Course 18.312: Algebraic Combinatorics

In-Class Exam # 2

April 17, 2009

Open notes. Closed Friends and Enemies. No calculators, computers, I-pods, or Zunes. Please explain your reasoning or method, even for computational problems. You may do the problems in any order. There is a total of 100 points. Good Luck.

0) (5 points) Please state a tentative title of your final project or a one-two sentence description.

1) Consider the full binary tree T containing 4 leaves (seven vertices in all). Consider coloring of T up to isomorphism. The symmetry group of T is a group of order 8.

(15 points) a) What is the cycle-index polynomial of G acting on the vertices of T?

Solution: Label the leaves as 1-4, the height one vertices as 5-6, and the root as 7.

We have four reflections of the lowest branches :

$$(1)(2)(3)(4)(5)(6)(7),\ (12)(3)(4)(5)(6)(7),\ (1)(2)(34)(5)(6)(7),\ (12)(34)(5)(6)(7)$$

and these four reflections composed with transposition of the topmost branches:

$$(13)(24)(56)(7),\quad (1423)(56)(7),\quad (1324)(56)(7),\quad (14)(23)(56)(7).$$

Thus

$$Z_G = \frac{1}{8}\bigg(z_1^7 + 2z_1^5 z_2 + z_1^3 z_2^2 + 2z_1 z_2^3 + 2z_1 z_2 z_4\bigg).$$

(10 points) b) In how many ways can the vertices of T be colored in n colors up to reflective symmetry?

Solution: We must plug in n into z_1 through z_4. We obtain an answer of

$$\frac{1}{8}\left(n^7 + 2n^6 + n^5 + 2n^4 + 2n^3\right).$$

2) (20 points) In how many ways can we begin with the empty partition $\emptyset$, then **add** $2n$ squares one at a time (always keeping a partition), then **remove** n squares at a time, then **add** n squares at a time, and finally **remove** $2n$ squares one at a time, ending up at $\emptyset$?

Solution: This is equal to the number of Hasse walks from $\emptyset$ to $\emptyset$ using the word $D^{2n}U^nD^nU^{2n}$.

$$\alpha(D^{2n}U^nD^nU^{2n}, \emptyset) = f^{\emptyset} \prod_{i \in S_w} (b_i - a_i)$$

according to Theorem 8.4.

Here $f^{\emptyset} = 1$ and $S_w = \{2n+1, 2n+2, \ldots, 3n\} \cup \{4n+1, 4n+2, \ldots, 6n\}$. For $i \in \{2n+1, 2n+2, \ldots, 3n\}$, we have

$$\begin{aligned} a_i &= i - 2n - 1 \\ b_i &= 2n. \end{aligned}$$

For $i \in \{4n+1, 4n+2, \ldots, 6n\}$,

$$\begin{aligned} a_i &= n + (i - 4n - 1) \\ b_i &= 3n. \end{aligned}$$

Consequently,

$$b_i - a_i = \begin{cases} 4n + 1 - i & \text{for } i \in \{2n+1, 2n+2, \ldots, 3n\} \ \text{ and} \\ 6n + 1 - i & \text{for } i \in \{4n+1, 4n+2, \ldots, 6n\} \end{cases}.$$

In conclusion,

$$\begin{aligned} \alpha(D^{2n}U^nD^nU^{2n}, \emptyset) &= 1 \cdot (2n(2n-1)\cdots(n+1))(2n(2n-1)\cdots 1) \\ &= \frac{(2n)!(2n)!}{n!}. \end{aligned}$$

3) Let G be a regular loopless (undirected) graph of degree d with p vertices and q edges.

(5 points) a) Find a simple relation between p, q, and d.

Solution: Since each of the p vertices has degree d, and every edge connects two vertices, we have

$$2q = dp.$$

(5 points) b) Express the biggest eigenvalue of the adjacency matrix A of G in terms of p, q, and d.

(You may use the fact from matrix theory that if M is a $p \times p$ matrix whose entries are nonnegative real numbers, and if z is a column vector of positive real numbers such that $Mz = \lambda z$, then λ is the largest (in absolute value) eigenvalue of M.)

Solution: The adjacency matrix $A(G)$ only has nonnegative integers as entries, and has the vector $z = [1, 1, 1, \ldots, 1]^T$ as an eigenvector of eigenvalue d. Thus d is the largest eigenvalue of $A(G)$.

(5 points) c) Suppose that G has no multiple edges. Express the number of closed walks in G of length two in terms of p, q, and d.

Solution: Since G has no multiple edges, a closed walk of length two must walk along an edge back and forth. Since this can be accomplished in two directions for each edge, the number of closed walks of length two is $2q$.

(10 points) d) Suppose that G has no multiple edges and that the number of closed walks in G of length ℓ is given by

$$4^\ell + 5(-2)^\ell + 3 \cdot 2^\ell.$$

Find the number $\kappa(G)$ of spanning trees of G. (Don't forget that A may have some eigenvalues equal to 0.) For full credit, give a purely numerical answer, not involving p, q, or d but leaving exponents in your expression is okay.

Solution: Since the number of closed walks of length ℓ equals

$$4^\ell + 5(-2)^\ell + 3 \cdot 2^\ell,$$

the adjacency matrix has eigenvalue 4 once, (-2) five times, 2 three times, and 0 $(p - 9)$ times.

From part (b), we may conclude that the degree of G equals the eigenvalue with largest absolute value, i.e. $d = 4$.

Since G is regular of degree $d = 4$, the Laplacian matrix L satisfies $L = 4I - A$, and the eigenvalues of the Laplacian are thus

$$\{0, 6, 6, 6, 6, 6, 2, 2, 2, 4, 4, 4, \ldots, 4\}$$

where 4 appears as an eigenvalue precisely $(p - 9)$ times. The number of spanning trees is $\frac{1}{p} \times$ (the product of the non-zero eigenvalues), hence

$$\kappa(G) = \frac{1}{p} 6^5 2^3 4^{p-9}.$$

However, we can actually solve for p given the above information. We know that $dp = 2q =$ the sum of the squares of the eigenvalues of $A(G)$ Hence, $4^2 + 5(-2)^2 + 3 \cdot 2^2 = 4p$ and we conclude $p = 12$.

$$\kappa(G) = \frac{1}{12} 6^5 2^3 4^3 = 6^4 2^2 4^3 = 2^{12} 3^4.$$

4) Let $f(n)$ denote the number of permutations in the symmetric group S_n, all of whose cycles have length divisible by three.

(15 points) a) Let

$$F(x) = \sum_{n=0}^{\infty} f(n) \frac{x^n}{n!}.$$

Find a simple expression for $F(x)$. For full credit, your answer should not involve any summation symbols (or their equivalent), logarithms, or the function e^x.

Solution: We use Theorem 7.13 which says that

$$\sum_{\ell \geq 0} Z_{S_\ell}(z_1, z_2, \ldots) x^\ell = \exp\left(\sum_{k \geq 1} z_k \frac{x^k}{k} \right).$$

Counting permutations which cycles only divisible by 3 means we let $z_k = 0$ unless $3|k$, in which case we let $z_{3m} = 1$. Consequently,

$$F(x) = \exp\left(\sum_{k \geq 1} \frac{x^{3k}}{3k} \right) = \exp\left(\frac{1}{3} \sum_{k \geq 1} \frac{(x^3)^k}{k} \right).$$

Hence,

$$F(x) = \exp\left(\frac{1}{3} \log\left(\frac{1}{1 - x^3} \right) \right) = (1 - x^3)^{-1/3}.$$

(Bonus 5 points) b) Use part (a) to find a formula for $f(n)$. The answer should be expressed in terms of a binomial coefficient $\binom{r}{s}$ where r need not be an integer, but s is a nonnegative intger.

Solution:
$$\frac{1}{(1-x^3)^{1/3}} = \sum_{\ell\geq 0} \binom{\frac{1}{3}+\ell-1}{\ell} x^{3\ell},$$
thus $f(n)=0$ if n is not divisible by 3 and
$$f(3\ell) = \binom{\frac{1}{3}+\ell-1}{\ell}.$$

5) (5 points) a) Write the elementary symmetric function e_{41} as a sum of h_λ's.

Solution: We use the duality of the e_i's and h_i's which state that $e_k = (-1)^{k-1}h_k + \sum_{i=1}^{k-1}(-1)^{i-1}h_ie_{k-i}$.

$$\begin{aligned}
e_1 &= h_1\\
e_2 &= h_1e_1 - h_2 = h_1^2 - h_2\\
e_3 &= h_1e_2 - h_2e_1 + h_3 = h_1(h_1^2-h_2) - h_2h_1 + h_3 = h_1^3 - 2h_2h_1 + h_3\\
e_4 &= h_1e_3 - h_2e_2 + h_3e_1 - h_4 = h_1(h_1^3 - 2h_2h_1 + h_3) - h_2(h_1^2-h_2) + h_3h_1 - h_4\\
&= h_1^4 - 3h_2h_1^2 + 2h_3h_1 - h_2^2 - h_4.
\end{aligned}$$

In conclusion,
$$e_{41} = e_4 \cdot e_1 = (h_1^4 - 3h_2h_1^2 + 2h_3h_1 - h_2^2 - h_4)h_1 = h_{11111} - 3h_{2111} + 2h_{311} + h_{221} - h_{41}.$$

(5 points) b) Write the Schur function $s_2 - s_{11}$ as a symmetric polynomial in the variables $\{x_1, x_2, x_3\}$.

Solution:

$$s_2(x_1,x_2,x_3) = h_2(x_1,x_2,x_3) = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2.$$
$$s_{11}(x_1,x_2,x_3) = e_2(x_1,x_2,x_3) = x_1x_2 + x_1x_3 + x_2x_3.$$

Thus
$$(s_2 - s_{11})(x_1,x_2,x_3) = x_1^2 + x_2^2 + x_3^2.$$

(Bonus 10 points) c) For $k \geq 2$, write the Schur function $s_k - s_{k-1,1}$ as a symmetric polynomial in the variables $\{x_1, x_2\}$.

Solution: By the definition in terms of SSYT, we see that $s_k = h_k$ and thus is the sum of all monomials of degree k in the variables x_1 and x_2. We evaluate $s_{k-1,1}$ by the Jacobi-Trudi identity as det $\begin{bmatrix} h_{k-1} & h_k \\ 1 & h_1 \end{bmatrix} = h_{k-1,1} - h_k$. Consequently,

$$\begin{aligned} & (s_k - s_{k-1,1})(x_1, x_2) = (2h_k - h_{k-1})(x_1, x_2) \\ = \; & 2(x_1^k + x_1^{k-1}x_2 + \cdots + x_1x_2^{k-1} + x_2^k) - (x_1^{k-1} + x_1^{k-2}x_2 + \cdots + x_1x_2^{k-2} + x_2^{k-1})(x_1 + x_2) \\ = \; & x_1^k + x_2^k. \end{aligned}$$

Remark: In fact, the symmetric polynomial $\sum_{i=0}^{k}(-1)^i s_{[k-i,1^i]}$ in the variables $\{x_1, x_2, x_3, \ldots, x_n\}$ is equal to $x_1^k + x_2^k + \cdots + x_n^k = p_k(x_1, x_2, \ldots, x_n)$.

To see this in the case of two variables, it sufficed to look at

$$s_k - s_{k-1,1}$$

since the rest of the terms in the sum correspond to Schur functions with more than two parts, hence are zero.