Course 18.312: Algebraic Combinatorics

Solution Set # 4

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1) (10 points) Fix some positive integer $k \geq 2$. Give an algebraic or a bijective proof that the number of partitions of n in which every part appears at most $(k-1)$ times equals the number of partitions where every part is not divisible by k.

(**Bonus 5 points**) Give both an algebraic and a bijective proof of this result.

Algebraic Proof: Let $a(n)$ = the number of partitions of n in which every part appears at most $(k-1)$ times. Let $b(n)$ = the number of partitions of n where each part is not divisible by k. Then

$$\sum_{n\geq 0} a(n)x^n = \prod_{j=1}^{\infty}(1 + x^j + x^{2j} + \cdots + x^{(k-1)j})$$

and

$$\begin{aligned}\sum_{n\geq 0} b(n)x^n &= \frac{1}{1-x}\frac{1}{1-x^2}\cdots\frac{1}{1-x^{k-1}}\frac{1}{1-x^{k+1}}\cdots\frac{1}{1-x^{2k-1}}\frac{1}{1-x^{2k+1}} \\ &= \prod_{i=1}^{\infty}\frac{1}{1-x^i}\cdot\prod_{j=1}^{\infty}(1-x^{kj}).\end{aligned}$$

Since

$$\frac{1-(x^j)^k}{1-(x^j)} = 1 + (x^j) + (x^j)^2 + \cdots + (x^j)^{k-1},$$

the equality of generating functions $A(x)$ and $B(x)$ follows.

Bijective Proof: We provide a bijective map Θ from partitions of "Type B" to partitions of "Type A". Assume that $\lambda = [r^{m_r}, (r-1)^{m_{r-1}}, \ldots, 2^{m_2}, 1^{m_1}]$, where $m_i = 0$ if k divides i.

We use the k-ary expansion of m_i:

$$m_i = c_{1,i}k^{\epsilon_{1,i}} + c_{2,i}k^{\epsilon_{2,i}} + \cdots + c_{j,i}k^{\epsilon_{j,i}}$$

where $\epsilon_{1,i} > \epsilon_{2,i} > \cdots > \epsilon_{j,i} \geq 0$ and $c_{p,i} \in \{1, 2, \ldots, k-1\}$.

We then define $\Theta(\lambda)$ to be the weakly decreasing rearrangement of the multiset

$$\bigcup_{i=1}^{r} \bigcup_{p=1}^{j} \left\{ i \cdot k^{\epsilon_{p,i}}, \ldots, i \cdot k^{\epsilon_{p,i}}, \right\}$$

where each $i \cdot k^{\epsilon_{p,i}}$ appears precisely $c_{p,i}$ times.

Analogous to Sylvester's proof of Euler's Theorem, we see that $|\Theta(\lambda)| = |\lambda|$, $\Theta(\lambda)$ has no part more than $(k-1)$ times, and Θ is bijective since a positive integer m can be written uniquely as $m = k^{\epsilon} m'$ where $k \not| m'$.

Note: This result is due to to Glaisher.

2) A partition λ is **self-conjugate** if $\lambda^T = \lambda$. Let $sc(n)$ denote the number of partitions of n which are self-conjugate.

(10 points) Deduce and prove an expression for the generating function of the sequence $\{sc(n)\}_{n \geq 1}$ as a simple infinite sum of rational expressions.

Hint: Enumerate partitions by the size of their **Durfee square**, which is defined to be the largest square in the northwest quadrant of a partition. For example, the partition $[5, 5, 4, 3, 2]$ has a Durfee square of size 3-by-3.

Any partition λ can be decomposed into a concatenation of a k-by-k Durfee square and two partitions μ, ν (μ in the northeast, ν in the southwest).

If $\lambda^T = \lambda$, then $\nu = \mu^T$. Thus it suffices to enumerate partitions with a k-by-k Durfee square by choosing a single partition μ with no part bigger than k.

Thus

$$\sum_{n \geq 1} sc(n) x^n = \sum_{k \geq 1} \frac{x^{k^2}}{\prod_{i=1}^{k} (1 - x^{2i})}$$

where the factors of x^{k^2} in the numerator come from the contribution of the k-by-k Durfee square to the size of the partition and the product

$$\frac{1}{\prod_{i=1}^{k} (1 - x^{2i})} = \prod_{i=1}^{k} (1 + (x^2)^i + (x^2)^{2i} + \ldots)$$

is exactly the generating function of pairs of identical partitions with at most k parts. Pick m_i for each $i \in \{1, 2, \ldots, k\}$ $m_i = 0$ for $i > k$.

3) Let $r(n)$ denote the number of partitions of n whose parts differ by at least 2 and $s(n)$ denote the number of those where, in addition, the part 1 does not appear.

(10 points) Prove that $\sum_{n\geq 0} r(n)x^n = \sum_{k\geq 0} \frac{x^{k^2}}{\prod_{i=1}^{k}(1-x^i)}$.

(**Bonus 5 points**) Prove that $\sum_{n\geq 0} r(n)x^n = \prod_{j=0}^{\infty} \frac{1}{(1-x^{5j+1})(1-x^{5j+4})}$.

(10 points) Prove that $\sum_{n\geq 0} s(n)x^n = \sum_{k\geq 0} \frac{x^{k^2+k}}{\prod_{i=1}^{k}(1-x^i)}$.

(**Bonus 5 points**) Prove that $\sum_{n\geq 0} s(n)x^n = \prod_{j=0}^{\infty} \frac{1}{(1-x^{5j+2})(1-x^{5j+3})}$.

If λ has k (nonzero) parts, then λ can be decomposed into a sum

$$\lambda = [\mu_1, \mu_2, \ldots, \mu_k] + [2k-1, 2k-3, \ldots, 3, 1]$$

where μ is a partition written in weakly decreasing order with at most k (nonzero) parts. (i.e. when we write μ as $[\mu_1, \mu_2, \ldots, \mu_k]$ then some of the μ_i's might be zero.) Since we are shifting the partition μ to get λ in this way, it follows that λ has no parts whose difference is 0 or 1 and all partitions with this restriction can be built like this.

Consequently, we sum over all positive integers k and get

$$\sum_{k\geq 0} r(n)x^n = \sum_{k\geq 0} \frac{x^{k^2}}{\prod_{i=1}^{k}(1-x^i)}$$

where the factor of $\frac{1}{\prod_{i=1}^{k}(1-x^i)}$ enumerates partitions μ with at most k parts, and the factor of $x^{1+3+5+\cdots+(2k-1)} = x^{k^2}$ corresponds to the number of squares added during the shift.

Analogously, if we require 1 to not be a part, we decompose such partitions as $\lambda = [\mu_1, \mu_2, \ldots, \mu_k] + [2k, 2k-2, \ldots, 4, 2]$ and

$$\sum_{k\geq 0} s(n)x^n = \sum_{k\geq 0} \frac{x^{k^2+k}}{\prod_{i=1}^{k}(1-x^i)}$$

where the factor of $x^{2+4+6+\cdots+(2k)} = x^{k^2+k}$ corresponds to the number of squares added during this second type of shift.

Note: The infinite product formulas described in the associated bonus problems are known as the Rogers-Ramanujan identities.

4) (5 points) Give an example of a finite graded poset P with the Sperner property, together with a group G acting on P, such that the quotient poset P/G is not Sperner. (**Hint:** By Theorem 5.9, P cannot be a boolean poset.)

   There are many possible solutions to this problem. See **attachment** for one such solution.

   Poset P is Sperner since no antichain of size > 4 can be constructed. To see this in more detail, firstly no antichain of that size can contain element b since b is only incomparable with a and c.

   Similarly, the largest antichain containing a or c is $\{a, c, w, z\}$ which has size 4 the size of the largest level of P.

   Let $G = \mathbb{Z}_2$ act on P by switching x and y. Then P/G is not Sperner since it contains an antichain

$$\{\{a\}, \{c\}, \{w\}, \{z\}\}$$

   which is larger than the size of the largest level.

5) Let q be a prime power, and let $\mathbb{F}_q$ denote the finite field with q elements. Let $V_n(q) = \mathbb{F}_q^n$, the n-dimensional vector space over $\mathbb{F}_q$ of n-tuples of elements $\mathbb{F}_q$. Let $B_n(q)$ denote the poset of all subspaces of $V_n(q)$, ordered by inclusion. It's easy to see that $B_n(q)$ is graded of rank n, the rank of a subspace of $V_n(q)$ being its dimension.

   (5 points) Draw the Hasse diagram of $B_3(2)$. (**Hint:** It has 16 elements.)

   See **attachment**. Notice that each element of rank one is covered by three elements, and each element of rank two is covered by three elements.

   (5 points) Compute the Möbius function $\mu(\{0\}, V)$ for each element $V \in B_3(2)$.

   $\mu(\{0\}, \{0\}) = 1$. If we let V_i have rank (i.e. dim) i, then $\mu(\{0\}, V_1) = -1$, $\mu(\{0\}, V_2) = 2$, and $\mu(\{0\}, V_3) = -8$. Here V_3 equals all of $\mathbb{F}_2^3$.

   (10 points) Show that the number of elements of $B_n(q)$ of rank k is given by the q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)}.$$

   (**Hint:** One way to do this is to count in two ways the number of k-tuples $(v_1, \ldots, v_k)$ of linearly independent elements from $\mathbb{F}_q^n$. (1) First choose v_1, then

v_2, etc. and (2) First choose the subspace W spanned by $v_1, \ldots, v_k$, and then choose v_1, v_2, etc.)

First Count: Notice that if V is an n-dimensional space over $\mathbb{F}_q$, so V has precisely q^n elements. Choose v_1 in $q^n - 1$ ways, i.e. any nonzero element of V. Then v_2 can be chosen to be any element of V not in the one dimensional subspace spanned by v_1. Hence, there are $(q^n - q)$ such choices for v_2. Similarly, v_3 can be any element of V not in the two-dimensional subspace spanned by v_1 and v_2, thus there are $(q^n - q^2)$ choices for v_3. Continuing, there are

$$\begin{aligned}(q^n-1)(q^n-q)\cdots(q^n-q^k) &= q^{0+1+2+\cdots+k}(q^n-1)(q^{n-1}-1)\cdots(q-1)\\ &= q^{\binom{k+1}{2}}(q^n-1)(q^{n-1}-1)\cdots(q-1)\end{aligned}$$

ways to pick k linearly independent vectors in $\mathbb{F}_q^n$.

Second Count: On the other hand, two different choices $\{v_1, v_2, \ldots, v_k\}$ and $\{v_1', v_2', \ldots, v_k'\}$ can determine the same subspace W. In fact, if W is a k-dimensional space over $\mathbb{F}_q$, W is isomorphic to $\mathbb{F}_q^k$ and we can pick k linearly independent elements in W, i.e. a basis for W, in

$$(q^k-1)(q^k-q)\cdots(q^k-q^{k-1}) = q^{\binom{k+1}{2}}(q^k-1)(q^{k-1}-1)\cdots(q-1)$$

ways. Dividing by this multiplicity, and we see that there are precisely $\begin{bmatrix} n \\ k \end{bmatrix}$ different k-dimensional subspaces of $\mathbb{F}_q^n$.

(5 points) Show that $B_n(q)$ is rank-symmetric. (**Hint:** Use part (iii).)

This follows immediately from Propositions 6.2 and 6.6 which says that $L(m, n)$ is a rank-symmetric poset, which has $\begin{bmatrix} n \\ k \end{bmatrix}$ as its rank-generating function. Alternatively, we can see this from Example 6.4 which shows that $\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix}$.

(5 points) Show that every element $x \in B_n(q)_k$ covers $[k] = 1 + q + \cdots + q^{k-1}$ elements and is covered by $[n-k] = 1 + q + \cdots + q^{n-k-1}$ elements.

For $V \in B_n(q)_k$, the number of elements that x covers is precisely the number of $(k-1)$-dimensional subspaces contained in V. Since $V \cong \mathbb{F}_q^k$, this number is $\begin{bmatrix} k \\ k-1 \end{bmatrix} = [k] = (1 + q + q^2 + \cdots + q^{k-1}) = \frac{1-q^k}{1-q}$.

For the other calculation, we use the dual poset, we see that V covers $\begin{bmatrix} n-k \\ n-k-1 \end{bmatrix} = [n-k] = (1+q+q^2+\cdots+q^{n-k-1}) = \frac{1-q^{n-k}}{1-q}$ elements in $B_n(q)^*$. Consequently, V is covered by $\frac{1-q^{n-k}}{1-q}$ elements in $B_n(q)_{k+1}$.

(5 points) Show that if V has dimension k ($0 \leq k \leq n$), then the interval $[0, V]$ in $B_n(q)$ is isomorphic to the poset $B_k(q)$.

Since all dimension k subspaces over a field are isomorphic, it follows that we can think of subspaces of V ($V \subset \mathbb{F}_q^n$ has dim k) as subspaces of $\mathbb{F}_q^k$. It follows that the poset structure is isomorphic and the interval $[\{0\}, V]$ is isomorphic to $B_k(q)$.

(**Bonus 5 points**) Deduce and prove a formula for the values of the Möbius function $\mu(\{0\}, V)$ for general $B_n(q)$.

Since an interval $[\{0\}, V]$ is isomorphic to $B_k(q)$, $\mu(\{0\}, V)$ only depends on the dimension of V. Furthermore, μ can be inductively computed as long as it is known how many elements W reside in the interval below V and the values of $\mu(\{0\}, W)$ for all subspaces W.

We show that

$$\mu(\{0\}, V\} = (-1)^{\text{ dim } V} q^{\binom{\text{ dim } V}{2}}.$$

The base case is clear enough. Let $n = \text{dim } V$.

By using the symmetry of the q-binomial coefficient, we can turn the identity of Lemma 6.5,

$$\begin{bmatrix} n \\ (n-k) \end{bmatrix} = \begin{bmatrix} n-1 \\ (n-k) \end{bmatrix} + q^{n-(n-k)} \begin{bmatrix} n-1 \\ (n-k)-1 \end{bmatrix},$$

into

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}.$$

Consequently, we see that

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} = 0$$

since the left-hand-side becomes

$$\begin{aligned} &\sum_{k=0}^{n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} (-1)^k q^{\binom{k}{2}} + \sum_{k=0}^{n} \begin{bmatrix} n-1 \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}+k} \\ = &\sum_{j=0}^{n-1} \begin{bmatrix} n-1 \\ j \end{bmatrix} (-1)^{j+1} q^{\binom{j+1}{2}} + \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix} (-1)^k q^{\binom{k+1}{2}}. \end{aligned}$$

However these two summands are the same except for sign, thus the entire sum is zero.

If we assume that $\mu(\{0\}, W) = (-1)^k q^{\binom{k}{2}}$ for any W of dimension $k < n$ and we require

$$\begin{aligned} 0 &= \sum_{W \subset V} \mu(\{0\}, W\} \\ &= \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} + \mu(\{0\}, V), \end{aligned}$$

then $\mu(\{0\}, V) = (-1)^n q^{\binom{n}{2}}$.

Define operators $U_i : \mathbb{R}B_n(q)_i \to \mathbb{R}B_n(q)_{i+1}$ and $D_i : \mathbb{R}B_n(q)_i \to \mathbb{R}B_n(q)_{i-1}$ by

$$U_i(x) = \sum_{y \in B_n(q)_{i+1}, y > x} y \quad \text{and} \quad D_i(x) = \sum_{z \in B_n(q)_{i-1}, z < x} z.$$

(10 points) Show that $D_{i+1}U_i - U_{i-1}D_i = ([n-i] - [i])I_i$.

Let $y \in B_n(q)_i$, $y \neq x$. The coefficient of y in the expansion of $D_{i+1}U_i(x)$ is precisely the number of subpsaces $z \in B_n(q)_{i+1}$ such that $x < \cdot z$ and $y < \cdot z$. If $Span(x, y)$ has dimension $(i+1)$ then there is only one choice of z (namely $z = x + y$). Otherwise, there is no such z.

On the other hand, the coefficient of x in $D_{i+1}U_i(x)$ is the number of subspaces, which is $[n-i]$ from above.

By analogous logic, the coefficient of $y \neq x$ in the expansion of $U_{i-1}D_i(x)$ is 1 if $x \cap y \in B_n(q)_{i-1}$ and 0 otherwise. Since we are working with subspaces, we have the identity

$$dim x + dim y = dim(Span(x, y)) + dim(x, y).$$

Thus $dim(Span(x, y)) = i+1$ if and only if $dim(x \cap y) = i-1$ so the coefficient of $D_{i+1}U_i - U_{i-1}D_i(x)|_y$ is zero for $y \neq x$.

Since the coefficient of x in $U_{i-1}D_i(x)$ is $[i]$ we obtain that

$$D_{i+1}U_i - U_{i-1}D_i = ([n-i] - [i])I_i$$

as desired.

(10 points) Deduce that $B_n(q)$ is rank-unimodal and Sperner.

For $i < n/2$, $[n-i] - [i]$ is a nonzero polynomial in q with nonnegative coefficients. Thus evaluating this quantity at q a positive prime power yields a positive integer. The proof is then analogous to the proof of Theorem 4.7 and Corollary 4.8.