) - 1) \\ &= p - k. \end{aligned}$$

Since $\dim \mathcal{B} + \dim \mathcal{C} = \dim \mathbb{R}^E = q$ by equation (67) and Theorem 11.3, we have

$$\dim \mathcal{C} = q - (p - k) = q - p + k.$$

(It is also possible to determine $\dim \mathcal{C}$ by a direct argument similar to our determination of $\dim \mathcal{B}$.) □

The number $q - p + k$ (which should be thought of as the number of independent cycles in D) is called the *cyclomatic number* of D (or of its undirected version G, since the direction of the edges have no effect).

Our next goal is to describe explicit bases of $\mathcal{C}$ and $\mathcal{B}$. We begin by defining a *forest* to be an undirected graph without circuits. Thus a forest is a disjoint union of trees. We extend the definition of forest to directed graphs by ignoring the arrows, i.e., a directed graph is a forest if it has no circuits as an undirected graph. Equivalently [why?], $\dim \mathcal{C} = 0$.

Pick a maximal forest T of $D = (V, E)$. Thus T restricted to each component of D is a spanning tree. If e is an edge of D not in T, then it is easy to see that $T \cup e$ contains a unique circuit C_e.

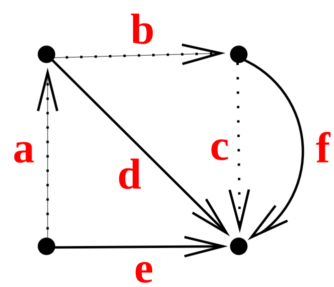
11.9 Theorem. *Let T be as above. Then the set S of circulations f_{C_e}, as e ranges over all edges of D not in T, is a basis for the cycle space $\mathcal{C}$.*

Proof. The circulations f_{C_e} are linearly independent, since for each $e \in E(D) - E(T)$ only f_{C_e} doesn't vanish on e. Moreover,

$$\#S = \#E(D) - \#E(T) = q - p + k = \dim \mathcal{C},$$

so S is a basis. $\square$

11.10 Example. Let D be the digraph shown below, with the edges a, b, c of T shown by dotted lines.



Orient each circuit C_t in the direction of the added edge, i.e., $f_{C_t}(t) = 1$. Then the basis matrix $\boldsymbol{C}$ of $\mathcal{C}$ corresponding to the basis $f_{C_d}, f_{C_e}, f_{C_f}$ is given by

$$\boldsymbol{C} = \begin{bmatrix} 0 & -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}. \tag{69}$$

We next want to find a basis for the bond space $\mathcal{B}$ analogous to that of Theorem 11.9.

11.11 Lemma. *Let T be a maximal forest of $D = (V, E)$. Let $T^* = D - E(T)$ (the digraph obtained from D by removing the edges of T), called a* cotree *if D is connected. Let e be an edge of T. Then $E(T^*) \cup e$ contains a unique bond.*

Proof. Removing $E(T^*)$ from D leaves a maximal forest T, so removing one further edge e disconnects some component of D. Hence $E(T^*) \cup e$ contains a bond B. It remains to show that B is unique. Removing e from

T breaks some component of T into two connected graphs T_1 and T_2 with vertex sets S and $\bar{S}$. It follows [why?] that we must have $B = [S, \bar{S}]$, so B is unique. $\square$

Let T be a maximal forest of the digraph D, and let e be an edge of T. By the previous lemma, $E(T^*) \cup e$ contains a unique bond B_e. Let g_{B_e} be the corresponding element of the bond space $\mathcal{B}$, chosen for definiteness so that $g_{B_e}(e) = 1$.

11.12 Theorem. *The set of functions g_{B_e}, as e ranges over all edges of T, is a basis for the bond space $\mathcal{B}$.*

Proof. The functions g_{B_e} are linearly independent, since only g_{B_e} is nonzero on $e \in E(T)$. Since

$$\#E(T) = p - k = \dim \mathcal{B},$$

it follows that the g_{B_e}'s are a basis for $\mathcal{B}$. $\square$

11.13 Example. Let D and T be as in the previous diagram. Thus a basis for $\mathcal{B}$ is given by the functions $g_{B_a}, g_{B_b}, g_{B_c}$. The corresponding basis matrix is given by

$$\boldsymbol{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Note that the rows of $\boldsymbol{B}$ are orthogonal to the rows of the matrix $\boldsymbol{C}$ of equation (69), in accordance with Theorem 11.3. Equivalently, $\boldsymbol{B}\boldsymbol{C}^t = \boldsymbol{0}$, the 3×3 zero matrix. (In general, $\boldsymbol{B}\boldsymbol{C}^t$ will have $q - p + k$ rows and $p - k$ columns. Here it is just a coincidence that these two numbers are equal.)

The basis matrices $\boldsymbol{C}_T$ and $\boldsymbol{B}_T$ of $\mathcal{C}$ and $\mathcal{B}$ obtained from a maximal forest T have an important property. A real matrix $m \times n$ matrix A with $m \leq n$ is said to be *unimodular* if every $m \times m$ submatrix has determinant 0, 1, or -1. For instance, the adjacency matrix $\boldsymbol{M}(D)$ of a digraph D is unimodular, as proved in Lemma 9.7 (by showing that the expansion of the determinant of a full submatrix has at most one nonzero term).

11.14 Theorem. *Let T be a maximal forest of D. Then the basis matrices $\boldsymbol{C}_T$ of $\mathcal{C}$ and $\boldsymbol{B}_T$ of $\mathcal{B}$ are unimodular.*

Proof. First consider the case $\boldsymbol{C}_T$. Let $\boldsymbol{P}$ be a full submatrix of $\boldsymbol{C}$ (so $\boldsymbol{P}$ has $q-p+k$ rows and columns). Assume $\det \boldsymbol{P} \neq 0$. We need to show $\det \boldsymbol{P} = \pm 1$. Since $\det \boldsymbol{P} \neq 0$, it follows from Theorem 11.7(ii) that $\boldsymbol{P} = \boldsymbol{C}_T[T_1^*]$ for the complement T_1^* of some maximal forest T_1. Note that the rows of the matrix $\boldsymbol{C}_T[T_1^*]$ are indexed by T^* and the columns by T_1^*. Similarly the rows of the basis matrix $\boldsymbol{C}_{T_1}$ are indexed by T_1^* and the columns by E (the set of all edges of D). Hence it makes sense to define the matrix product

$$\boldsymbol{Z} = \boldsymbol{C}_T[T_1^*]\boldsymbol{C}_{T_1},$$

a matrix whose rows are indexed by T^* and columns by E.

Note that the matrix $\boldsymbol{Z}$ is a basis matrix for the cycle space $\mathcal{C}$ since its rows are linear combinations of the rows of the basis matrix $C^*_{T_1}$, and it has full rank since the matrix $\boldsymbol{C}_T[T_1^*]$ is invertible. Now $\boldsymbol{C}_{T_1}[T_1^*] = I_{T_1^*}$ (the identity matrix indexed by T_1^*), so $\boldsymbol{Z}[T_1^*] = \boldsymbol{C}_T[T_1^*]$. Thus $\boldsymbol{Z}$ agrees with the basis matrix $\boldsymbol{C}_T$ in columns T_1^*. Hence the rows of $\boldsymbol{Z} - \boldsymbol{C}_T$ are circulations supported on a subset of T_1. Since T_1 is acyclic, it follows from Lemma 11.4 that the only such circulation is identically 0, so $\boldsymbol{Z} = \boldsymbol{C}_T$.

We have just shown that

$$\boldsymbol{C}_T[T_1^*]\boldsymbol{C}_{T_1} = \boldsymbol{C}_T.$$

Restricting both sides to T^*, we obtain

$$\boldsymbol{C}_T[T_1^*]\boldsymbol{C}_{T_1}[T^*] = \boldsymbol{C}_T[T^*] = I_{T^*}.$$

Taking determinants yields

$$\det(\boldsymbol{C}_T[T_1^*])\det(\boldsymbol{C}_{T_1}[T^*]) = 1.$$

Since all the matrices we have been considering have integer entries, the above determinants are integers. Hence

$$\det \boldsymbol{C}_T[T_1^*] = \pm 1,$$

as was to be proved. (This proof is due to Tutte in 1965.)

A similar proof works for $\boldsymbol{B}_T$. $\square$

11.3 Electrical networks.

We will give a brief indication of the connection between the above discussion and the theory of electrical networks. Let D be a digraph, which for convenience we assume is *connected* and *loopless*. Suppose that at each edge e there is a voltage (potential difference) V_e from $\text{init}(e)$ to $\text{fin}(e)$, and a current I_e in the direction of e (so a negative current I_e indicates a current of $|I_e|$ in the direction opposite to e). Think of V and I as functions on the edges, i.e., as elements of the vector space $\mathbb{R}^E$. There are three fundamental laws relating the quantities V_e and I_e.

Kirchhoff's First Law. $I \in \mathcal{C}_D$. *In other words, the current flowing into a vertex equals the current flowing out. In symbols,*

$$\sum_{\substack{e \\ \text{init}(e)=v}} I_e = \sum_{\substack{e \\ \text{fin}(e)=v}} I_e,$$

for all vertices $v \in V$.

Kirchhoff's Second Law. $V \in \mathcal{C}_D^{\perp} = \mathcal{B}$. *In other words, the sum of the voltages around any circuit (called* loops *by electrical engineers), taking into account orientations, is* 0.

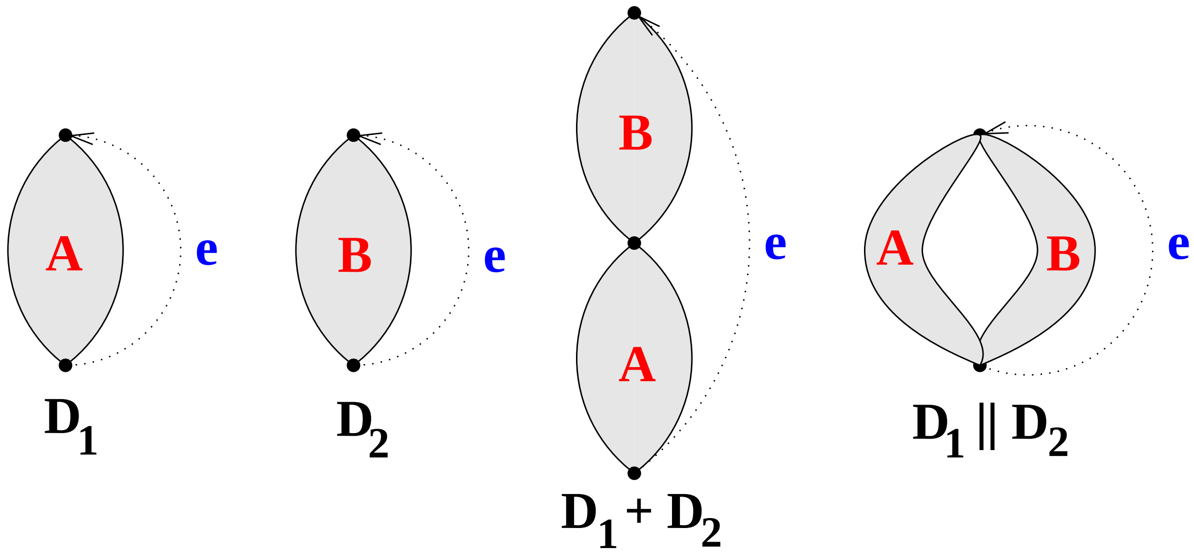
Ohm's Law. *If edge e has resistance $R_e > 0$, then $V_e = I_e R_e$.*

The central problem of electrical network theory dealing with the above three laws[5] is the following: Which of the $3q$ quantities V_e, I_e, R_e need to be specified to uniquely determine all the others, and how can we find or stipulate the solution in a fast and elegant way? We will be concerned here only with a special case, perhaps the most important special case in practical applications. Namely, suppose we apply a voltage V_q at edge e_q, with resistances $R_1, \ldots, R_{q-1}$ at the other edges $e_1, \ldots, e_{q-1}$. Let V_i, I_i be the voltage and current at edge e_i. We would like to express each V_i and I_i in terms of V_q and $R_1, \ldots, R_{q-1}$. (By "physical intuition" there should be a unique solution, since we can actually build a network meeting the specifications of

[5] Of course the situation becomes much more complicated when one introduces *dynamic* network elements like capacitors, alternating current, etc.

the problem.) Note that if we have quantities V_i, I_i, R_i satisfying the three network laws above, then for any scalar α the quantities $\alpha V_i, \alpha I_i, R_i$ are also a solution. This means that we might as well assume that $V_q = 1$, since we can always multiply all voltages and currents afterwards by whatever value we want V_q to be.

As an illustration of a simple method of computing the total resistance of a network, the following diagram illustrates the notion of a *series connection* $D_1 + D_2$ and a *parallel connection* $D_1 \parallel D_2$ of two networks D_1 and D_2 with a distinguished edge e at which a voltage is applied.

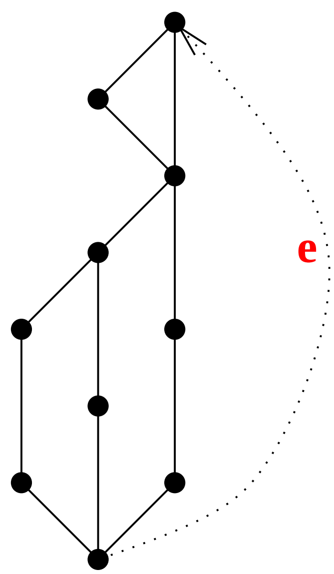


If $R(D)$ denotes the total resistance $-V_e/I_e$ of the network D together with the distinguished edge e, then it is well-known and easy to deduce from the three network Laws that

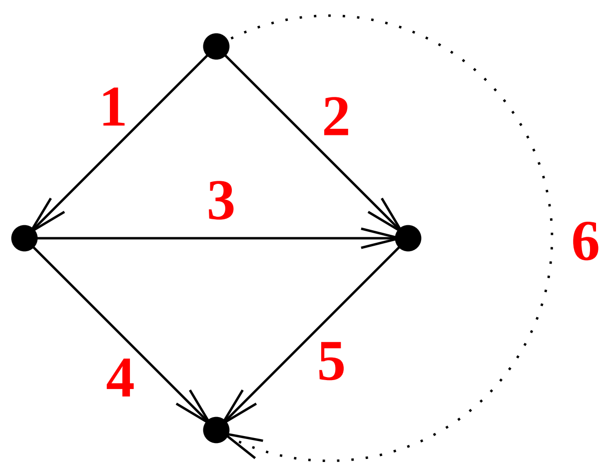
$$R(D_1 + D_2) = R(D_1) + R(D_2)$$

$$\frac{1}{R(D_1 \parallel D_2)} = \frac{1}{R(D_1)} + \frac{1}{R(D_2)}.$$

A network that is built up from a single edge by a sequence of series and parallel connections is called a *series-parallel network*. An example is the following, with the distinguished edge e shown by a broken line from bottom to top.



The simplest network which is not a series-parallel network and has no multiple edges (as an undirected graph) is called the *Wheatstone bridge* and is illustrated below. (The direction of the arrows has been chosen arbitrarily.) We will use this network as our main example in the discussion that follows.



We now return to an arbitrary connected loopless digraph D, with currents I_i, voltages V_i, and resistances R_i at the edges e_i. Recall that we are fixing $V_q = 1$ and $R_1, \ldots, R_{q-1}$. Let T be a spanning tree of D. Since I is a current if and only if it is orthogonal to the bond space $\mathcal{B}$ (Theorem 11.3 and

Kirchhoff's First Law), it follows that any basis for $\mathcal{B}$ defines a complete and minimal set of linear relations satisfied by the I_i's (namely, the relation that I is orthogonal to the basis elements). In particular, the basis matrix $\boldsymbol{C}_T$ defines such a set of relations. For example, if D is the Wheatstone bridge shown above and if $T = \{e_1, e_2, e_5\}$, then we obtain the following relations by adding the edges e_1, e_2, e_5 of T in turn to T^*.

$$\begin{aligned} I_1 - I_3 - I_4 &= 0 \\ I_2 + I_3 + I_4 + I_6 &= 0 \\ I_4 + I_5 + I_6 &= 0 \end{aligned} \tag{70}$$

These three ($= p - 1$) equations give all the relations satisfied by the I_i's alone, and the equations are linearly independent.

Similary if V is a voltage then it is orthogonal to the cycle space $\mathcal{C}$. Thus any basis for $\mathcal{C}$ defines a complete and minimal set of linear relations satisfied by the V_i's (namely, the relation that V is orthogonal to the basis elements). In particular, the basis matrix C_T defines such a set of relations. Continuing our example, we obtain the following relations by adding the edges e_3, e_4, e_6 of T^* in turn to T.

$$\begin{aligned} V_1 - V_2 + V_3 &= 0 \\ V_1 - V_2 + V_4 - V_5 &= 0 \\ V_2 + V_5 &= 1, \end{aligned} \tag{71}$$

These three ($= q - p + k$) equations give all the relations satisfied by the V_i's alone, and the equations are linearly independent.

In addition, Ohm's Law gives the $q-1$ equations $V_i = R_i I_i$, $1 \le i \le q-1$. We have a total of $(p - k) + (q - p + k) + (q - 1) = 2q - 1$ equations in the $2q-1$ unknowns I_i ($1 \le i \le q$) and V_i ($1 \le i \le q-1$). Moreover, it is easy to see that these $2q - 1$ equations are linearly independent, using the fact that we already know that just the equations involving the I_i's alone are linearly independent, and similarly the V_i's. Hence this system of $2q - 1$ equations in $2q - 1$ unknowns has a unique solution. We have now reduced the problem to straightforward linear algebra. However, it is possible to describe the solution explicitly. We will be content here with giving a formula just for the total resistance $R(D) = -V_q/I_q = -1/I_q$.

Write the $2q-1$ equations in the form of a $(2q-1)\times 2q$ matrix $\boldsymbol{K}$. The columns of the matrix are indexed by $I_1, I_2, \ldots, I_q, V_1, V_2, \ldots, V_q$. The last column V_q of the matrix keeps track of the constant terms of the equations. The rows of $\boldsymbol{K}$ are given first by the equations among the I_i's, then the V_i's, and finally Ohm's Law. For our example of the Wheatstone bridge, we obtain the matrix

$$
\boldsymbol{K} = \begin{array}{|ccccc|c|ccccc|c|}
I_1 & I_2 & I_3 & I_4 & I_5 & I_6 & V_1 & V_2 & V_3 & V_4 & V_5 & V_6 \\
\hline
1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 \\
\hline
R_1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & R_2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & R_3 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & R_4 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & R_5 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{array}
$$

We want to solve for I_q by Cramer's rule. Call the submatrix consisting of all but the last column X. Let Y be the result of replacing the I_q column of X by the last column of $\boldsymbol{K}$. Cramer's rule then asserts that

$$I_q = \frac{\det Y}{\det X}.$$

We evaluate $\det X$ by taking a Laplace expansion along the first $p-1$ rows. In other words,

$$\det X = \sum_S \pm \det(X[[p-1], S]) \cdot \det(X[[p-1]^c, \bar{S}]), \tag{72}$$

where (a) S indexes all $(p-1)$-element subsets of the columns, (b) $X[[p-1], S]$ denotes the submatrix of X consisting of entries in the first $p-1$ rows and in the columns S, (c) $X[[p-1]^c, \bar{S}]$ denotes the submatrix of X consisting of entries in the last $2q-p$ rows and in the columns other than S. In order for $\det(X[[p-1],S]) \neq 0$, we must choose $S = \{I_{i_1}, \ldots, I_{i_{p-1}}\}$, where $\{e_{i_1}, \ldots, e_{i_{p-1}}\}$ is a spanning tree T_1 (by Theorem 11.7(i)). In this case, $\det(X[[p-1],S]) = \pm 1$ by Theorem 11.14. If $I_q \notin S$, then the I_q column of $X[[p-1]^c, \bar{S}]$ will be zero. Hence to get a nonzero term in (72), we must have $e_q \in S$. The matrix $X[[p-1]^c, \bar{S}]$ will have one nonzero entry in each of the first $q-p+1$ columns, namely, the resistances R_j where e_j is not an edge of T_1. This accounts for $q-p+1$ entries from the last $q-1$ rows of $X[[p-1]^c, \bar{S}]$. The remaining $p-2$ of the last $q-1$ rows have available only one nonzero entry each, a -1 in the columns indexed by V_j where e_j is an edge of T_1 other than e_q. Hence we need to choose $q-p+1$ remaining entries from rows p through q and columns indexed by V_j for e_j not edge of T_1. By Theorems 11.7(ii) and 11.14, this remaining submatrix has determinant ± 1. It follows that

$$\det(X[[p-1],S]) \cdot \det(X[[p-1]^c, \bar{S}]) = \pm \prod_{e_j \notin E(T_1)} R_j.$$

Hence by (72), we get

$$\det X = \sum_{T_1} \pm \left(\prod_{e_j \notin E(T_1)} R_j \right), \tag{73}$$

where T_1 ranges over all spanning trees of D containing e_q. A careful analysis of the signs[6] shows that all signs in (73) are plus, so we finally arrive at the remarkable formula

$$\det X = \sum_{\substack{\text{spanning trees } T_1 \\ \text{containing } e_q}} \prod_{e_j \notin E(T_1)} R_j.$$

For example, if D is the Wheatstone bridge as above, and if we abbreviate $R_1 = a$, $R_2 = b$, $R_3 = c$, $R_4 = d$, $R_5 = e$, then

$$\det X = abc + abd + abe + ace + ade + bcd + bde + cde.$$

Now suppose we replace column I_q in X by column V_q in the matrix $\boldsymbol{K}$, obtaining the matrix Y. There is a unique nonzero entry in the new column,

[6] To be inserted.

so it must be chosen in any nonzero term in the expansion of $\det Y$. The argument now goes just as it did for $\det X$, except we have to choose S to correspond to a spanning tree T_1 that *doesn't* contain e_q. We therefore obtain

$$\det Y = \sum_{\substack{\text{spanning trees } T_1 \\ \text{not containing } e_q}} \prod_{\substack{e_j \notin E(T_1) \\ e_j \neq e_q}} R_j.$$

For example, for the Wheatstone bridge we get

$$\det Y = ac + ad + ae + bc + bd + be + cd + ce.$$

Recall that $I_q = \det(Y)/\det(X)$ and that the total resistance of the network is $1/I_q$. Putting everything together gives our main result on electrical networks.

11.15 Theorem. *In the situation described above, the total resistance of the network is given by*

$$R(D) = \frac{1}{I_q} = -\frac{\displaystyle\sum_{\substack{\text{spanning trees } T_1 \\ \text{containing } e_q}} \prod_{e_j \notin E(T_1)} R_j}{\displaystyle\sum_{\substack{\text{spanning trees } T_1 \\ \text{not containing } e_q}} \prod_{\substack{e_j \notin E(T_1) \\ e_j \neq e_q}} R_j}.$$

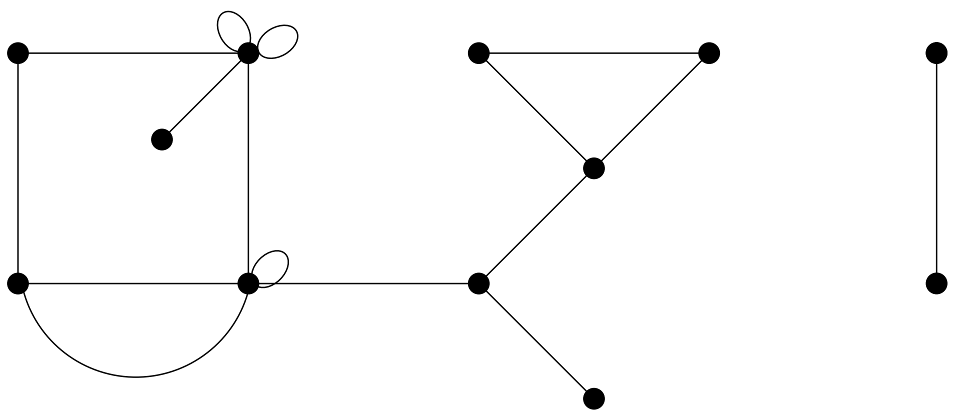
11.16 Corollary. *If the resistances $R_1, \ldots, R_{q-1}$ are all equal to one, then the total resistance of the network is given by*

$$R(D) = \frac{1}{I_q} = \frac{\text{number of spanning trees containing } e_q}{\text{number of spanning trees not containing } e_q}.$$

In particular, if $R_1 = \cdots = R_{q-1} = 1$, then the total resistance, when reduced to lowest terms a/b, has the curious property that the number $\kappa(D)$ of spanning trees of D is divisible by $a + b$.

11.4 Planar graphs (sketch).

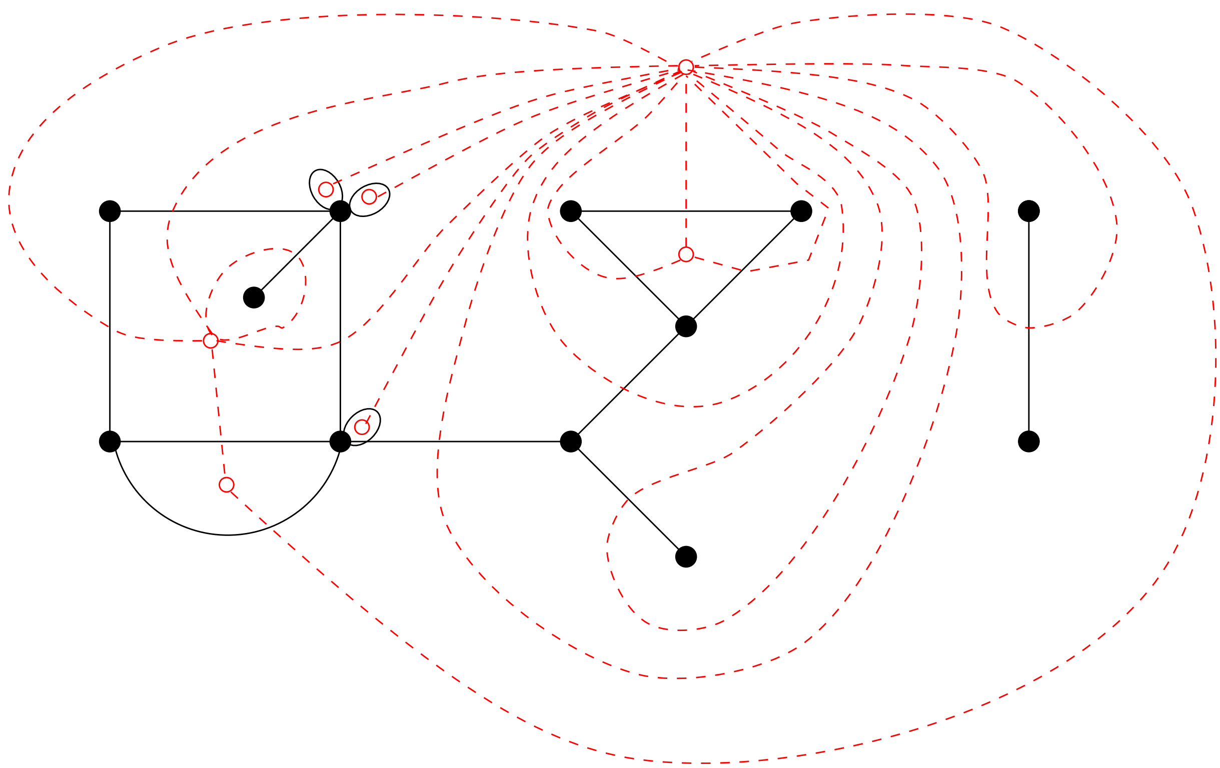
A graph G is *planar* if it can be drawn in the plane $\mathbb{R}^2$ without crossing edges. A drawing of G in this way is called a *planar embedding*.



If the vertices and edges of a planar embedding of G are removed from $\mathbb{R}^2$, then we obtain a disjoint union of open sets, called the *regions* (or *faces*) of G. (More precisely, these open sets are the regions of the planar embedding of G. Often we will not bother to distinguish between a planar graph and a planar embedding if no confusion should result.) Let $R = R(G)$ be the set of regions of G, and as usual $V(G)$ and $E(G)$ denote the set of vertices and edges of G, respectively.

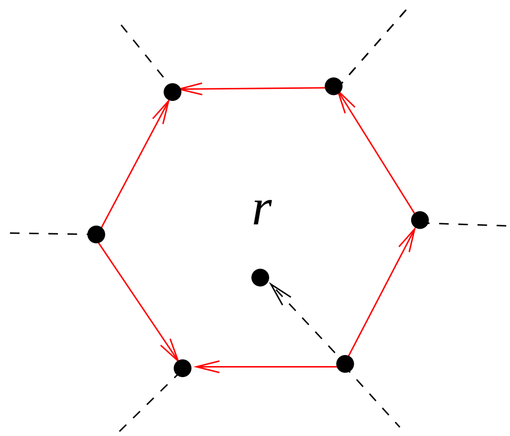
NOTE. If G is simple (no loops or multiple edges) then it can be shown that there exists a planar embedding with edges as straight lines and with regions (regarding as the sequence of vertices and edges obtained by walking around the boundaries of the regions) preserved.

The *dual* G^* of the planar embedded graph G has vertex set $R(G)$ and edge set $E^*(G) = \{e^* : e \in E(G)\}$. If e is an edge of G, then let r and r' be the regions on its two sides. (Possibly $r = r'$; there are five such edges in the example above.) Then define e^* to connect r and r'. We can always draw G^* to be planar, letting e and e^* intersect once. If G is connected then every region of G^* contains exactly one nonisolated vertex of G and $G^{**} \cong G$.

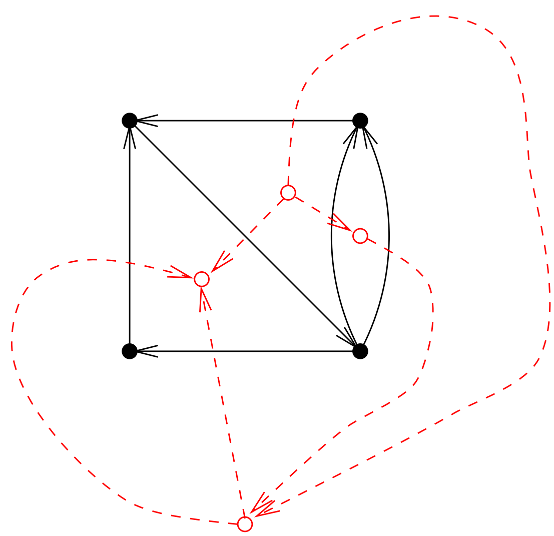


11.17 Example. Let G consist of two disjoint edges. Then G^* has one vertex and two loops, while G^{**} is a three-vertex path. The unbounded region of G^* contains two vertices of G, and $G^{**} \not\cong G$.

Orient the edges of the planar graph G in any way to get a digraph D. Let r be an interior (i.e., bounded) region of D. An *outside edge* of r is an edge e such that r lies on one side of the edge, and a *different* region lies on the other side. The outside edges of any interior region r define a circulation (shown as solid edges in the diagram below), and these circulations (as r ranges over all interior regions of D) form a basis for the cycle space $\mathcal{C}_G$ of G.



Given the orientation D of G, orient the edges of G^* as follows: as we walk along e in the direction of its orientation, e^* points to our *right*.



11.18 Theorem. *Let $f : E(G) \to \mathbb{R}$. Define $f^* : E(G^*) \to \mathbb{R}$ by $f^*(e^*) = f(e)$. Then*

$$f \in \mathcal{B}_G \quad \Leftrightarrow f^* \in \mathcal{C}_{G^*}$$
$$f \in \mathcal{C}_G \quad \Leftrightarrow f^* \in \mathcal{B}_{G^*}.$$

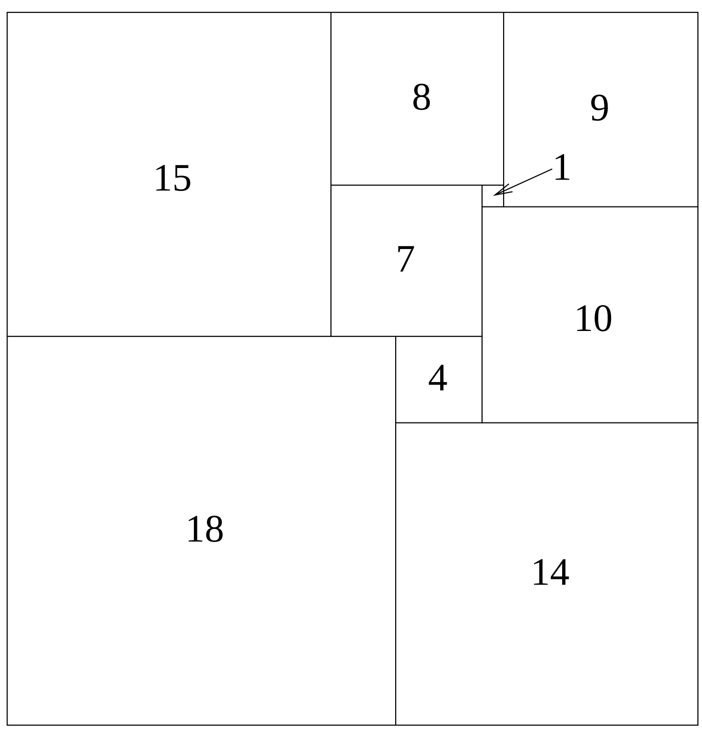
11.19 Proposition. *S is the set of edges of a spanning tree T of G if and only if $S^* = \{e^* : e \in S\}$ is the set of edges of a cotree T^* of G^*.*

11.20 Corollary. $\kappa(G) = \kappa(G^*)$

For nonplanar graphs there is still a notion of a "dual" object, but it is no longer a graph but rather something called a *matroid*. Matroid theory is a flourishing subject which may be regarded as a combinatorial abstraction of linear algebra.

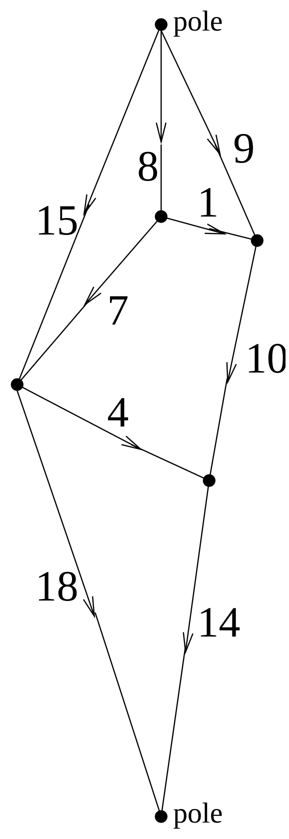
11.5 Squaring the square.

A *squared rectangle* is a rectangle partitioned into finitely many (but more than one) squares. A squared rectangle is *perfect* if all the squares are of different sizes. The earliest squared rectangle was found in 1936; its size is 33×32 and consists of nine squares:



The question then arose: does there exist a perfect squared square? An isolated example with 55 squares was found by Sprague in 1939. Then Brooks, Smith, Stone, and Tutte developed a network theory approach which we now explain.

The *Smith diagram D* of a squared rectangle is a directed graph whose vertices are the horizontal line segments of the squared rectangle and whose squares are the edges, directed from top to bottom. The top vertex (corresponding to the top edge of the rectangle being squared) and the bottom vertex (corresponding to the bottom edge) are called *poles*. Label each edge by the side length of the square to which it corresponds. The figure below shows the Smith diagram of the (perfect) squared rectangle above.



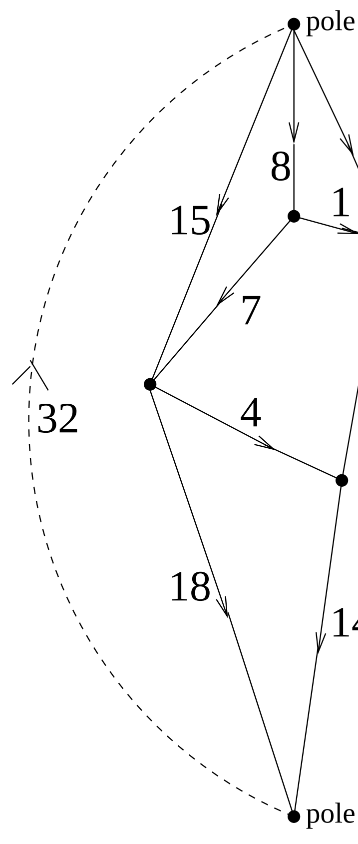
The following result concerning Smith diagrams is straightforward to verify.

11.21 Theorem.

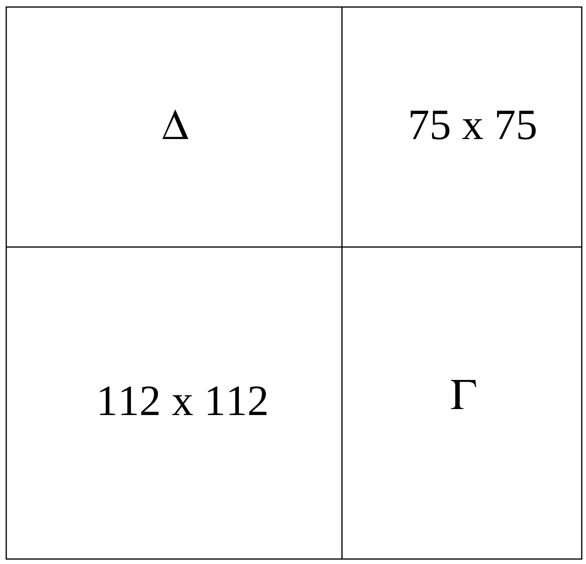
(a) *If we set I_e and V_e equal to the label of edge e, then Kirchhoff's two laws hold (so $R_e = 1$) except at the poles.*

(b) *The Smith diagram is planar and can be drawn without separation of poles. Joining the poles by an edge from the bottom to the top gives a 3-connected graph, i.e., a connected graph that remains connected when one or two vertices are removed.*

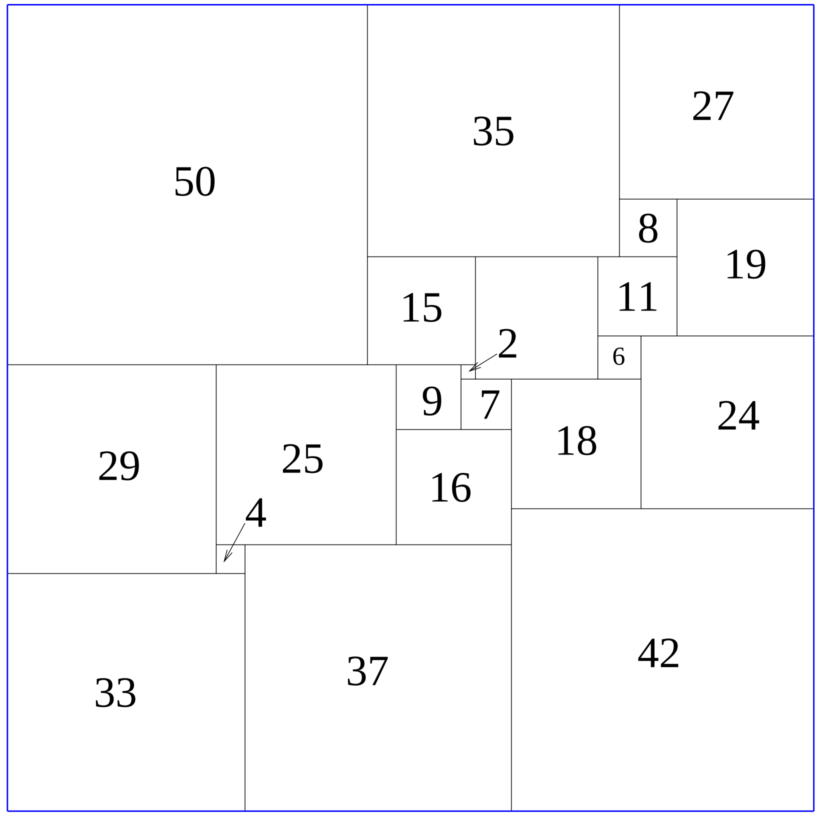
Call the 3-connected graph of Theorem 11.21 the *extended* Smith diagram of the squared rectangle. If we label the new edge e_1 between poles by the horizontal length b of the squared rectangle and set $V_{e_1} = I_{e_1} = b$, then Kirchhoff's two laws hold at *all* vertices.



We therefore have a recipe for searching for perfect squared rectangles and squares: start listing all 3-connected planar graphs. Then choose an edge e_1 to apply a voltage V_1. Put a resistance $R_e = 1$ at the remaining edges e. Solve for I_e $(= V_e)$ to get a squared rectangle, and hope that one of these will be a square. One example Γ found by Brooks et al. was a 112×75 rectangle with 14 squares. It was given to Brooks' mother as a jigsaw puzzle, and she found a different solution Δ! We therefore have found a squared square (though not perfect):



Building on this idea, Brooks et al. finally found two 422×593 perfect rectangles with thirteen squares, all 26 squares being of different sizes. Putting them together as above gives a perfect squared square. This example has two defects: (a) it contains a smaller perfect squared rectangle (and is therefore not *simple*), and (b) it contains a "cross" (four squares meeting a point). They eventually found a perfect squared square with 69 squares without either of these defects. It is now known (thanks to computers) that the smallest order (number of squares) of a perfect squared square is 21. It is unique and happens to be simple and crossfree. See the figure below. It is known that the number (up to symmetry) of simple perfect squared squares of order n for $n \geq 21$ is $1, 8, 12, 26, 160, 441, 1152, \ldots$.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