# Course 18.312: Algebraic Combinatorics

Solution Set # 3

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1) (5 points) Find the unique four-element poset which is not a series-parallel poset.

The poset $P_N = \{1, 2, 3, 4\}$ where $1 > 2$, $2 < 3$ and $3 > 4$. (The Hasse diagram looks like the letter "N".)

**Solution 1:** To see that this is not a series-parallel poset, we note that since $P_N$ is connected, it must have the form $A \oplus B$ where $\oplus$ is ordinal sum. Since no element is less than or greater than three of the others $A$ and $B$ must both be posets of size two; and the fact that there is no maximal chain of size 3 means that $A$ nor $B$ can be $P_1 \oplus P_1$. However $(P_1 + P_1) \oplus (P_1 + P_1)$ is a different poset.

$$(P_1 + P_1) \oplus (P_1 + P_1) = \{1, 2, 3, 4\} \text{ with } 1 < 3,\ 1 < 4,\ 2 < 3,\ 2 < 4$$

**Solution 2:** Alternatively, since it was given that there is a unique poset of size four that is not series-parallel, we can also show that the other 15 posets are series-parallel.

We see the possibilities are:

$$\begin{array}{rl} P_1 + P_1 + P_1 + P_1 & (P_1 \oplus P_1) + P_1 + P_1 \\ (P_1 \oplus P_1 \oplus P_1) + P_1 & (P_1 \oplus P_1) + (P_1 \oplus P_1) \\ ((P_1 + P_1) \oplus P_1) + P_1 & (P_1 \oplus (P_1 + P_1)) + P_1 \\ (P_1 + P_1) \oplus (P_1 + P_1) & ((P_1 \oplus P_1) + P_1) \oplus P_1 \\ P_1 \oplus ((P_1 \oplus P_1) + P_1) & P_1 \oplus (P_1 + P_1) \oplus P_1 \\ (P_1 + P_1) \oplus P_1 \oplus P_1 & P_1 \oplus P_1 \oplus (P_1 + P_1) \end{array}$$

$$(P_1 + P_1 + P_1) \oplus P_1 \qquad P_1 \oplus (P_1 + P_1 + P_1)$$
$$P_1 \oplus P_1 \oplus P_1 \oplus P_1$$

We leave it to the reader to match this up these expressions to the other 15 posets.

(10 points) Prove that the map $\mu : (B_n)_i \to (B_n)_{i+1}$ defined on page 26 of Section 4 of the notes (and in class on Wednesday) is injective when $i < n/2$, and thus an order matching.

**Because of the unexpected difficulty of this problem, this problem was graded out of 15 points, rather than the original 10 points (giving students the chance of up to 5 possible bonus points)**

Given two elements $S$ and $T$ of $(B_n)_i$ (i.e. $i$-element subsets) we see that $\mu(S) = \mu(T) \in (B_n)_{i+1}$ is possible if and only if there exists $s, t$ $(s \neq t) \in \{1, 2, \ldots, n\}$ such that $S \cup \{t\} = \mu(S) = \mu(T) = T \cup \{s\}$. In other words, if there exists $s$ and $t$ such that $S \cap T = S \setminus \{s\} = T \setminus \{t\}$.

Thus the corresponding $+/-$'s sequences for $S$ and $T$ must look identical except for the locations indexed by $s$ and $t$. Without loss of generality, assume that $s < t$ and thus $seq(S)$ contains a $+$ in the $s$th spot and a $-$ in the $t$th spot, while $seq(T)$ contains a $-$ in the $s$th spot and a $+$ in the $t$th spot. $seq(S)$ and $seq(T)$ look identical elsewhere.

$$\begin{aligned} seq(S) &= * * * * * * * + * * * * * * - * * * \\ seq(T) &= * * * * * * * - * * * * * * + * * * \end{aligned}$$

We prove the injectivity by induction on $t - s$. We first show that $t - s \geq 2$. Consider the $(t-1)$th position of $seq(S)$. This character cannot be a $+$ sign since then the $-$ sign in the $t$th position of $seq(S)$ would then be matched.

Thus we know that the $(t-1)$th position of both $seq(S)$ and $seq(T)$ must be a $-$ sign. If $t - s = 2$, we have a contradiction since the $-$ sign in position $(t-1) = (s+1)$ of $seq(S)$ is matched with the $+$ sign in position $s$. However, there is no such $+$ sign in $seq(T)$ and if there were an earlier $+$ sign to match up with the $-$ sign in the $(t-1)$st position, it could be used in $seq(S)$ to match up the $-$ sign in the $t$th position; thus leading to a contradiction.

Thus assume $t - s > 2$ and consider the $(t-2)$nd position. If it is a $+$ sign, then the $-$ sign in position $(t-1)$ in both $seq(S)$ and $seq(T)$ matches with it and we can remove these two characters, essentially shrinking the region between the $s$th and $t$th positions. Thus we get a contradiction by induction in this case.

On the other hand, if the $(t-2)$nd position is a $-$ sign, we continue until we find the last $+$ sign in the positions labeled $\{s+1, s+2, \ldots, t-1\}$. As long as we can find one, say it is in the $i$th position, we then match up the $+$ sign in the $i$th position with the $-$ sign in the $(i+1)$st position, and again shrink the region between the $s$th and $t$th position, and obtain a contradiction by induction.

Lastly if all of the characters in positions $\{s+1, s+2, \ldots, t-1\}$ are $-$ signs, then they can only match up with $+$ signs preceding the $(s+1)$st position. However, $seq(T)$ has one less $+$ sign than $seq(S)$ in this region, thus at least one of the $-$ signs in positions $\{s+1, s+2, \ldots, t-1\}$ will not be able to be matched up. Thus the $-$ sign in position $s$ of $seq(T)$ cannot be the last unmatched $-$ sign, which is a contradiction.

2) (5 points) Show that for $(x, y) \leq (x', y') \in P \times Q$ that

$$\mu_{P\times Q}\Big((x,y),(x',y')\Big) = \mu_P(x,x')\mu_Q(y,y').$$

We use the definition of the Mobius function that $\mu$ is the unique function satisfying

$$\sum_{y\in[x,z]} \mu(x,y) = \delta_{xz},$$

for all intervals $[x, z]$. Here $\delta_{xz} = 1$ if $x = z$ and equals 0 otherwise.

Since $(x, x') \leq_{P\times Q} (z, z')$ if and only if $x \leq_P z$ And $x' \leq_Q z'$ it follows that the Mobius function on $P \times Q$ satisfies

$$\begin{aligned}\sum_{(y,y')\in[(x,x'),(z,z')]} \mu((x,x'),(y,y')) &= \sum_{y\in[x,z],\ \ y'\in[x',z']} \mu((x,x'),(y,y')) \\ &= \delta_{(x,z),(x',z')} = \delta_{x,z}\delta_{x',z'}.\end{aligned}$$

We assume by induction on the number of elements in the interval $[x, z]\times[x', z']$ (notice that the base case where the intervals are $[x, x] = \{x\}$ and $[z, z] = \{z\}$ is clear) that $\mu_{P\times Q}((x, x'),(y, y')) = \mu_P(x, y)\mu_Q(x', y')$.

We thus obtain that

$$\begin{aligned}
\Big(\sum_{y\in[x,z]} \mu_P(x,y)\Big)\Big(\sum_{y'\in[x',z']} \mu_Q(x',y')\Big) &= \delta_{x,z}\delta_{x',z'} \\
&= \sum_{y\in[x,z],\quad y'\in[x',z']} \mu_{P\times Q}\Big((x,x'),(y,y')\Big) \\
&= \sum_{x<y\le z,\quad y'\in[x',z']} \mu_P(x,y)\mu_Q(x',y') \\
&+ \sum_{x\le y'<z'} \mu_P(x,z)\mu_Q(x',y') \\
&+ \mu_{P\times Q}\Big((x,z),(x',z')\Big).
\end{aligned}$$

We conclude that $\mu_{P\times Q}\Big((x,x'),(z,z')\Big) = \mu_P(x,z)\mu_Q(x',z')$.

(10 points) Show that the Möbius function on the boolean poset $B_n$ is given by the formula

$$\mu_{B_n}(S,T) = (-1)^{|T|-|S|}$$

whenever $S \subset T \subset \{1,2,\ldots,n\}$.

(**Hint:**) Show that the boolean poset $B_n$ is isomorphic to the poset $(C_2)^n = C_2 \times \ldots \times C_2$.

A subset $S$ (i.e. an element of $B_n$) can be represented by an ordered $n$-tuple $(a_1,a_2,\ldots,a_n)$ where each $a_i \in \{0,1\}$, and $a_i = 1$ if and only if $i \in S$. Furthermore, $S \le T$ iff every element of $S$ is also an element of $T$, meaning that the sequence corresponding to $T$ has a one in every position where $S$ does. Consequently the image of this map is the set $\{0,1\}^n$ with the desired poset structure. It is easy to see that $\mu_{C_2}(0,0) = 1 = \mu_{C_2}(1,1)$ and $\mu_{C_2}(0,1) = -1$. By the multiplicativity of the Möbius function, we obtain that $\mu_{B_n}(S,T) = (-1)^{\#\text{ elements which } T \text{ contains that } S \text{ does not}}$.

(5 points) Show that if positive integer $n$ factors as $n = p_1^{e_1}\cdots p_m^{e_m}$, (the $p_i$'s are prime) then the poset of divisors of $n$, $D_n$ is isomorphic to $C_{e_1+1}\times\cdots\times C_{e_m+1}$.

We map an integer $d \in D_n$ to $C_{e_1+1}\times\cdots\times C_{e_m+1}$ by $\phi(d) = (a_1,\ldots,a_m)$ where $a_i$ equals the highest power of $p_i$ which divides $d$. Since $d|n$ and $n = p_1^{e_1}\cdots p_m^{e_m}$, it follows that $0 \le a_i \le e_i$. Furthermore, $d|d'$ if and only if each power $a_i$ corresponding to $d$ is less than or equal to the corresponding exponent of $p_i$ in $d'$.

(10 points) Show that $\mu_{D_n}(1,d)$ agrees with the the number theoretic Möbius function $\hat{\mu}(d)$ defined in Lecture Notes from February 18th.

$\mu_{C_n}(0,0) = 1$ implies that $\mu_{C_n}(0,1) = -1$ which yields $\mu_{C_n}(0,k) = 0$ for $k \geq 2$.

Thus $\mu_{D_n}(1,d) = 0$ whenever $d$ contains a prime to a power of two or more, and otherwise is $(-1)^{\#\text{ distinct primes in its factorization}}$. This agrees with the definition of $\hat{\mu}(d)$.

3) Let $G = \{1, \pi\}$ be a group of order two (with identity element 1). Let $G$ act on $\{1,2,3,4\}$ by $\pi \cdot 1 = 1$, $\pi \cdot 2 = 3$, $\pi \cdot 3 = 2$, $\pi \cdot 4 = 4$.

(5 points) Draw the Hasse diagram of the quotient poset $B_4/G$.

**See attachment.**

(5 points) What is the size of the largest antichain? List all antichains of this size.

This poset is Sperner by Theorem 5.9, so the size of the largest antichain is the same as the size of the largest level. In this case, the level itself, $\{\{23\}, \{12, 13\}, \{24, 34\}, \{14\}\}$ is the only antichain of this size. This follows by inspection: for example the orbit $\{123\}$ is incomparable only to $\{234\}, \{124, 134\}, \{24, 34\}, \{14\}$, and $\{4\}$ and no matter which of these five other elements we add, we can add at most one more element to retain an antichain.

(10 points) Draw the Hasse diagram and do the same computations for the action $\pi \cdot 1 = 4$, $\pi \cdot 2 = 3$, $\pi \cdot 3 = 2$, $\pi \cdot 4 = 1$.

**See attachment for Hasse diagram.** Again this poset has the Sperner property and the level $\{\{14\}, \{12, 34\}, \{13, 24\}, \{23\}\}$, an antichain of size 4 is of maximal size. There are no other antichains of this size. This is even easier to see in this case. We try to build an antichain of size 4 that contains the orbit $\{124, 134\}$ and see that there are only two elements left which are incomparable with $\{124, 134\}$. By symmetry, the argument is analogous if we try to build an antichain with another element of level 1 or 3.

**Note:** Some students incorrectly asserted that there is only one antichain of maximal size by Sperner's Theorem. Sperner's Theorem does not imply uniqueness of a maximal antichain. For example, consider the boolean poset $B_n$ for $n$ odd.

4) A $(0,1)$-necklace of length $n$ and weight $i$ is a circular arrangement of $i$ 1's and $(n-i)$ 0's. For instance the $(0,1)$-necklaces of lengths 6 and weight 3 are (writing a circular arrangement linearly) 000111, 001011, 010011, and 010101. (Cylcic shifts of a linear word represent the same necklace, e.g., 000111 is the same as 110001.) Let $N_n$ denote the set of all $(0,1)$-necklaces of length $n$. Define a partial order on $N_n$ by letting $u \leq v$ if we can obtain $v$ from $u$ by changing some 0's to 1's. It is easy to see (you may assume it) that $N_n$ is graded of rank $n$, with the rank of a necklace being its weight.

(15 pts) Show that $N_n$ is rank-symmetric, rank-unimodal, and Sperner.

**Hint:** Show that $N_n \cong B_n/G$ for a suitable group $G$.

Let $G$ be the cyclic group of order $n$, generated by the $n$-cycle $\sigma$ defined as $\sigma(1) = 2, \sigma(2) = 3, \ldots, \sigma(n-1) = n, \sigma(n) = 1$. A subset $S$ of $\{1, 2, \ldots, n\}$ corresponds to a sequence $a_1 a_2 \cdots a_n$ where $a_i \in \{0, 1\}$ by letting $a_i = 1$ if $i \in S$ and $a_i = 0$ otherwise. Thus as $G$ acts on $B_n$, $G$ also acts on the set of $(0,1)$-sequences by

$$\sigma \cdot a_1 a_2 \cdots a_n = a_2 a_3 \cdots a_n a_1.$$

Thus two subsets (resp. sequences) are in the same $G$-orbits iff they are cyclic shifts of one another, which is equivalent to saying that they define the same necklace. Moreover, $x \leq y$ in $B_n/G$ if and only if the corresponding sequences have 0's that can be changed to 1's. Thus $N_n \cong B_n/G$ and by Theorem 5.9, $N_n$ is rank-symmetric, rank-unimodal, and Sperner.

5) Define the **shift** of a linear word $a_1 a_2 \ldots a_n$ to be the linear word $a_n a_1 a_2 \ldots a_{n-1}$, and define the **period** to be the smallest number of shifts needed to return to the original word. For example, the period of 010101 is 2, the period of 101101 is 3, while the period of 001011 is 6.

$$(\#\text{ number of necklaces of length } n) = \sum_{d|n} c_d \; (\#\text{ strings of period } d)$$

for some choice of $c_d$'s.

(5 points) Show this identity with the proper $c_d$'s filled in.

For $k \in \mathbb{Z}$, let $\overline{k}$ denote $k$'s image in $\{1, 2, \ldots, n\}$ modulo $n$. Note that $a_1 a_2 \ldots a_n$ and $a_{\overline{1+k}} a_{\overline{2+k}} \cdots a_{\overline{n+k}}$ denote the same necklace for all integers $k$. If $a_1 a_2 \cdots a_n$ is a string of period $d$, then $a_{\overline{1+d}} a_{\overline{2+d}} \cdots a_{\overline{n+d}}$ is also the same string as $a_1a_2\cdots a_n$, but for $k \in \{0,1,\ldots,d-1\}$, the $a_{\overline{1+k}}a_{\overline{2+k}}\cdots a_{\overline{n+k}}$'s are different strings.

Hence we have proved the above identity with

$$c_d = \frac{1}{d}.$$

(5 points) What is a closed expression for $\sum_{d|n}(\#\text{ strings of period } d)$ ?

Any string of length $n$ has a a period which divides $n$. Thus this expression is equal to the total number of $(0,1)$-strings of length $n$, which is

$$2^n.$$

(10 points) Use Möbius inversion to obtain a closed formula for the number of necklaces of length $n$.

Let $S(d)$ denote the number of strings of length $n$ of period $d$. (In fact $S(d)$ is the same quantity regardless of what $n$ is, as long as $d|n$.) We use Möbius inversion on the expressions

$$2^n = \sum_{d|n} S(d)$$

to obtain

$$S(d) = \sum_{e|d} \hat{\mu}(e)2^{d/e} = \sum_{e'|d} \hat{\mu}\Big(\frac{d}{e'}\Big)2^{e'}.$$

(Here we have let $e' = \frac{d}{e}$.)

Substituting this expression into the above, we get

$$N_n = \sum_{d|n}\sum_{e'|d} \frac{1}{d}\hat{\mu}\Big(\frac{d}{e'}\Big)2^{e'}.$$

Bonus points were awarded for simplifying this double-sum into a single sum using algebra and the Euler $\phi$ function:

$$N_n = \sum_{d|n} \frac{1}{d}\Big(\sum_{e'|d} \hat{\mu}\Big(\frac{d}{e'}\Big)2^{e'}\Big)$$

$$
\begin{aligned}
&= \sum_{e'|n} 2^{e'} \Big( \sum_{d \quad s.t. \quad e'|d, \quad d|n} \frac{1}{d} \hat{\mu}\Big(\frac{d}{e'}\Big)\Big) \\
&= \sum_{e'|n} 2^{e'} \Big( \sum_{d'|(\frac{n}{e'})} \frac{1}{d'e'} \hat{\mu}(d')\Big) \\
&= \frac{1}{n} \sum_{e'|n} 2^{e'} \Big( \sum_{d'|(\frac{n}{e'})} \frac{n/e'}{d'} \hat{\mu}(d')\Big) \\
&= \frac{1}{n} \sum_{e'|n} 2^{e'} \phi\Big(\frac{n}{e'}\Big).
\end{aligned}
$$

An alternate solution yields the identity

$$
N_n = \frac{1}{n} \sum_{k=1}^{n} 2^{gcd(k,n)}.
$$