Course 18.312: Algebraic Combinatorics

Solution Set # 2

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1) Start with n coins heads up. Choose a coin at random(each equally likely) and turn it over. Do this a total of ℓ times.

(10 points) What is the probability that all coins will have heads up? This set of n coins can be modeled as ${\mathbb{Z}_2}^n$. We correspond coins to components of ${\mathbb{Z}_2}^n$ and for each coin 0 and 1 denote heads and tails respectively. Therefore switching one coin is equivalent to switching one component of $v \in {\mathbb{Z}_2}^n$. This is the exact definition of the edges of C_n. So flipping coins ℓ times is the same thing as a walk of length ℓ in C_n. Thus a walk that goes from all heads to all heads is a closed walk of length ℓ. Let A be the adjacency matrix of C_n. The number of closed walks of length ℓ from u to u can be computed as follows:

$$(A^\ell)_{uu} = \frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}(n-2i)^\ell \tag{1}$$

Since there are n^ℓ that one could flip ℓ times, so the probability is

$$\frac{1}{n^\ell}\frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}(n-2i)^\ell \tag{2}$$

(10 points)What is the probability that all coins will have tails up?

Same as previous part, except now we compute the probability that all coins have tails up. The number of ways to get from all heads to all tails can be computed as follows:

$$A^\ell{}_{\hat{0}\hat{1}} = \sum_{i=0}^{n}\sum_{j=0}^{n}(-1)^j\binom{k}{j}\binom{n-k}{i-j}(n-2i)^\ell\Big|_{k=n} = \frac{1}{2^n}\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-2i)^\ell \tag{3}$$

Note that $\binom{0}{i-j} = 0$ unless $i = j$, in which case $\binom{0}{i-j} = 1$. And dividing by n^ℓ again, the probability is

$$\frac{1}{n^\ell}\frac{1}{2^n}\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-2i)^\ell \tag{4}$$

(10 points) If we turn over two coins at a time, what is the probability that all coins will have heads up?

First, we need to find the eigenvalues of graph G with vertex set $\mathbb{Z}_2{}^n$ and edges connecting vertices differing in exactly two coordinates. We use the Radon transform. For $\Gamma = \{\delta_i + \delta_j : 1 \leq i < j \leq n\}$ and $\lambda_u = \sum_{1\leq i<j\leq n}(-1)^{u\cdot(\delta_i+\delta_j)}$ for $u \in \mathbb{Z}_2^n$. If the Hamming weight of u is $W(u)$, then $u \cdot (\delta_i + \delta_j) = 0$ for $\binom{W(u)}{2} + \binom{n-W(u)}{2}$ of the possible values of (i, j) and $u \cdot (\delta_i + \delta_j) = 1$ for the $W(u)(n - W(u))$ remaining values. Thus the eigenvalues are:

$$\lambda_u = \binom{n}{2} - 2W(u)[n - W(u)] \tag{5}$$

Since there are $\binom{n}{i}$ vertices with hamming weight i, the number of closed walks is equal to the sum of ℓth power of eigenvalues, can be computed as follows:

$$\sum_{i=0}^{n}\binom{n}{i}[\binom{n}{2} - 2i(n-i)]^\ell \tag{6}$$

Thus the probability is

$$\frac{1}{2^n\binom{n}{2}^\ell}\sum_{i=0}^{n}\binom{n}{i}\left(\binom{n}{2} - 2i(n-i)\right)^\ell.$$

2) The Perron-Frobenius Theorem asserts the following: Let M be a $p \times p$ matrix of nonnegative real numbers, such that some power of M has all positive entries. Then there is a unique eigenvalue λ of M of maximum absolute value, and λ is real and positive. Moreover, the eigenvector v corresponding to λ is the unique eigenvector of M (up to scalar multiplication) all of whose entries are positive. Now let G be a (finite) graph with vertices $v_1, \cdots, v_p$. Assume that some power of the "probability matrix" $\mathbf{M}(G)$ defined in Section 3 has positive entries. (It's not hard to see that this is equivalent to G being connected and containing at least one cycle of odd length, but you do not need

to show this.) Let d_k denote the degree (number of incident edges) of vertex v_k. Let $D = d_1 + d_2 + \ldots + d_p = 2q - r$, where G has q edges and r loops. Start at any vertex of G and do a random walk on the vertices of G as defined in Section 3. Let $p_k(l)$ denote the probability of ending up at vertex v_k after ℓ steps.

(15 points)Assuming the Perron-Frobenius Theorem, show that $\lim_{l\to\infty} p_k(l) = d_k/D$.

Let us first prove that $[d_1/D \cdots d_p/D]$ is a left eigenvector of $M(G)$.

$$[d_1/D \cdots d_p/D]M(G) = [\frac{m_{11}d_1 + \cdots + m_{1p}d_p}{D} \cdots \frac{m_{p1}d_1 + \cdots + m_{pp}d_p}{D}] \tag{7}$$

Since

$$m_{ij} = \frac{(A(G)_{ij})}{d_j} \tag{8}$$

So we have

$$\frac{m_{i1}d_1 + \cdots + m_{ip}d_p}{D} = \sum_{j=1}^{p} \frac{(A(G))_{ij}.d_j}{D \cdot d_j} = \frac{\sum_{j=1}^{p}(A(G))_{ij}}{D} \tag{9}$$

On the other hand,

$$\sum_{j=1}^{p}(A(G))_{ij} = d_i,$$

so

$$[d_1/D \cdots d_p/D]M(G) = [d_1/D \cdots d_p/D].$$

Therefore $v_1 = [d_1/D \cdots d_p/D]$ is the unique eigenvector with all positive entries for eigenvalue $\lambda_1 = 1$, and $\lambda_1 = 1$ is the unique eigenvalue with an absolute value that high. Now suppose $v_1, ..., v_p$ form an eigenbasis. Therefore, we can write each e_i as a unique linear combination of $v_1, ..., v_p$. $e_i = c_{1i}v_1 + \cdots + c_{pi}v_p$. We know that $e_i \cdot M(G)^\ell = [s_1 \cdots s_p]$, where s_k is the prob of v_i to v_k in ℓ steps.

Note that we can replace e_i with $c_{1i}[d_1/D \cdots d_p/D] + c_{2i}v_2 + \cdots c_{pi}v_p$. Therefore,

$$[c_{1i}[d_1/D \cdots d_p/D] + c_{2i}v_2 + \cdots c_{pi}v_p]M(G)^\ell = c_{1i}1^\ell[d_1/D \cdots d_p/D] + c_{2i}{\lambda_2}^\ell v_2 + \cdots c_{pi}{\lambda_p}^\ell v_p \tag{10}$$

Since $|\lambda_i| < 1$, so $\lim_{l\to\infty}{\lambda_i}^\ell = 0$ for $i = 2, \cdots, p$.

Therefore, the probability that we go from v_i to v_j is the jth coordinate of $c_{1i}[d_1/D \cdots d_p/D]$. Hence, it suffices to determine $c_{1i} = 1$ and then $\lim_{l \to \infty} = \frac{d_k}{D}$.

Now, let's assume $c_{i1} \neq 1$ for one or more $i \in [p]$. Then $\sum_{k=1}^{p} \lim_{l \to \infty} p_k(l) = c_{1i} \frac{d_1 + \cdots + d_p}{D} = c_{1i}$.

Since $\sum_{k=1}^{p} \lim_{l \to \infty} p_k(l) = 1$ and $D = d_1 + d_2 + \cdots + d_p$, we conclude that $c_{1i} = 1$ for all i's. And this complete the proof of $\lim_{l \to \infty} p_k(l) = d_k/D$.

(5 points) If G is connected, regular, and bipartite, show that $M(G)$ has an eigenvalue of -1. Why is -1 not an eigenvalue of $M(G)$ if G is regular, connected, but not bipartite?

The graph G is regular, $A(G)$ and $M(G)$ have the same eigenvalues. Since G is a regular connected bipartite graph, each part has the same number of vertices n. We order vertices of G so that vertices of second part W appear after vertices of first part U: $u_1, \cdots, u_n, w_1, \cdots, w_n$. Let's $A(G)$ be the adjacency matrix of graph G. If we multiply $A(G)$ with vector $v = [1 \cdots 1 - 1 \cdots - 1]^T$ which has n 1 and n -1 entries. It's easy to see that $A(G)v = -v$. Therefore -1 is an eigenvalue of this graph and v is the associated eigenvector. Note that $A(G)$ and $M(G)$ have the same eigenvalues.

Now we would like explain why is -1 not an eigenvalue of $M(G)$ if G is regular, connected, but not bipartite? The reason is non bipartite graph has odd cycle and it's connected. Easily it can be seen that this graph satisfy Perron-Frobenius Theorem assertion. This matrix has an eigenvalue of maximum absolute value $+1$. So it cannot have the eigenvalue -1 as well, since there is unique eigenvalue λ of M of maximum absolute value.

3) (5 points) Draw Hasse diagrams of the 16 nonisomorphic four-element posets. (Bonus 5 points): Draw also the 63 five-element posets) see attachment

   (10 points) Draw the Hasse diagram for the poset of nonisomorphic simple graphs with 4 vertices (with the subgraph ordering). What is the size of the largest antichain? How may antichains have this size? See the attachment.

4) Let P be a finite poset and $f : P \to P$ an order-preserving bijection. I.e. f is a bijection (one-to-one and onto), and if $x \leq y$ in P then $f(x) \leq f(y)$.

   (10 points) Show that f is an automorphism of P, i.e. f^{-1} is order-preserving.

   We label the vertices of poset P, by $x_1, \cdots, x_n$. Thus we have the $f(x_i) = y_i$.

Let us first prove the following facts:

1. For every $1 \leq i \leq n$, there exists m_i such that $f^{m_i}(y_i) = y_i$.

   Since P is finite, for each y_i there exits $a_i > b_i$, such that $f^{a_i}(y_i) = f^{b_i}y_i$. Just set $m_i = a_i - b_i$.

2. There exists $k \in \mathbb{N}$, such that $f^k(y_i) = x_i$ for $1 \leq i \leq n$.

   Set $k = m_1 m_3 \cdots m_n - 1$. Therefore, for all i, $f^{k+1}(y_i) = y_i$. So, $f^k(y_i) = x_i = f^{-1}(y_i)$, for all $y_i \in P$

By induction we can see f^n for positive integers n as well as $f^k = f^{-1}$ is order preserving. Hence f is an automorphism.

Another solution: For each $x, y \in P$, set $g(x, y)$ to be $(f(x), f(y))$. Note that g maps the subset C of comparable pairs (x, y) (namely, $x \leq y$ in P) in $P \times P$ injectively to itself. As C is finite, then g must be bijective. Hence, all comparable pairs are images of g. This implies that f is an automorphism.

(5 points) Show that the last statement is false if we do not assume that P is finite.

We define P to be a countable collection of copies of $\mathbb{Z}$. In other words, $P = \mathbb{N} \times \mathbb{Z}$ with the following order relation:

$(x, y) \leq (w, z)$ if and only if $x \leq w$. We define the function f as follows: $f(0, x) = (0, 2x)$,$f(1, x) = (1, 2x + 1)$ and $f(y, x) = (y - 1, x)$ for $y \geq 2$. Therefore $f^{-1}(y, x) = (y + 1, x)$ for $y \geq 1$, $f^{-1}(0, 2x) = (0, x)$ and $f^{-1}(0, 2x + 1) = (1, x)$.

It's not hard to see that $(0, x) \leq (0, y), (z, y)$, for all $x, y \in \mathbb{Z}$ and $z \in \mathbb{N}$. Therefore, $f(0, x) = (0, 2x)$ is less than or equal to all elements of P. For all $x, y \in \mathbb{Z}$, $(1, x) > (0, y)$ and $f(1, x) = (1, 2x + 1) > f(0, y)$.

$(2, x) \geq (2, w)$=$, (1, y), (0, z)$. We see that $f(2, x) = (1, x)$ where $(1, x) \geq f(2, w)(=), f(1, y), f(0, z)$. Since f preserves order for (x, y) with $x = 0, 1, 2$, and for $x > 2$, f just moves x to $x - 1$. By induction we see that f is order preserving.

In this case f^{-1} is not order preserving: Consider $(0, 1) \leq (0, 2)$. $f^{-1}((0, 1)) = (1, 0) \not\leq (0, 1) = f^{-1}((0, 2))$. We are done.

5) We say that a graph G is regular if all vertices of G have the same degree; G is known as integral if its adjacency matrix has only integral eigenvalues;

and $\overline{G}$, the complement of simple graph G, is the simple graph such that $V(\overline{G}) = V(G)$ and $\{u,v\} \in E(\overline{G})$ if and only if $\{u,v\} \in E(G)$.

(5 points) Show that if G is a simple and regular integral graph, then so is $\overline{G}$.

Method 1)Note that for regular graph G of degree r, if v is an eigenvector of $A(G)$ with eigenvalue λ_v, then $A(\overline{G})v = (J - I - A(G))v = Jv - v - \lambda_v v$. However $Jv = nv$ and $Av = rv$ if $[1,1,\dots,1]^T$ and $Jv = 0$ if v is orthogonal to $[1,1,\dots,1]^T$, i.e. any other eigenvector. Thus v is also an eigenvector of $\overline{G}$, with eigenvalue $n-1-r$ or $-1-\lambda_v$, depending on whether v is a scalar multiple of, or orthogonal to, $[1,1,\dots,1]^T$. So, if G is a simple and regular integral graph, then so is $\overline{G}$.

Method 2)Since the characteristic polynomial $P_{\overline{G}}(\lambda)$ of the complement $\overline{G}$ of a regular graph G on n vertices of degree r can be expressed as

$$P_{\overline{G}}(\lambda) = (-1)^n \frac{\lambda - n + r + 1}{\lambda + r + 1} P_G(-\lambda - 1);$$

we see that the complement of an integral regular graph must be integral

The disconnected union of G and H, denoted as $G \sqcup H$, has vertex set $V(G \sqcup H) = V(G) \cup V(H)$ and edge set $E(G \sqcup H) = E(G) \cup E(H)$.

(5 points) Show that if G and H are integral graphs, then so is $G \sqcup H$. If we order vertices of H to be bigger that vertices of G and consider the the adjacency matrix of $G \sqcup H$. It's easy to see that $A(G \sqcup H) = A(G) \oplus A(H)$. Therefore, $P_{G \sqcup H}(\lambda) = P_G(\lambda) P_H(\lambda)$. The spectrum of $G \sqcup H$ is the union of the spectra of its components G and H. So, if G and H are integral graphs, then is $G \sqcup H$.

We say that a graph S_{n+1} is an n-star if the vertices are in bijection with $\{0,1,2,\cdots,n\}$ and the edges of S_{n+1} are of the form $\{\{0,1\},\{0,2\},\{0,3\}\cdots,\{0,n\}\}$. (10 points) For what $n \geq 1$ is S_{n+1} an integral graph?

The complete bipartite graph $K_{m,n}$ has $\sqrt{mn}; 0^{m+n-2}; -\sqrt{mn}$ as its eigenvalues, it is integral if and only if mn is a perfect square. Thus, the stars $K_{1,n} = S_{n+1}$ is integral if and only if $n = p^2$ for some integer p, we get an infinite series of integral graphs.

Bonus Problems : We say that a graph Z_n is an n-cycle if the vertices are in bijection with $1, 2, \cdots, n$ and Z_n has n edges: $\{\{1,2\},\{2,3\},\cdots,\{n-1,n\},\{n,1\}\}$. (Notice that Z_1 and Z_2 are not simple graphs.)

(Bonus 5 points) For what $n \geq 2$ is Z_n an integral graph? (Hint: The eigenvalues of $A(Z_n)$ will involve trigonometric quantities or roots of unity.)

The adjacency matrix of a cycle graph looks like

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{bmatrix}$$

and one of its eigenvalues is $2\cos(2\pi/n) = e^{2\pi/n} + e^{-2\pi/n}$ with eigenvector $[1, e^{2\pi/n}, e^{4\pi/n}, e^{6\pi/n}, \dots, e^{2\pi(n-1)/n}]$ since $A(Z_n)$ is circulant and $e^{2\pi/n}$ is an nth root of unity. (In fact all the other eigenvectors are also of the form $[1, \omega, \omega^2, \dots, \omega^{n-1}]$ where $\omega^n = 1$.) Since $2\cos(2\pi/n)$ is an integer iff $n = 2, 3, 4, 6$ and it is easy to check that the other eigenvalues for $n = 2, 3, 4, 6$ are also integers, it follows that Z_2, Z_3, Z_4, and Z_6 are the only examples of cyclic graphs with integral eigenvalues.

(Bonus 5 points) There are 8 connected and simple integral graphs with at most 5 vertices. Find these and give brief reasoning as to why each has integral eigenvalues. (Hint: Seven out of eight of these are familiar graphs.

Please see http://mathworld.wolfram.com/IntegralGraph.html

Regarding to what we see before, complete graph $K_1, \cdots, K_5$, star S_5, and the cycle Z_4 are integral graphs. The last one is a little bit more challenging.

(Bonus 5 points) Find the six connected and simple integral graphs with 6 vertices, and give brief reasoning as to why each has integral eigenvalues. (Hint: Five out of six of these 6-vertex integral graphs are regular and the other 6-vertex integral graph is a tree.)

Please see http://mathworld.wolfram.com/IntegralGraph.html

We know that K_6, C_6 and it's complement, 1-skeleton of octahedron the complement of 3 disjoint diagonals are integral graph as well as the star S_6.

Here is a survey on integral graphs:

`http://matematika.etf.bg.ac.yu/publikacije/pub/P13(02)/rrad06.pdf`