Course 18.312: Algebraic Combinatorics

Solution Set # 10

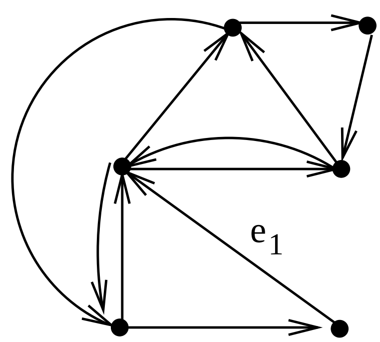
Due Friday May 1, 2009

You may discuss the homework with other students in the class, but please write the names of your collaborators at the top of your assignment. Please be advised that you should not just obtain the solution from another source. Please explain your reasoning to receive full credit, even for computational questions.

1) Let D be the digraph pictured in the figure depicted below.

   (10 points) How many Euluerian tours are there in D starting with edge e_1?

   (**Note:** You may use a computer algebra package for this problem, but make sure to describe your calculations.)



Solution: By counting in- and out-degrees, we see that D is connected and balanced. Thus, by Corollary 10.5, the number of Eulerian tours starting with edge e_1 is given by

$$\epsilon(D, e_1) = (\det L_0(D)) \prod_{u \in V} (\text{outdeg}(u) - 1)!$$

where $L_0(D)$ is the Laplacian matrix corresponding to digraph D with the row and column corresponding to $\text{init}(e_1)$ deleted. Let us label the six vertices in left-to-right order from top-to-bottom.

$$L(D) = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 3 & -1 & -1 & 0 \\ -1 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}.$$

Under this labeling, $v_6 = \text{init}(e_1)$ and so

$$L_0(D) = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{bmatrix},$$

which by using a computer algebra package such as Maple has a determinant of 6.

We also know that $\det(L_0(G))$ should be the number of oriented spanning trees rooted at v_6 which can also be enumerated in this example.

Thus $\epsilon(D, e_1) = 6 \cdot (2-1)!(1-1)!(3-1)!(2-1)!(2-1)! = 12$.

2) Let $G = P_6$, the path graph on vertices $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ with v_i adjacent to v_{i-1} and v_{i+1} for $2 \leq i \leq 5$. Consider the initial configuration C which contains N chips on vertex v_1 and zero chips on all other vertices.

(5 points) For what values of N does the firing process go on infinitely, and for what values of N does the firing process terminate?

Solution: For the path graph P_6, $|E| = 5$ and $|V| = 6$. By the Theorem of Björner-Lovász-Shor, if $N > 2|E| - |V| = 4$ then the game is infinite. If $N < |E| = 5$ then the game reduces to a unique stable configuration. In this case, it is impossible for $5 = |E| \leq N \leq 2|E| - |V| = 4$ and so we do not have to worry about case of (c) of the theorem where we need to worry about the initial configuration.

Thus if $N \in \{0, 1, 2, 3, 4\}$, the game is finite and if $N \geq 5$, the game is infinite.

Remark: In more generality, if G is path graph P_n, $|E| = n-1$ and $|V| = n$ and thus when $N > 2(n-1) - n = n-2$ then game is infinite and if $N < |E| = n-1$, then the game is finite, and case (c) is not relevant again. As a side note, this case of a path graph was one of the original motivations of Björner-Lovász-Shor for defining the chip-firing game. This originally appeared in “Balancing vectors in the max norm" by Joel Spencer in Combinatorica **6** (1986), pp. 55-65.

(10 points) For the values of N for which the process terminates, describe the possible resulting configurations.

Solution: If $N = 0$, the initial configuration is already stable.

If $N = 1$, vertex v_1 can fire and then the resulting state of

$$[0, 1, 0, 0, 0, 0]$$

is stable.

If $N = 2$, vertex v_1 can fire twice, followed by v_2 firing. Then v_1 can fire again and the resulting state of

$$[0, 1, 1, 0, 0, 0]$$

is stable.

If $N = 3$, vertex v_1 can fire three times, followed by v_2 firing. Then v_1 can fire again and then the resulting state is

$$[0, 2, 1, 0, 0, 0].$$

We then let v_2 fire, followed by v_1 and v_3 firing to obtain

$$[0, 1, 2, 0, 0, 0].$$

We then let v_3 fire, followed by v_1 and v_2 firing yielding the stable configuration

$$[0, 1, 1, 1, 0, 0].$$

We may continue in this way, simulating the firing sequence until reaching a stable configuration. However, instead we prove and use the following claim that holds for general path graphs. Note that we already saw in the solution to the last problem that for $G = P_n$, the configuration with N chips on vertex v_1 and zero on the remaining vertices stabilizes if and only if

$$N \in \{0, 1, 2, \ldots, n-2\}.$$

Claim: There is a unique stable chip configuration $C = [C_1, C_2, \dots, C_n]$ on path graph P_n such that $\sum_{i=1}^{n} C_i = n - 2$.

Proof: A chip configuration C on P_n is stable if and only if $C_1 = 0$, $C_n = 0$, and $C_i \in \{0, 1\}$ for $2 \leq i \leq n-1$. Thus $[4, 0, 0, 0, 0, 0]$ must stabilize to

$$[0, 1, 1, 1, 1, 0].$$

3) Let $G = C_n$ be a cycle graph on n vertices and let v_0 be one of G's vertices.

(5 points) How many critical configurations does C_n have, letting v_0 be the sink vertex?

Solution: This is equal to the number of spanning trees of C_n which can be obtained by deleting any single edge of C_n. Thus the number of critical configurations is n, the number of edges in C_n.

(10 points) Describe the critical configurations of C_n. (**Hint:** To get started, try writing down the critical configurations for small n.)

Let $V(C_n) = \{v_0, v_1, \dots, v_{n-1}\}$ where v_0 is adjacent to v_1 and v_{n-1} while v_i adjacent to v_{i-1} and v_{i+1} for $2 \leq i \leq n-1$.

Since each vertex has degree two, a configuration C of C_n is stable (with respect to sink v_0) if and only $C_i \in \{0, 1\}$ for all $v \in V(G) \setminus \{v_0\}$.

We thus need to find which of these configurations are recurrent. Since the larger the number of chips in C, the closer C is to an unstable configuration, we begin by considering C_{max} such that $C_{max}(v_i) = 1$ for all $v \in V(G) \setminus \{v_0\}$. We let v_0 fire and then v_1 and v_{n-1} have two chips each and are ready to fire. Letting v_1 and v_{n-1} fire, and then v_2 and v_{n-2} each have two chips and v_1, v_{n-1} have no chips. Continuing inductively, we let the pairs (v_2, v_{n-2}), (v_3, v_{n-3}) fire until we get a configuration such that $C(v_i) = 1$ for all but the central most four (or five) vertices furthest away from v_0. In particular, the configuration on the four or five central vertices look like $[0, 2, 2, 0]$ or $[0, 2, 1, 2, 0]$ depending on parity. We let the second and third (resp. and fourth) of these vertices fire and we obtain $[1, 0, 3, 0] \rightarrow [1, 1, 1, 1]$ and $[1, 0, 2, 2, 0] \rightarrow [1, 1, 0, 3, 0] \rightarrow [1, 1, 1, 1, 1]$ and so we see that the stable chip-configuration C_{max} recurs and hence is critical.

We also show that for all $i \in \{1, 2, \dots, n-1\}$ the configuration $C^{(i)}$ is critical where $C^{(i)}$ is defined by $C^{(i)}(v_i) = 0$ and $C^{(i)}(v_j) = 1$ for all $j \neq i$. To see that $C^{(i)}$ is recurrent, we let v_0 fire and then the pairs (v_1, v_{n-1}), (v_2, v_{n-2}), $\dots$, (v_k, v_{n-k}) fire until $i = k+1$ or $i = (n-k)-1$. We then obtain a chip-configuration of the form

$$\begin{aligned}[C_1, C_2, \dots, C_n] &= [1,1,1,\dots,1,0,2,0,1,1,1,1\dots,1,2,0,1,1,\dots,1] \\ &\rightarrow [1,1,\dots,1,1,0,1,1,1,1,1,\dots,1,2,0,1,1,\dots,1].\end{aligned}$$

We then continue firing the vertex with two chips, which will move clockwise the unique vertex with two chips. We then see that we recover $C^{(i)}$.

$$\begin{aligned}&[1,1,\dots,1,1,0,1,2,0,1,1\dots,1,1,1,1,1,\dots,1] \\ \rightarrow\ &[1,1,\dots,1,1,0,2,0,1,1,1\dots,1,1,1,1,1,\dots,1] \\ \rightarrow\ &[1,1,\dots,1,1,1,0,1,1,1,1\dots,1,1,1,1,1,\dots,1]\end{aligned}$$

Thus

$$C_{max}, C^{(1)}, C^{(2)}, \dots, C^{(n-1)}$$

are all distinct critical configurations hence we have found the n critical configurations of cyclic graph C_n.

Remark. See the paper "The Chip Firing Game on n-Cycles" by Janice Jeffs and Suzanne Seager in Graphs and Combinatorics (1995) 11 : 59-67 for more on this example.

(10 points) What is the critical group $K(C_n, v_0)$, written as a product of cyclic groups?

Solution: If n does not contain a power of a prime, it is clear that the answer is $\mathbb{Z}/n\mathbb{Z}$.

By looking at the case $n = 4$, we see that $K(C_4) = \mathbb{Z}/4\mathbb{Z}$ (rather than $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$).

It can be seen that for general n $K(C_n) = \mathbb{Z}/n\mathbb{Z}$. One solution involves looking at the Smith Normal Form of $L_0(C_n)$.

The Laplacian matrix $L(C_n)$ has the form

$$\begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 \\ -1 & 0 & 0 & 0 & \dots & -1 & 2 \end{bmatrix}$$

so letting v_0 be the first row and column, we see that

$$L_0(C_n) = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{bmatrix}.$$

We cyclically rotate the rows and multiply all rows by (-1) without changing the Smith Normal Form to obtain

$$L_0(C_n) \sim \begin{bmatrix} 1 & -2 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & -2 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & 1 & -2 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & -2 \\ -2 & 1 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}.$$

We then add $2R_1$ to R_{n-1} followed by $3R_2$ to R_{n-1}, etc. to make this matrix upper triangular with ones as the first $(n-1)$ diagonal entries. At this point, we can add multiples of columns to obtain a diagonal matrix with ones as the first $(n-1)$ diagonal entries. We conclude that $\mathbb{Z}^{n-1}/\text{Im } L_0(C_n) \cong \mathbb{Z}/n\mathbb{Z}$, as desired.

A second way to prove this is to find a generator of $K(C_n)$ which has order n. Firstly, the critical configuration $C_{max} = [1, 1, 1 \dots, 1]$ is the identity element since $[2, 2, \dots, 2]$ can be shown to stabilize to $[1, 1, 1, \dots, 1]$. Then, we see that critical configuration $C^{(1)} = [0, 1, 1, \dots, 1]$ will be a generator of $K(C_n)$ of order n, but to see this involves discovering an inductive pattern in the chip-firing moves to conclude that $[0, n, n, \dots, n]$ stabilizes to $[1, 1, 1, \dots, 1]$ while $[0, k, k, \dots, k]$ stabilizes to $C^{(k)}$ for all $1 \leq k \leq n-1$.