Course 18.312: Algebraic Combinatorics

Solution Set # 9

Due Tax Day, Wednesday April 15, 2009

You may discuss the homework with other students in the class, but please write the names of your collaborators at the top of your assignment. Please be advised that you should not just obtain the solution from another source. Please explain your reasoning to receive full credit, even for computational questions.

1) Let $f(n)$ be the number of graphs G on the vertex set $\{1, 2, \ldots, n\}$ such that every connected component of G is isomorphic to a path P_i with $i \geq 1$ vertices. (P_1 is a single vertex and P_2 is a single edge.) Set $f(0) = 1$.

   (10 points) Find $E_f(x) = \sum_{n \geq 0} f(n) \frac{x^n}{n!}$. (For full credit, your final answer should not contain any infinite sums.)

   This is problem 5.15 (d) in Stanley's Enumerative Combinatorics, Vol. 2.

   We let $g(i)$ be the number of paths P_i (up to isomorphism) on the vertex set $\{1, 2, \ldots, i\}$, and

$$E_g(x) = \sum_{i \geq 1} \frac{g(i)}{i!} x^i.$$

   Then it follows from the **exponential formula** and the above constructive definition of the family of graphs counted by $f(n)$ that

$$E_f(x) = \exp(E_g(x)).$$

   It suffices to calculate $g(i)$ for each $i \geq 1$.

   It is clear that $g(1) = 1$, and for $i \geq 2$, we pick one of the $i!$ linear orderings of $\{1, 2, \ldots, i\}$. (This is as if we are coloring i objects in a line with i distinct colors.) The only symmetry is the $\mathbb{Z}/2\mathbb{Z}$-symmetry of reflecting the entire line, i.e. $1 \leftrightarrow i$, $2 \leftrightarrow (i-1)$, etc., and this transposition has no fixed points.)

We conclude that

$$\begin{aligned} E_f(x) &= \exp\left(x + \sum_{i\geq 2} \frac{i!}{2}\frac{x^i}{i!}\right) = \exp\left(x + \frac{1}{2}\sum_{i\geq 2} x^i\right) \\ &= \exp\left(x + \frac{1}{2}\left(\frac{x^2}{1-x}\right)\right) = \exp\left(\frac{x}{2} + \frac{x}{2(1-x)}\right). \end{aligned}$$

2) (5 points) a) Write $\frac{a+bx}{c+dx}$ as a formal power series $F(x) = a_1x + a_2x^2 + a_3x^3 + \ldots$. In other words, write a closed formula for a_i.

Following the Errata: First we let $a = 0$ and then conclude that

$$\frac{bx}{c+dx} = \frac{(b/c)x}{1-(-d/c)x} = \left(\frac{b}{c}\right)\sum_{i\geq 1}\left(\frac{-d}{c}\right)^{i-1} x^i.$$

Thus we let

$$a_i = \left(\frac{b}{c}\right)\left(\frac{-d}{c}\right)^{i-1} = \frac{(-1)^{i-1}\, b\, d^{i-1}}{c^i}.$$

For a more general solution, we can assume no restriction on a and we obtain $a_0 = \frac{a}{c}$ and $a_i = \left(\frac{a}{c} - \frac{b}{d}\right)\left(\frac{-d}{c}\right)^i$. However, then unless $a = 0$, we cannot use Lagrange inversion.

(10 points) b) If we let $F^{\langle -1\rangle}(x) = b_1x + b_2x^2 + b_3x^3 + \ldots$, use Lagrange inversion to give a closed formula for b_i.

By the Lagrange Inversion Formula, $[x^n]F^{\langle -1\rangle}(x) = \frac{1}{n}[x^{n-1}]\left(\frac{x}{F(x)}\right)^n$. We continue following the errata and assume that $a = 0$. Thus

$$\begin{aligned} b_n &= \frac{1}{n}[x^{n-1}]\left(\frac{x(c+dx)}{bx}\right)^n = \frac{1}{n}[x^{n-1}]\left(\frac{c}{b} + \frac{d}{b}x\right)^n \\ &= \frac{1}{n}\binom{n}{n-1}\left(\frac{c}{b}\right)\left(\frac{d}{b}\right)^{n-1} = \left(\frac{c}{b}\right)\left(\frac{d}{b}\right)^{n-1}. \end{aligned}$$

(5 points) c) Using part (b) or otherwise, what is a closed form expression for $F^{\langle -1\rangle}(x)$?

The most general solution occurs when $ad - bc \neq 0$. Then we get $F^{\langle -1\rangle}(x) = \frac{-a+cx}{b-dx}$ under this hypothesis. It is easy to check that

$$\frac{a + b\left(\frac{-a+cx}{b-dx}\right)}{c + d\left(\frac{-a+cx}{b-dx}\right)} = \frac{a(b-dx) + b(-a+cx)}{c(b-dx) + d(-a+cx)} = \frac{(-ad+bc)x}{-ad+bc} = x.$$

Notice that letting $a = 0$ and $F^{\langle -1\rangle}(x) = \frac{cx}{b-dx}$, we recover the formula for $[x^n]F^{\langle -1\rangle}(x)$ from part (b).

3) Let $L = L(K_{rs})$ be the Laplacian matrix of the complete bipartite graph K_{rs}.

(5 points) a) Find a simple upper bound on $rank(L - rI)$. Deduce a lower bound on the number of eigenvalues of L equal to r.

Matrix $L(K_{rs})$ can be decomposed into blocks as

$$\begin{bmatrix} sI_r & -J_{rs} \\ -J_{sr} & rI_s \end{bmatrix}$$

where J_{ab} denotes the $a \times b$ rectangular matrix consisting of all 1's and I_n is the $n \times n$ identity matrix. Consequently, the last s rows of matrix $(L(K_{rs}) - rI)$ are identical and so the nullspace of $(L(K_{rs}) - rI)$ has dimension at least $(s-1)$. Hence a simple upper bound on the rank of $(L(K_{rs}) - rI)$ is $(r+s)-(s-1) = r+1$. Hence, $(L(K_{rs}) - rI)$ has at least $(s-1)$ eigenvalues equal to zero and thus the Laplacian $L(K_{rs})$ has at least $(s-1)$ eigenvalues equal to r.

(5 points) b) Assume $r \neq s$, and do the same as (a) for s instead of r.

By analogous reasoning, $L(K_{rs})$ has at least $(r-1)$ eigenvalues equal to s.

(5 points) c) Find the remaining eigenvalues of L.

(**Hint:** Use the fact that the rows of L sum to 0 and compute the trace of L.)

Since K_{rs} has $r+s$ vertices, there are $(r+s)-(s-1)-(r-1) = 2$ left unaccounted for. However, since the rows of $L(K_{rs})$ sum to 0 (in fact this is true for any Laplacian matrix), we have eigenvector $[1, 1, \ldots, 1]^T$ with eigenvalue 0. Finally, the trace of $L(K_{rs})$ is the sum $\sum_{i=1}^{r+s} \deg(v_i) = rs + sr$.

In conclusion, $\lambda_1 + \lambda_2 + \cdots + \lambda_{r+s} = rs + sr = 2rs$ and since we know all but one eigenvalue, say λ_{r+s}, we have $(s-1)r + (r-1)s + 0 + \lambda_{r+s} = 2rs$ implies that the last eigenvalue is $(r+s)$.

(5 points) d) Use (a)-(c) to compute $\kappa(K_{rs})$, the number of spanning trees of K_{rs}.

By the Matrix Tree Theorem,

$$\begin{aligned} \kappa(K_{rs}) &= \frac{1}{\#V(K_{rs})} \prod_{\lambda_i \text{ is a non-zero eigenvalue of } L(K_{rs})} \lambda_i = \frac{1}{r+s}(r^{s-1}s^{r-1}(r+s)) \\ &= r^{s-1}s^{r-1}. \end{aligned}$$

(**Bonus 5 points**) e) Give a combinatorial proof of the formula for $\kappa(K_{rs})$.

There are several possible combinatorial proofs. One example uses an analogue of the Prüfer code that we used to prove that the number of all labeled trees on p vertices (i.e. all spanning trees of K_p) was p^{p-2}.