Course 18.312: Algebraic Combinatorics

Solution Set # 1

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1) Let graph G be the graph pictured in Figure 1, i.e. G is the 1-skeleton of an octahedron.

(15 points) Compute the eigenvalues of the adjancency matrix for graph G.

There are several ways to compute these eigenvalues; one of which includes calculating the characteristic polynomial $\det(I - xA(G))$ and then factoring. We describe here three methods for finding all the eigenvalues with limited computations.

All three methods start the same way. The adjacency matrix is a 6-by-6 matrix which has rank 3, in fact three of its six rows are simply copies of the other three and the remaining three rows are linearly independent. Thus we know that three of the six eigenvalues must be 0, and the other three are nonzero.

Additionally, since G is a regular graph, i.e. all vertices have the same degree, one of $A(G)$'s eigenvalues must be the common degree of all of the vertices, which is 4 in this case. To see this, show that $[1, 1, 1, 1, 1, 1]^T$ is an eigenvector of any regular graph.

Here are three approaches to get the remaining two eigenvalues:

Method 1) We showed in class that $Tr(A(G)^\ell)$ equals the total number of closed walks of length ℓ. Applying this fact to the cases $\ell = 1$ and $\ell = 2$ we get the equations

$$\lambda_1 + \lambda_2 + 4 + 0 + 0 + 0 = 0,$$

since G has no loops, and

$$\lambda_1^2 + \lambda_2^2 + 4^2 + 0^2 + 0^2 + 0^2 = 24,$$

since the number of closed walks of length 2 in a simple graph (no loops or multiple edges) is twice the number of edges. For every edge $\{u, v\}$, choose either endpoint and then walk along the edge and come back immediately. No other closed walks of length 2 exist.

We therefore have to solve a quadratic equation, and we obtain

$$\lambda_1 = -2 = \lambda_2.$$

Method 2) One can also solve this problem by utilizing symmetries of the graph.

Since Row 1 = Row 3, Row 2 = Row 4, and Row 5 = Row 6, any eigenvector (with nonzero eigenvalue) of $A(G)$ must be of the form $[x, y, x, y, z, z]^T$. Consequently, we need only satisfy the linear system

$$\begin{aligned} 0x + 2y + 2z &= \lambda x \\ 2x + 0y + 2z &= \lambda y \\ 2x + 2y + 0z &= \lambda z \end{aligned}$$

and thus computing the determinant of the 3-by-3 matrix $\begin{bmatrix} \lambda & -2 & -2 \\ -2 & \lambda & -2 \\ -2 & -2 & \lambda \end{bmatrix}$, which equals

$$\lambda^3 - 12\lambda + 16.$$

This cubic can be factored using the rational root test or by using the fact that $\lambda = 4$ is an eigenvalue since G is regular of degree 4 which reduces this problem to factoring a quadratic.

Method 3) We can explicitly write down a basis of eigenvectors, for example:

$$\begin{array}{rll} [1,0,-1,0,0,0]^T & \text{has eigenvalue} & 0 \\ [0,1,0,-1,0,0]^T & \text{has eigenvalue} & 0 \\ [0,0,0,0,1,-1]^T & \text{has eigenvalue} & 0 \\ [1,1,1,1,1,1]^T & \text{has eigenvalue} & 4 \\ [1,-1,1,-1,0,0]^T & \text{has eigenvalue} & -2 \\ [1,0,1,0,-1,-1]^T & \text{has eigenvalue} & -2. \end{array}$$

It suffices to show that the two eigenvectors with eigenvalue -2 and three eigenvectors with eigenvalue 0 are linearly independent (although we already know there are three linearly indepedent of the latter because $rank\ A(G) = 3$).

(5 points) Compute the number of closed walks of length 5 in G.

We compute $Tr(A(G)^5) = (-2)^5 + (-2)^5 + (4)^5 + 0^5 + 0^5 + 0^5 = 960$.

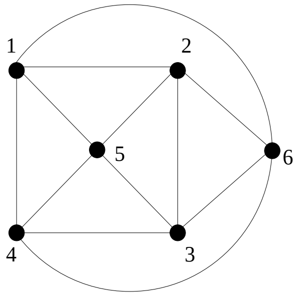

Figure 1: Graph G

2) Let K_{rs} denote the complete bipartite graph, defined on $r + s$ vertices $\{v_1, v_2, \dots, v_r, w_1, \dots, w_s\}$, with an edge between v_i and w_j for $1 \leq i \leq r$ and $1 \leq j \leq s$.

(10 points) By combinatorial reasoning, give a closed formula for the number of closed walks of length k in K_{rs}.

Let $V(K_{rs}) = V \sqcup W$ where $V = \{v_1, v_2, \dots, v_r\}$ and $W = \{w_1, w_2, \dots, w_s\}$ denote the two parts of graph K_{rs}.

We first observe that a walk lands in part V or W of the graph depending on the parity of the number of steps. Thus if ℓ is odd, the number of closed walks equals zero.

So let $\ell = 2k$. Since each vertex v_i has degree s and each vertex w_i has degree r, the number of closed walks starting from v_i of length $\ell = 2k$ is $s^k r^{k-1}$ and the number of closed walks starting from w_i of length $\ell = 2k$ is $r^k s^{k-1}$.

Thus the total number of closed walks starting from any vertex of V is $r(s^k r^{k-1}) = s^k r^k$ and similarly the number of closed walks starting from any vertex of W is also $s^k r^k$ and we obtain that there are $2(rs)^k = 2\sqrt{rs}^{\ell}$ closed walks of length ℓ when ℓ is even.

(10 points) Deduce from this formula the eigenvalues for the adjancency matrix of K_{rs}.

We need to find $r + s$ λ_i's such that

$$\lambda_1^\ell + \dots \lambda_{r+s}^\ell = \begin{cases} 2\sqrt{rs}^{\ell} & \text{if } \ell \text{ is even} \\ 0 & \text{if } \ell \text{ is odd} \end{cases} .$$

By Lemma 1.7 of the notes, such λ_i's are unique. Consequently, two of the eigenvalues of $A(K_{rs})$ must square to rs and the remaining $r+s-2$ must be zero. Since the sum of the eigenvalues is zero, the two nonzero eigenvalues must be additive inverses, and we conclude that the eigenvalues of $A(K_{rs})$ are

$$\{\sqrt{rs}, -\sqrt{rs}, 0, 0, \ldots, 0\}.$$

3) Recall that $\binom{n}{k}$, binomial coefficient, counts the number of subsets of $\{1, 2, \ldots, n\}$ of size k.

   (10 points) Give an algebraic proof of the identity, i.e. use generating functions:

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2. \tag{1}$$

   We notice that the left-hand-side is the coefficient of x^n in the expression

$$\sum_{k=0}^{\infty} \binom{2n}{k} x^k = (1+x)^{2n} = (1+x)^n(1+x)^n = \left(\sum_{i=0}^{\infty} \binom{n}{i}\right)\left(\sum_{j=0}^{\infty} \binom{n}{j}\right). \tag{2}$$

   However, the coefficient of x_n in the product $A(X)B(X)$ is exactly the convolution $\sum_{k=0}^{n} a_k b_{n-k}$. Applying this fact to the above case, we get that the coefficient of x^n in expression (2) is $\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$. However, since $\binom{n}{n-k} = \binom{n}{k}$ and $\binom{n}{k} = 0$ if $k > n$, we have the result.

   (10 points) Give a combinatorial proof of identity (1).

   We divide the set $\{1, 2, \ldots, 2n\}$ into two equal subsets S_1 and S_2 arbitrarily, e.g. $S_1 = \{1, 2, \ldots, n\}$ and $S_2 = \{n+1, n+2, \ldots, 2n\}$. Then $\binom{2n}{n}$ counts the number of ways to choose n elements out of $\{1, 2, \ldots, 2n\}$. The right-hand-side counts the number of ways to choose integer k between 1 and n, and then choose a k element subset T_1 (resp. T_2) of set S_1 (resp. S_2). We then form a subest of $\{1, 2, 3, \ldots, 2n\}$ by taking T_1 and the complement $S_2 \setminus T_2$. Since all subsets of $\{1, 2, \ldots, 2n\}$ can be chosen uniquely this way, we have shown the identity.

4) Define a sequence of integers, $\{L_n\}$, by the initial conditions $L_0 = 2$, $L_1 = 1$, and the recurrence $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.

   Note: The sequence described in this problem are known as the **Lucas** numbers and have many nice properties reminiscent of the Fibonacci numbers.

(10 points) Give a rational expression for the generating function

$$L(x) = \sum_{n=0}^{\infty} L_n x^n.$$

$$\begin{aligned} L(x) = L_0 + L_1 x + \sum_{n=2}^{\infty} L_n x^n &= L_0 + L_1 x + \sum_{n=2}^{\infty} L_{n-1} x^n + \sum_{n=2}^{\infty} L_{n-2} x^n \\ &= L_0 + L_1 x + \Big(xL(x) - L_0 x \Big) + x^2 L(x) \\ &= 2 + x + xL(x) - 2x + x^2 L(x). \end{aligned}$$

Thus,

$$L(x) = \frac{2-x}{1-x-x^2}.$$

(10 points) Show that $L(x)$ has the form

$$\frac{1}{1+\lambda_1 x} + \frac{1}{1+\lambda_2 x}.$$

What are λ_1 and λ_2?

$$\begin{aligned} \frac{2-x}{1-x-x^2} &= \frac{A}{1+\lambda_1 x} + \frac{B}{1+\lambda_2 x} \\ &= \frac{(A+B) + (A\lambda_2 + B\lambda_1)x}{1-x-x^2} \end{aligned}$$

The two roots of $1-x-x^2$ are $\frac{-1\pm\sqrt{5}}{2}$, so the negative inverse roots are $\frac{-1\mp\sqrt{5}}{2}$. (This is a coincidence since $\frac{1+\sqrt{5}}{2}$ is the golden ratio.) Solving $A+B=2$ and $A(\frac{-1+\sqrt{5}}{2}) + B(\frac{-1-\sqrt{5}}{2}) = -1$, we get $A=B=1$.

(5 points) Use this expression for $L(x)$ to obtain a closed formula for L_n.

We rewrite $L(x) = \frac{1}{1+\lambda_1 x} + \frac{1}{1+\lambda_2 x} = \frac{1}{1-\alpha_1 x} + \frac{1}{1-\alpha_2 x}$ by letting $\alpha_1 = -\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\alpha_2 = -\lambda_2 = \frac{1-\sqrt{5}}{2}$. We thus obtain $L(x) = \sum_{i=0}^{\infty}(\alpha_1)^i x^i + \sum_{j=0}^{\infty}(\alpha_2)^j x^j$, it follows that L_n, the coefficient of x^n in $L(x)$ is

$$\alpha_1^n + \alpha_2^n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

(15 points) Prove that the integer sequence $\{L_n\}$ has the following combinatorial interpretation: L_n equals the number of subsets S of $\{a_1, a_2, \ldots, a_n\}$ such that

1. a_i, a_{i+1} are not both in S
2. a_1 and a_n are not both in S.

There are several ways to do this problem. Here is one:

Recall from class that F_n equals the number of subsets S of $\{a_1, a_2, \ldots, a_{n-1}\}$ such that a_i and a_{i+1} are not both in S.

(Here is a quick proof: $F_1 = 1 = \#\{\ \emptyset\ \}$, $F_2 = 2 = \#\{\ \emptyset, \{1\}\ \}$, and let $\hat{F}_n$ be the number of subsets S described above. Then $\hat{F}_n = \hat{F}_{n-1} + \hat{F}_{n-2}$ by distinguishing between the caes $a_{n-1} \notin S$ and $a_{n-1} \in S$, so the $\hat{F}_n$'s satisfy the same recurrence as the F_n's.)

Simiarly we let $\hat{L}_n$ denote the number of such subsets S of $\{a_1, a_2, \ldots, a_n\}$ with properties (1) and (2). We again distinguish between the cases $a_n \notin S$ and $a_n \in S$ and find that

$$\hat{L}_n = \hat{F}_n + \hat{F}_{n-2} = F_n + F_{n-2}.$$

Consequently we see that

$$\begin{aligned} \hat{L}_n &= (F_{n-1} + F_{n-2}) + (F_{n-3} + F_{n-4}) \\ &= (F_{n-1} + F_{n-3}) + (F_{n-2} + F_{n-4}) \\ &= \hat{L}_{n-1} + \hat{L}_{n-2}. \end{aligned}$$

and so the $\hat{L}_n$'s satisfy the same recurrence as the L_n's. After checking that $\hat{L}_1 = 2 = L_1$ and $\hat{L}_2 = 3 = L_2$, we are done.

Bonus) (10 points) Define a sequence of integers, $\{P_n\}$ by the inital conditions $P_1 = 1$, $P_2 = 2$, and the recurrence $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 3$.

To what real number does the sequence

$$\Big\{(P_1 + P_2)/P_2,\ (P_2 + P_3)/P_3,\ (P_3 + P_4)/P_4,\ \ldots\Big\}$$

converge?

Since the P_n's satisfy $P_n - 2P_{n-1} - P_{n-2} = 0$, the denominator of the corresponding generating function must be $x^2 - 2x - 1$. This is also known as the **characteristic polynomial** of the sequence. This polynomial factors as

$$x^2 - 2x - 1 = (1 - \lambda_1 x)(1 - \lambda_2 x)$$

with $\lambda_{1,2} = 1 \pm \sqrt{2}$.

Consequently, $P_n = A(1+\sqrt{2})^n + B(1-\sqrt{2})^n$ for some constants A and B and since $\lim_{n\to\infty}(1-\sqrt{2})^n = 0$ we get $\lim_{n\to\infty} \frac{P_{n-1}}{P_n} = \frac{1}{\lambda_1} = \sqrt{2}-1$. Consequently, the desired sequence converges to $\sqrt{2}$.

Note: The numbers satisfying this sequence are known as **Pell** numbers and are important in number theory. For example, they can be used to generate Pythagorean triples involving consecutive integers: if $x^2 + (x+1)^2 = z^2$ then z is a Pell number of odd index.