Course 18.312: Algebraic Combinatorics

Solution Set # 7

Due Wednesday April 1, 2009

You may discuss the homework with other students in the class, but please write the names of your collaborators at the top of your assignment. Please be advised that you should not just obtain the solution from another source. Please explain your reasoning to receive full credit, even for computational questions.

1) An **involution** is a permutation π with the property that π^2 is the identity.

(15 points) Give a formula for the number of involutions of $\{1, 2, \ldots, n\}$ in terms of f^λ's.

Solution: We use the RSK algorithm to solve this problem. Let $\pi \in S_n$. By the **Symmetry Theorem**, if

$$RSK(\pi) = (P, Q) \quad \text{then} \quad RSK(\pi^{-1}) = (Q, P).$$

Thus, if $\pi^2 = 1$ (i.e. $\pi^{-1} = \pi$), it follows that $RSK(\pi) = (P, P)$ where P is some Standard Young Tableau of shape $\lambda \vdash n$.

Since RSK is a bijection, it also follows that for any Standard Young Tableaux P of size n, $RSK^{-1}(P, P)$ must be an involution so

$$\text{The number of involutions of } \{1, 2, \ldots, n\} \quad = \sum_{\lambda \vdash n} f_\lambda.$$

2) (10 points) Show that

$$A(x) = \sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}.$$

Solution: We use the generalized binomial Theorem which says that the RHS equals

$$
\begin{aligned}
(1-4x)^{-1/2} &= \sum_{n=0}^{\infty} \binom{-1/2}{n} (-4x)^n \\
&= \sum_{n=0}^{\infty} \frac{(-1/2)(-3/2)(-5/2)\cdots(-1/2-n+1)}{n!}(-2)^n \cdot 2^n\, x^n \\
&= \sum_{n=0}^{\infty} \frac{(1)(3)(5)\cdots(2n-1)}{n!} \cdot 2^n\, x^n \\
&= \sum_{n=0}^{\infty} \frac{(1)(3)(5)\cdots(2n-1)}{n!} \cdot \frac{(2)(4)\cdots(2n)}{n!}\, x^n,
\end{aligned}
$$

which equals the LHS.

This can also be obtained by integrating $SA(x)$, and then using the generating function for the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$.

(5 points) Give a quadratic equation that $A(x)$ satisfies.

Solution: By squaring both sides and multiplying, we get

$$(1-4x)\ A(x)^2 - 1 = 0.$$

This is the special case of an example/exercise from Stanley's Enumerative Combinatorics Vol. 2. See Example 6.2.7 or Exercise 6.13. In general, the series $y = \sum_{n\geq 0}\binom{kn}{n}x^n$ satisfies

$$k^k x\ y^k - (y-1)\Big((k-1)y+1\Big)^{k-1} = 0.$$

Notice that we recover the above quadratic for $A(x)$ by letting $k=2$.

(Bonus 5 points) Give a cubic equation that $B(x) = \sum_{n=0}^{\infty}\binom{3n}{n}x^n$ satisfies.

Solution: Using the above for $k=3$, we get that $B(x)$ satisfies

$$
\begin{aligned}
27x\ B(x)^3 - (B(x)-1)\Big(2B(x)+1\Big)^2 &= \\
(27x-4)B(x)^3 + 3B(x) + 1 &= 0.
\end{aligned}
$$

3) (5 points) Let $\ell - i - j = 2m$ and

$$b_{ij}(\ell) = \frac{\ell!}{2^m i! j! m!}.$$

Show that the $b_{ij}(\ell)$'s satisfy the recurrence

$$b_{ij}(\ell+1) = b_{i,j-1}(\ell) + (i+1)b_{i+1,j}(\ell) + b_{i-1,j}(\ell).$$

Solution: Firstly, if $\ell + 1 - i - j$ is odd then this equation simplifies to the tautology $0 = 0$. Thus, assume $\ell + 1 - i - j$ is even and let $m' = \frac{\ell+1-i-j}{2}$.

The RHS equals

$$\frac{\ell!}{2^{m'} i!(j-1)!(m')!} + (i+1)\frac{\ell!}{2^{m'+1}(i+1)!j!(m'+1)!} + \frac{\ell!}{2^{m'}(i-1)!j!(m')!},$$

which simplifies to

$$\frac{\ell!}{2^{m'} i!j!(m')!} \cdot \Big(j + 2m' + i\Big) = \frac{(\ell+1)!}{2^{m'} i!j!(m')!} = b_{i,j}(\ell+1).$$

Let λ be a partition of 9. Let $c(\lambda)$ be the number of ways to **delete** a square, **insert** a square, **delete** a square, and finally **insert** a square to return to partition λ. (Note: at each step when we delete or add a square, we assume that the resulting shape is still a partition.)

(10 points) What is $\sum_{\lambda \vdash 9} c(\lambda)$?

Solution: We use Corollary 8.10 of Stanley's notes which tells us that

$$\sum_{\lambda \vdash 9} c(\lambda) = \sum_{s=1}^{j} \Big(p(j-s) - p(j-s-1)\Big) s^m$$

for the case $j = 9$ and $m = 2$.

From class, we have seen that $p(0) = 1$, $p(1) = 1$, $p(2) = 2$, $p(3) = 3$, $p(4) = 5$, $p(5) = 7$, $p(6) = 11$. We can find $p(7)$ and $p(8)$ in many places or count partitions directly, so although it is not necessary, for more fun we shall use Euler's pentagonal formula to compute these quantities:

$$\begin{aligned} p(7) &= p(6) + p(5) - p(2) - p(0) = 15 \\ p(8) &= p(7) + p(6) - p(3) - p(1) = 22 \end{aligned}$$

Thus the number of ways to delete, insert, delete, insert is:

$$\begin{aligned}
& \Big(p(8)-p(7)\Big)1^2+\Big(p(7)-p(6)\Big)2^2+\Big(p(6)-p(5)\Big)3^2+\Big(p(5)-p(4)\Big)4^2 \\
+ \quad & \Big(p(4)-p(3)\Big)5^2+\Big(p(3)-p(2)\Big)6^2+\Big(p(2)-p(1)\Big)7^2+\Big(p(1)-p(0)\Big)8^2 \\
+ \quad & \Big(p(0)\Big)9^2 \quad = \quad 7\cdot 1^2+4\cdot 2^2+4\cdot 3^2+2\cdot 4^2+2\cdot 5^2+6^2+7^2+9^2 \\
= \quad & 307.
\end{aligned}$$

4) In this problem, we give a proof of the hook formula for f^λ, the number of SYT of shape λ, using a sequence of algebraic identities. Let $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k]$ be a partition of n with at most k parts, and $\ell_i = \lambda_i + k - i$ for each $1 \le i \le k$.

(10 points) Using the combinatorial interpretation of f^λ as the number of SYT's of shape λ, show the recursion

$$f^{[\lambda_1,\ldots,\lambda_k]} = \sum_{r=1}^{k} f^{[\lambda_1,\ldots,\lambda_r-1,\ldots,\lambda_k]}$$

where we define $f^{[\lambda_1,\ldots,\lambda_r-1,\ldots,\lambda_k]} = 0$ if sequence $[\lambda_1,\ldots,\lambda_r-1,\ldots,\lambda_k] = 0$ is not weakly decreasing.

Solution: We can subtract one from λ_r and still have a weakly decreasing sequence exactly when there is a corner in the rth row of the Young Diagram corresponding to λ. Thus, this identity is an algebraic way of writing

$$f^\lambda = \sum_{\mu \,<\!\cdot\, \lambda} f^\mu,$$

where λ covers μ in Young's lattice.

Let $\Delta(x_1,\ldots,x_k) = \prod_{1\le i<j\le k}(x_i - x_j)$ (the Vandermonde determinant)

(10 points) Show that

$$\sum_{r=1}^{k} x_r \Delta(x_1,\ldots,x_r+t,\ldots,x_k) = (x_1+x_2+\cdots+x_k+\binom{k}{2}t)\cdot\Delta(x_1,\ldots,x_k). \quad (1)$$

Hint 1: Show that both the left-hand and right-hand sides are antisymmetric; i.e. if we switch x_i and x_j each expression becomes its negative.

Hint 2: Evaluate each side at the values $x_i = k - i$ and $t = 1$.

Solution: We follow the hint, assuming $i < j$, and use the fact that

$$\Delta(x)|_{x_i \leftrightarrow x_j} = -\Delta(x).$$

Consequently we see that the LHS becomes (under $x_i \leftrightarrow x_j$)

$$\sum_{r=1,\quad r\neq i,j}^{k} -x_r\Delta(x_1,\ldots,x_r+t,\ldots,x_k) \;+\; x_i\Delta(x_1,\ldots,x_j,\ldots,x_i+t,\ldots,x_k) \;+\; x_j\Delta(x_1,\ldots,x_j+t,\ldots,x_i,\ldots,x_k),$$

which after re-ordering equals

$$\sum_{r=1,\quad r\neq i,j}^{k} -x_r\Delta(x_1,\ldots,x_r+t,\ldots,x_k) \;-\; x_i\Delta(x_1,\ldots,x_i+t,\ldots,x_j,\ldots,x_k) \;-\; x_j\Delta(x_1,\ldots,x_i,\ldots,x_j+t,\ldots,x_k),$$

the negative of the LHS.

For the RHS, we use antisymmetry of $\Delta(x)$ and symmetry of

$$(x_1+x_2+\cdots+x_k+\binom{k}{2})$$

to obtain antisymmetry of the RHS.

Since the LHS is antisymmetric, it follows that this LHS must equal

$$f(x_1,x_2,\ldots,x_k,t)\cdot\Delta(x_1,\ldots,x_k),$$

where f is symmetric in the variables x_1 through x_k. Furthermore, since the LHS is homogeneous, comparing total degrees, we see that f is linear, thus of the form

$$a(x_1+x_2+\cdots+x_n)+bt.$$

To see precisely which function to use for f, we substitue in the specific values for the x_i's and t.

Firstly, if $t=0$, the LHS $=(x_1+\cdots+x_k)\Delta(x_1,\ldots,x_k)$ hence $a=1$.

Plugging in $x_i=k-i$ and $t=1$, we get

$$\begin{aligned}
& \sum_{i=1}^{k}(k-i)\Delta(k-1,\ldots,k-(i-1),k-i-1,k-(i+1),\ldots,2,1,0) \\
= \;& (k-1)\Delta(k,k-2,\ldots,1,0) = (k-1)\left(\prod_{i=2}^{k}(k-(k-i))\right)\cdot\Delta(k-2,k-3,\ldots,1,0) \\
= \;& (k-1)(2\cdot 3\cdots k)\Delta(k-2,k-3,\ldots,1,0)
\end{aligned}$$

since the terms of the sum on the left equal zero unless $i = 1$. Under this same substituion, the RHS equals

$$\begin{aligned} &\Big((1 + 2 + \cdots + (k-1)) + b\Big) \cdot \Delta(k-1, k-2, \ldots, 1, 0) \\ = \quad &\left(\binom{k}{2} + b\right)(1 \cdot 2 \cdots (k-1)) \cdot \Delta(k-2, k-3, \ldots, 1, 0). \end{aligned}$$

We divide both sides by $\Delta(k-2, \ldots, 1, 0)(k-1)!$ and get

$$(k-1) \cdot k = \binom{k}{2} + b,$$

and thus $b = \binom{k}{2}$.

Note: One can also proceed by letting $t = -1$.

(10 points) Letting $\ell_i = \lambda_i + k - i$ and $|\lambda| = n$, use (1) to show

$$n \cdot \Delta(\ell_1, \ldots, \ell_k) = \sum_{r=1}^{k} \ell_r \cdot \Delta(\ell_1, \ldots, \ell_r - 1, \ldots, \ell_k). \tag{2}$$

Solution: Letting $x_i = \ell_i$, $t = -1$, the RHS of (2) agrees with the LHS of (1). Since

$$\sum_{i=1}^{k} \ell_i + \binom{k}{2} = \sum_{i=1}^{k} \Big(\ell_i + (k-i)\Big) = \sum_{i=1}^{k} \lambda_i = n,$$

the LHS of (2) also equals the RHS of (1), and we have identity (1) implies identity (2).

Let $F(\ell_1, \ldots, \ell_k) = \frac{n! \cdot \Delta(\ell_1, \ldots, \ell_k)}{\ell_1! \ell_2! \cdots \ell_k!}$.

(10 points) Show that identity (2) is equivalent to

$$F(\ell_1, \ldots, \ell_k) = \sum_{r=1}^{k} F(\ell_1, \ldots, \ell_r - 1, \ldots, \ell_k).$$

Solution: We multiply both sides of the equation by $\frac{(n-1)!}{\ell_1! \ell_2! \cdots \ell_k!}$ and renormalize the terms on the RHS.

(10 points) Show that

$$F(\ell_1, \ldots, \ell_k) = \frac{n!}{\prod_{c \in \lambda} h_\lambda(c)}$$

Solution: We proceed by induction on k. If $k = 1$, the LHS equals

$$\frac{\lambda_1! \cdot \Delta(\emptyset)}{\left(\lambda_1 + (1-1)\right)} = 1$$

while the RHS equals $\frac{\lambda_1!}{\lambda_1!} = 1$ since the Young diagram of λ is a single row.

We now build up our partition placing rows on top of one another, since hook lengths can be counted starting from the last row and proceeding upwards. Assume by induction that

$$F(\ell_2, \ldots, \ell_k) = \frac{(n - \lambda_1)!}{\prod_{c \in [\lambda_2, \lambda_3, \ldots, \lambda_k]} h_{[\lambda_2, \lambda_3, \ldots, \lambda_k]}(c)}.$$

Adding the first row adds blocks whose hook lengths start with $\lambda_1 + (k-1)$ since this quantity is the size of the hook of the top-left square in a partition with k rows. As long as the first column is the same size as the second column, the hook length of the second left-most square in the top row will be one less, $\lambda_1 + (k-2)$. It will proceed in this way until the second column is shorter, which happens precisely when we move from column $(\lambda_k - \lambda_{k-1})$ to column $(\lambda_k - \lambda_{k-1} + 1)$ Thus we skip the value $(\lambda_1 - \lambda_k + (k-1))$. Analogous reasoning shows that the product of the hook lengths of the first row is precisely

$$\frac{(\lambda_1 + k - 1)!}{(\lambda_1 - \lambda_2 + 1)(\lambda_1 - \lambda_3 + 2) \cdots (\lambda_1 - \lambda_k + (k-1))} = \frac{\ell_1!}{\prod_{1 < j \leq k} (\ell_1 - \ell_j)}.$$

We conclude with the fact that

$$F(\ell_1, \ell_2, \ldots, \ell_k) = F(\ell_2, \ldots, \ell_k) \cdot \frac{\ell_1!}{\prod_{1 < j \leq k} (\ell_1 - \ell_j)}.$$

(5 points) Conclude that $f^\lambda = \frac{n!}{\prod_{c \in \lambda} h_\lambda(c)}$.

Solution: Since $F(k-1, k-2, \ldots, 1, 0) = 1 = f^{[0,0,\ldots,0]}$ and the $F(\ell_1, \ldots, \ell_k)$'s satisfy the same recurrence as the f^λ's, we have proven the hook formula.