# Course 18.312: Algebraic Combinatorics

## Information about the Final Project

## Due Monday May 11th, 2009

Recall that a paper is due at the end of the term. It should be on a topic related to course material, unless permission is given otherwise. Please give me in writing by April 17th the topic of your paper, and you are encouraged to come talk to me about your paper beforehand. The paper is not expected to contain original research, but it should have some element of novelity, such as unifying the material from several sources, simplifying a part of a proof, creating interesting examples, etc. Unless it consists primarily of original research, it should contain at least two references. There is no required length, but around 7-10 LaTeX pages would be reaonable. You are encouraged to, but not required to, type your paper using LaTeX.

Here is a list of some possible topics, with references at the end. Feel free to ask me more about any topic or for more references. The topics are listed roughly in the order that we encountered the related material in class. You may also discuss with me other possible topics not listed below.

1. Identities Involving Fibonacci and Lucas Numbers. Many possible sources including papers from Arthur Benjamin's website: `http://ww.math.hmc.edu/~benjamin` or the book [BQ03].

2. The transfer-matrix method (an application of counting walks in graphs to other problems in enumerative combinatorics. See [Sta86, Section 4.7] for basic information. Futher references are cited in [Sta86, pp. 260-264], and the exercises to Chapter 4 of [Sta86] contains some further examples.

3. The relation between properties of a graph and the eigenvalues of its adjacency or Laplacian matrix. Many possibilities, see [CDS95] or [Lo93, Section 11]

4. More on the Radon Transform, see [DG85]. Also a generalization of the Radon transform to Cayley Graphs and connections to character theory.

5. Mixing Times for Random Walks and the Cheeger Inequality. Some links from Fan Chung Graham's website, `www.math.ucsd.edu/~fan`
6. Describe the theory of a special kind of Posets. Numerous possibilities, some of which are Binomial Posets, Distributive Lattices, Face Posets of polytopes, or Posets arising from hyperplane arrangements. A good starting point is Section 3 of [Sta99]. Also see `http://math.mit.edu/~rstan/arrangements/arr.html`, Richard Stanley's lecture notes from Park City 2004.
7. Partition identities, such as the Rogers-Ramanujan identities. Some basic information appears in Section 2 of [BS]. A more extensive reference is [An76].
8. There are various proofs of the Hook Length Formula in the literature. One possible project would be to study one of these in depth, another would be to compare and contrast two different proofs. Some of the proofs include the probabilisitc approach of Greene-Nijenhuis-Wilf [GNW79] mentioned in class, a bijective proof of Novelli-Pak-Stoyanovskii [NPS97], or a recent algebraic proof by Bandlow `http://www.math.upenn.edu/~jbandlow/papers/hookFormula.pdf`
9. The full RSK algorithm for semi-standard Young tableaux, including proofs of the Symmetry Theory and other applications. See Chapter 7 of [Sta99], Chapter 3 of [Sag01], or Part I of [F97].
10. Variations of RSK and including RSK applied to Oscillating Tableaux (see for example `http://www.mat.univie.ac.at/~kratt/artikel/oscill.html` by C. Krattenthaler) or Skyline Fillings (see

    `http://www.davidson.edu/math/mason/nsrskfinal.pdf` by S. Mason). Another possibility is connection to balanced tableaux, see the next topic.
11. Reduced decompositions of permutations. These concern expressing a permutation as a product of a minimal number of adjacent transpositions $(i, i+1)$. This topic also is related to variations of RSK (balanced tableaux) and applications of the nilCoxeter algebra [FS94].
12. Differential Posets, extension of material from Section 8 of the lecture notes. Also see [Sta88] and [Sta90].
13. Skew-schur Functions. Many possibilities, including questions like how to write a skew-schur function as the sum of schur functions. Related to the problem of writing the product of schur functions as a sum of schur functions and

Littlewood-Richardson coefficients. See Chapter 7 of [Sta99] or Chapter 4 of [Sag01].

14. Algebraic Generating Functions. See Chapter 6 [Sta99] for a good starting place. Project could also explore different combinatorial interpretation of Catalan numbers beyond what we have seen in class.

**The following topics are related to future course material:**

15. Aspects of representation theory of $S_n$. Many connections, possible topics include the Murganham-Nakayama rule, the Branching rule. See Chapter 4 of [Sag01]. Also can discuss connections to representation theory of $GL_n$ such as Schur-Weyl duality.

16. MacMahon Master Theorem. A very important combinatorial result. See [KP07] for recent discussion of variants.

17. Zeta functions of algebraic curves. For those interested in connections to number theory or algebraic geometry, a beautiful subject where techniques of generating functions and symmetric functions can be used to study point-counting on curves.

18. Generalizations of Polya's theorem, especially de Brujin's theorem where there is a group also acting on the set of colors. A nice survey is due to de Brujin [deB64]. Many further results appear in [HP73].

19. Tree enumeration. A large topic, see for instance [Sta99, Ch. 5] or [Mo70] for the labeled case. For counting unlabeled trees using Polya's theorem, see for instance Chapter 4 of [HP73].

20. The Chromatic Polynomial, the Tutte Polynomial, the Jones Polynomial of Knot theory, or connections to acyclic orientations of a graph. See for example chapters 15-16 of [GR01].

21. Study further aspects of chip-firing games on graphs, possibly including the abelian sandpile model, the computation of critical groups of graphs, and $G$-parking functions. See for example chapters 13-14 of [GR01].

22. Alternatively, a study of classical parking functions and possibly their connections to $(q, t)$-Catalan numbers is another possible topic.

23. Basics on Polytopes or specific examples such as Transportation and Birkhoff Polytopes, Cylic Polytopes, Associahedra, or Permutohedra. See for example [Zieg95]. Can also look at Ehrhart Reciprocity, see for example Chapter 4 of [Sta86].

24. Tilings. There are many results involving algebra. See also Sections 7 and 8 of [BS]. The reference `http://math.mit.edu/~rstan/papers/tilings.ps` contains many further interesting results on tilings. See Conway or Thurston for related material.

25. Topics from Statisitical Mechanics such as Matching Enumeration, Pfaffians, the Ising Model, or aspects of the proof of the Alternating Sign Matrix Conjecture

26. Introduction to Cluster Algebras. Cluster algebras are quickly becoming a huge subject. Possible topics include descriptions of quivers and quiver mutation, classification of cluster algebras of finite type, or of small rank.

27. The Gessel-Viennot theory of evaluation of determinants based on the combinatorics of involutions. A brief introduction appears in [Sta86, Section 2.7]. Futher references are [GV85] and [S-W,Section 4.5] (Stanton-White - Constructive Combinatorics). Chapter 4 of this latter reference contains some further applications of involutions. Krattenthaler [Kra99] which also contains beautiful determinantal identities connected to lattice paths. There is also a recent generalization of Gessel-Viennot theory due to K. Talaska [Tal08]

# References

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