Course 18.312: Algebraic Combinatorics

Solution Set # 8

Due Wednesday April 8, 2009

You may discuss the homework with other students in the class, but please write the names of your collaborators at the top of your assignment. Please be advised that you should not just obtain the solution from another source. Please explain your reasoning to receive full credit, even for computational questions.

1) (10 points) Expand $h_{3,1}$ in terms of the e_λ's.

We use the fact that $h_{3,1} = h_3 \cdot h_1$ and use the relations

$$\begin{aligned} h_3 &= h_2e_1 - h_1e_2 + e_3 \\ h_2 &= h_1e_1 - e_2 \\ h_1 &= e_1 \end{aligned}$$

which become

$$\begin{aligned} h_1 &= e_1 \\ h_2 &= e_1^2 - e_2 \\ h_3 &= (e_1^2 - e_2)e_1 - e_1e_2 + e_3 \end{aligned}$$

Thus

$$h_{3,1} = e_1^4 - e_1^2e_2 - e_1^2e_2 + e_1e_3 = e_{1111} - 2e_{211} + e_{31}.$$

(10 points) Expand $e_{2,2}$ in terms of the p_λ's.

We use the Newton-Girard identities which say

$$2e_2 = e_1p_1 - p_2 = p_1^2 - p_2$$

(since $e_1 = p_1$).

Hence

$$e_{2,2} = e_2^2 = \frac{1}{4}(p_1^2 - p_2)^2 = \frac{1}{4}p_1^4 - \frac{1}{2}p_1^2p_2 + \frac{1}{4}p_2^2.$$

(10 points) Compute $s_{3,1,1}(x_1, x_2, x_3)$ as a polynomial in terms of $\{x_1, x_2, x_3\}$.

Method 1: We count with weights the number of semi-standard Young tableaux of shape $[3,1,1]$ using letters from the alphabet $\{1,2,3\}$. Since the first column has three rows, we have no choice but to fill in the first column with $1,2,3$ in decreasing order. Thus any such semi-standard tableaux looks like $\begin{matrix} 1 & A & B \\ 2 & & \\ 3 & & \end{matrix}$ where $1 \leq A \leq B \leq 3$. Thus we get possibilities:

$$\begin{matrix} 1 & 1 & 1 \\ 2 & & \\ 3 & & \end{matrix}, \quad \begin{matrix} 1 & 1 & 2 \\ 2 & & \\ 3 & & \end{matrix}, \quad \begin{matrix} 1 & 1 & 3 \\ 2 & & \\ 3 & & \end{matrix}, \quad \begin{matrix} 1 & 2 & 2 \\ 2 & & \\ 3 & & \end{matrix}, \quad \begin{matrix} 1 & 2 & 3 \\ 2 & & \\ 3 & & \end{matrix}, \quad \begin{matrix} 1 & 3 & 3 \\ 2 & & \\ 3 & & \end{matrix}$$

Summing up the weights yields

$$x_1^3x_2x_3 + x_1^2x_2^2x_3 + x_1^2x_2x_3^2 + x_1x_2^3x_3 + x_1x_2^2x_3^2 + x_1x_2x_3^3.$$

Method 2: We use the determinantal formula

$$s_{3,1,1}(x_1, x_2, x_3) = \frac{\det \begin{bmatrix} x_1^{3+2} & x_2^{3+2} & x_3^{3+2} \\ x_1^{1+1} & x_2^{1+1} & x_3^{1+1} \\ x_1^{1+0} & x_2^{1+0} & x_3^{1+0} \end{bmatrix}}{\det \begin{bmatrix} x_1^2 & x_2^2 & x_3^2 \\ x_1^1 & x_2^1 & x_3^1 \\ x_1^0 & x_2^0 & x_3^0 \end{bmatrix}},$$

which equals

$$\frac{x_1^5x_2^2x_3 - x_1^5x_2x_3^2 - x_1^2x_2^5x_3 + x_1^2x_2x_3^5 + x_1x_2^5x_3^2 - x_1x_2^2x_3^5}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)}.$$

After factoring and cancelling factors from the top and bottom, this resolves to

$$x_1^3x_2x_3 + x_1^2x_2^2x_3 + x_1^2x_2x_3^2 + x_1x_2^3x_3 + x_1x_2^2x_3^2 + x_1x_2x_3^3.$$

Method 3: We use the Jacobi-Trudi Identity first which yields

$$s_{[3,1,1]}(x_1, x_2, x_3) = \det \begin{bmatrix} h_{3-1+1} & h_{3-1+2} & h_{3-1+3} \\ h_{1-2+1} & h_{1-2+2} & h_{1-2+3} \\ h_{1-3+1} & h_{1-3+2} & h_{1-3+3} \end{bmatrix} = \det \begin{bmatrix} h_3 & h_4 & h_5 \\ 1 & h_1 & h_2 \\ 0 & 1 & h_1 \end{bmatrix} = h_3h_1^2 + h_5 - h_3h_2 - h_4h_1.$$

Letting $h_k = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq 3} x_{i_1}x_{i_2} \cdots x_{i_k}$, and simplifying yields the same answer as above.

2) **For this problem, you are encouraged to do computer experimentation.**

(5 points) Compute $s_{2,1}(x_1, x_2, x_3)$

We enumerate valid SSYT using of shape $[2,1]$ on letters $\{1,2,3\}$:

$$\begin{matrix}1 & 1\\ 2\end{matrix}\ ,\ \begin{matrix}1 & 1\\ 3\end{matrix}\ ,\ \begin{matrix}1 & 2\\ 2\end{matrix}\ ,\ \begin{matrix}1 & 2\\ 3\end{matrix}\ ,\ \begin{matrix}1 & 3\\ 2\end{matrix}\ ,\ \begin{matrix}1 & 3\\ 3\end{matrix}\ ,\ \begin{matrix}2 & 2\\ 3\end{matrix}\ ,\ \begin{matrix}2 & 3\\ 3\end{matrix}\ .$$

Hence

$$s_{2,1}(x_1,x_2,x_3) = x_1^2x_2+x_1^2x_3+x_1x_2^2+2x_1x_2x_3+x_1x_3^2+x_2^2x_3+x_2x_3^2 = (x_1+x_2)(x_1+x_3)(x_2+x_3).$$

(5 points) Compute $s_{3,2,1}(x_1, x_2, x_3, x_4)$

We either write a program to count SSYT and then try factoring the polynomial or use the determinantal definition:

$$\begin{aligned} s_{3,2,1}(x_1,x_2,x_3,x_4) &= \frac{\det\begin{bmatrix} x_1^{3+3} & x_2^{3+3} & x_3^{3+3} & x_4^{3+3} \\ x_1^{2+2} & x_2^{2+2} & x_3^{2+2} & x_4^{2+2} \\ x_1^{1+1} & x_2^{1+1} & x_3^{1+1} & x_4^{1+1} \\ x_1^{0+0} & x_2^{0+0} & x_3^{0+0} & x_4^{0+0} \end{bmatrix}}{\det\begin{bmatrix} x_1^3 & x_2^3 & x_3^3 & x_4^3 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^1 & x_2^1 & x_3^1 & x_4^1 \\ x_1^{0+0} & x_2^0 & x_3^0 & x_4^0 \end{bmatrix}}. \\ &= (x_1+x_2)(x_1+x_3)(x_1+x_4)(x_2+x_3)(x_2+x_4)(x_3+x_4). \end{aligned}$$

(**Bonus 5 points**) Do you have a conjecture for $s_{k,k-1,\ldots,3,2,1}(x_1, x_2, \ldots, x_{k+1})$? Prove it.

$$s_{k,k-1,\ldots,3,2,1}(x_1,x_2,\ldots,x_{k+1}) = \prod_{1\le i<j\le k+1}(x_i+x_j).$$

(5 points) Compute $f(x_1, x_2, x_3) = e_1(x_1, x_2, x_3)s_{2,1}(x_1, x_2, x_3)$ as a polynomial in terms of $\{x_1, x_2, x_3\}$.

$$\begin{aligned} f(x_1,x_2,x_3) &= (x_1+x_2+x_3)(x_1+x_2)(x_1+x_3)(x_2+x_3) \\ &= (x_1+x_2+x_3)(x_1^2x_2+x_1^2x_3+x_1x_2^2+2x_1x_2x_3+x_1x_3^2+x_2^2x_3+x_2x_3^2) \\ &= {x_1}^3x_2+{x_1}^3x_3+4\,{x_1}^2x_3x_2+2\,{x_1}^2{x_3}^2+2\,{x_1}^2{x_2}^2+4\,x_1{x_2}^2x_3+4\,x_1x_2{x_3}^2 \\ &+ \ x_1{x_2}^3+{x_2}^3x_3+2\,{x_2}^2{x_3}^2+x_1{x_3}^3+{x_3}^3x_2. \end{aligned}$$

(10 points) Write $f(x_1, x_2, x_3)$ as a $\mathbb{Z}$-linear combination of $s_4(x_1, x_2, x_3)$, $s_{31}(x_1, x_2, x_3)$, $s_{22}(x_1, x_2, x_3)$, $s_{211}(x_1, x_2, x_3)$, and $s_{1111}(x_1, x_2, x_3)$.

Hint:

$$\begin{aligned} s_{3,1}(x_1,x_2,x_3) &= 2x_1^2x_2x_3 + 2x_1x_2^2x_3 + 2x_1x_2x_3^2 + x_2^3x_3 + x_2^2x_3^2 + x_1^3x_2 \\ &+ x_2x_3^3 + x_1^2x_3^2 + x_1^2x_2^2 + x_1^3x_3 + x_1x_3^3 + x_1x_2^3 \end{aligned}$$

Firstly, $s_{1111}(x_1, x_2, x_3) = 0$ since there are more rows than variables.

$$s_4(x_1, x_2, x_3) = h_4(x_1, x_2, x_3) = x_1^4 + x_1^3x_2 + \cdots + x_3^4.$$

The expansion of $s_{3,1}$ is given.

$$s_{2,2}(x_1, x_2, x_3) = x_1^2x_2^2 + x_1^2x_2x_3 + x_1^2x_3^2 + x_1x_2^2x_3 + x_1x_2x_3^2 + x_2^2x_3^2$$

can easily be found by looking at SSYT of shape $[2, 2]$ on $\{1, 2, 3\}$. Analogously,

$$s_{2,1,1}(x_1, x_2, x_3) = x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2.$$

By using undetermined coefficients or otherwise, we see that

$$e_1s_{2,1} = s_{3,1} + s_{2,2} + s_{2,1,1} \tag{1}$$

on the alphabet $\{x_1, x_2, x_3\}$. (To be absolutely precise, we can add $Cs_{1,1,1,1}$ where C is a constant since $s_{1,1,1,1} = 0$ but this was not needed.)

Remark. Identity (1) is a special case of a more general identity known as the **Pieri Rule**, which is beyond the scope of this class. However, you may see from this example and others that

$$e_1s_\lambda = \sum_{\mu,\ \lambda <\cdot \mu} s_\mu.$$

The Pieri Rule handles the more general case of how to expand $e_ks_\lambda = \sum_\mu c_\mu s_\mu$.

3) Ten balls are stacked in a triangular array with 1 atop 2 atop 3 atop 4. (Think of billiards.) The triangular array is free to rotate.

(10 points) Find the generating function for the number of inequivalent colorings $r_1, r_2, \ldots, r_{10}$. (You do not need to simplify your answer.)

The symmetry group G on the triangular array is cyclic of order 3 specifically including the permutations

$$\{(1)(2)(3)(4)(5)(6)(7)(8)(9)(10), (1,7,10)(2,8,6)(3,4,9)(5), (1,10,7)(2,6,8)(3,9,4)(5)\}$$

if we label the billiard balls top-to-bottom, left-to-right.

Thus the cycle indicator polynomial is given by

$$Z_G = \frac{1}{3}(z_1^{10} + 2z_1z_3^3).$$

Hence by Pólya's theorem, the generating function F_G is given by

$$F_G = \frac{1}{3}\left((\sum r_i^{10}) + 2(\sum r_i)(\sum r_i^3)^3\right).$$

(5 points) How many inequivalent colorings have four periwinkle balls, three teal balls, and three celadon balls?

This question is asking for the coefficient of $r_1^4r_2^3r_3^3$ in F_G. The coefficient in $(\sum r_i)^{10}$ is the multinomial coefficient $\binom{10}{4,3,3} = \frac{10!}{4!\ 3!\ 3!} = 4200$. The coefficient of $r_1^4r_2^3r_3^3$ in $(\sum r_i)(\sum r_i^3)^3$ is the same as the coefficient of $r_1^3r_2^3r_3^3$ in $(\sum r_i^3)^3$ which is $3! = 6$. Thus the answer is

$$\frac{1}{3}(4200 + 2 \cdot 6) = 1404.$$

(5 points) How many inequivalent colorings have four burgundy balls, four fuchsia balls, and two taupe balls?

We now want the coefficient of $r_1^4r_2^4r_3^2$ in F_G, and fortunately, the coefficient of $r_1^4r_2^4r_3^2$ in $(\sum r_i)(\sum r_i^3)^3$ is zero. Thus the answer is

$$\frac{1}{3}\binom{10}{4,4,2} = 1050.$$

(Note that all three cyclic rotations of any coloring with four burgundy balls, four fuchsia balls, and two taupe balls are different hence why only the leading term of F_G has a nonzero contribution.)

4) For any finite group G of permutations of an ℓ-element set X, let $f(n)$ be the number of inequivalent (under the action of G) colorings of X with n colors.

(15 points) Find $\lim_{n\to\infty} f(n)/n^\ell$. Interpret your answer as saying that "most" colorings of X are asymmetric (have no symmetries).

The number of inequivalent colors, $f(n)$, equals the value $Z_G(n, n, \dots, n)$ so

$$f(n) = \frac{1}{|G|} \sum_{\pi \in G} n^{c(\pi)}$$

where $c(\pi)$ is the number of cycles in permutation π. If π is the identity permutation of G, $n^{c(\pi)} = n^\ell$. All other terms have fewer than ℓ cycles so their contributions are dwarfed by n^ℓ as $n \to \infty$. Hence

$$\begin{aligned}
\lim_{n \to \infty} \frac{f(n)}{n^\ell} &= \lim_{n \to \infty} \frac{1}{n^\ell} \frac{1}{|G|} \sum_{\pi \in G} n^{c(\pi)} \\
&= \frac{1}{|G|} \sum_{\pi \in G} \lim_{n \to \infty} \frac{1}{n^{\ell - c(\pi)}} \\
&= \frac{1}{|G|}.
\end{aligned}$$

If $a(n)$ is the number of asymmetric colorings with n colors, then each of the remaining $n^\ell - a(n)$ must be equivalent to at least one other coloring. We thus have the bound

$$f(n) \le \frac{a(n)}{|G|} + \frac{n^\ell - a(n)}{2|G|}.$$

Dividing by n^ℓ and letting $n \to \infty$, we obtain

$$\frac{1}{|G|} \le \lim_{n \to \infty} \frac{a(n)}{2|G| n^\ell} + \frac{1}{2|G|},$$

hence

$$1 \le \lim_{n \to \infty} \frac{a(n)}{2n^\ell} + \frac{1}{2}.$$

Since $a(n) \le n^\ell$, the number of all colorings with n colors, we conclude that

$$\lim_{n \to \infty} \frac{a(n)}{n^\ell} = 1.$$

Thus the fraction of all colorings which are asymmetric approaches one as $n \to \infty$.

5) Let $\{N_1, N_2, N_3, \dots\}$ be a sequence of positive integers with the property that

$$\exp\left(\sum_{k=1}^{\infty} \frac{N_k}{k} T^k \right) = \frac{(1 - a_1 T)(1 - a_2 T) \cdots (1 - a_m T)}{(1 - b_1 T)(1 - b_2 T) \cdots (1 - b_n T)}.$$

(10 points) Show that $N_k = b_1^k + b_2^k + \cdots + b_n^k - a_1^k - a_2^k - \cdots - a_m^k$.

We take the logarithm of both sides, thus obtaining

$$\sum_{k=1}^{\infty} \frac{N_k}{k} T^k = \sum_{i=1}^{m} \log(1 - a_i T) - \sum_{j=1}^{n} \log(1 - b_j T).$$

Taking the derviative of both sides yields

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{N_k}{T}^{k-1} &= \sum_{i=1}^{m} \frac{-a_i}{(1 - a_i T)} - \sum_{j=1}^{n} \log \frac{-b_j}{(1 - b_j T)} \\ &= \left(\sum_{i=1}^{m} (-a_i) \sum_{r=0}^{\infty} a_i^r T^r \right) - \left(\sum_{j=1}^{n} (-b_i) \sum_{r=0}^{\infty} b_i^r T^r \right) \end{aligned}$$

Taking the coefficient of T^{k-1} of both sides thus yields

$$N_k = \sum_{i=1}^{m} -a_i^k + \sum_{j=1}^{n} b_j^k$$

which is the desired identity.

Bonus) The names of one hundred prisoners are placed in one hundred wooden boxes, one name to a box, and the boxes are lined up on a table in a room. One by one, the prisoners are led into the room; they may look into up to fifty of the boxes to try to find their own name, but must leave the room exactly as it was. They are permitted no further communication after leaving the room. The prisoners have a chance to plot a strategy in advance, and they are going to need it, because unless *they all find their own names* they will all be executed.

(**Bonus 10 points**) There is a strategy that has a probability of success exceeding thirty percent - find it.

This beautiful problem has been popularized by Peter Winkler and is originally due to Peter Bro Miltersen and Anna Gal.