

# Course 18.312: Algebraic Combinatorics

## In-Class Exam # 1

Friday The Thirteenth, March 2009

Open books. Closed Friends and Enemies. No calculators, computers, I-pods, or Zunes. Please explain your reasoning or method, even for computational questions. You may do the problems in any order. Good Luck.

- 1) (10 points) Bijectively show that the number of partitions which are self-conjugate is the same as the number of partitions such that each part is odd and all of the parts are distinct.

**Solution:** We bend each odd part at its central square to turn each into a symmetric hook. For example, the odd part [7] corresponds to the hook [4, 1, 1, 1]. Since each of these odd parts are distinct these can be layered NW to SE so that the resulting Young diagram is has rows of weakly decreasing size (so that it is a partition) and self-conjugate due to the symmetry. Since a self-conjugate partition can always be uniquely constructed in this way, the process is reversible and we get a bijection. Notice for example, the partition [7, 5, 1] corresponds to the self-conjugate partition [4, 4, 3, 2].

(10 points) Use this to show a generating function identity.

**Hint:** Your identity should involve an infinite sum and an infinite product.

**Solution:** The generating function for the number of partitions with distinct parts, whose parts are all odd is given by  $\prod_{j=1}^{\infty} (1 + x^{2j+1})$  since for each odd number, we either include that odd part exactly zero or one time.

From HW 4, problem 2, we saw that the generating function for self-conjugate partitions was  $\sum_{k \geq 1} \frac{x^{k^2}}{\prod_{i=1}^k (1 - x^{2i})}$ . Thus we have combinatorial shown the algebraic identity

$$\prod_{j=1}^{\infty} (1 + x^{2j+1}) = \sum_{k \geq 1} \frac{x^{k^2}}{\prod_{i=1}^k (1 - x^{2i})}.$$

**Remark:** The number of partitions with odd distinct partitions is also in bijection with the number of **Lecture Hall partitions**. See George Andrews' book for more details.

- 2) (10 points) List all permutations  $\pi$  of  $\{1, 2, 3, 4\}$  such that  $RSK(\pi) = (P, Q)$  where  $sh(P) = [2, 2]$ .

**Solution:**

$$RSK^{-1} \left( \begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array}, \begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array} \right) = 3412,$$

$$RSK^{-1} \left( \begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array}, \begin{array}{cc} 1 & 3 \\ 2 & 4 \end{array} \right) = 3142,$$

$$RSK^{-1} \left( \begin{array}{cc} 1 & 3 \\ 2 & 4 \end{array}, \begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array} \right) = 2413,$$

$$RSK^{-1} \left( \begin{array}{cc} 1 & 3 \\ 2 & 4 \end{array}, \begin{array}{cc} 1 & 3 \\ 2 & 4 \end{array} \right) = 2143.$$

Observe that  $\{3412, 3142, 2413, 2143\}$  are exactly the permutations of  $\{1, 2, 3, 4\}$  that have a longest increasing sequence of size 2 and a longest decreasing sequence of size 2.

- 3) Define a family of graphs, we call them pinwheel graphs, as follows: Let  $PW_T$  be the graph on  $2T + 1$  vertices

$$u_0 \cup \{v_1, \dots, v_T\} \cup \{w_1, \dots, w_T\}$$

such that there is an edge between  $u_0$  and every other vertex and  $v_i$  is connected to  $w_j$  if and only if  $i = j$ .

(20 points) a) Compute the eigenvalues of the adjacency matrix  $A(PW_T)$  in terms of  $T$ .

**Hint:** One way to approach this problem is to use symmetry to find a large set of linearly independent eigenvectors with first entry zero, and use combinatorial formulas to deduce the remaining eigenvalues.

**Solution:** The adjacency matrix of  $PW_T$  is a  $(2T + 1)$ -by- $(2T + 1)$  matrix whose first row is  $[0, 1, 1, \dots, 1]$ , first column is  $[0, 1, 1, \dots, 1]^T$  and the remainder of the matrix is block-diagonal using  $T$  identical blocks of the form  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ . Accordingly, there are  $T$  linearly independent eigenvectors (with eigenvalue  $(-1)$ ) of the form  $[0, 1, -1, 0, 0, 0, \dots, 0, 0, 0]^T$ ,  $[0, 0, 0, 1, -1, 0, \dots, 0, 0, 0]$ ,

...,  $[0, 0, 0, 0, 0, 0, \dots, 0, 1, -1]^T$ . These are clearly independent since no two both contain a nonzero entry in the same row, and by antisymmetry, they each have eigenvalue  $(-1)$ .

Similarly, we find  $(T-1)$  linearly independent eigenvectors (with eigenvalue 1) of the form  $[0, 1, 1, -1, -1, 0, 0, 0, \dots, 0, 0, 0]^T$ ,  $[0, 1, 1, 0, 0, -1, -1, 0, \dots, 0, 0, 0]^T$ , ...,  $[0, 1, 1, 0, 0, 0, 0, 0, \dots, 0, -1, -1]$ . Thus we have found  $2T-1$  out of the  $2T+1$  eigenvalues. We use the fact that

$$\lambda_1 + \lambda_2 + \dots + \lambda_{2T+1} = \text{Tr}\{A(PW_T)\} = 0$$

and

$$\lambda_1^2 + \lambda_2^2 + \dots + \lambda_{2T+1}^2 = \text{Tr}\{A(PW_T)^2\} = 2e$$

where  $e = 3T$  equals the number of edges to obtain

$$\begin{aligned} T(-1) + (T-1)(1) + \lambda_{2T} + \lambda_{2T+1} &= 0 \\ T(-1)^2 + (T-1)(1)^2 + \lambda_{2T}^2 + \lambda_{2T+1}^2 &= 6T, \end{aligned}$$

and thus  $\lambda_{2T+1} = -\lambda_{2T} + 1$  and  $\lambda_{2T}^2 + (-\lambda_{2T} + 1)^2 = 4T + 1$ .  $\lambda_{2T}$  satisfies

$$2\lambda_{2T}^2 - 2\lambda_{2T} + (-4T) = 0$$

and we conclude

$$\{\lambda_{2T}, \lambda_{2T+1}\} = \left\{ \frac{1}{2} \pm \frac{\sqrt{8T+1}}{2} \right\}.$$

(5 points) b) For what values of  $T$  is  $PW_T$  an integral graph?

True if the radical  $8T+1$  is a perfect square since it would be an odd perfect square in this case.

**Extra points for the following observation:** This is equivalent to the condition that  $T$  is a triangular number, i.e. of the form  $T = \binom{n+1}{2} = 1 + 2 + 3 + \dots + n$  for some  $n \geq 1$ .

In particular, one can see that if  $T = \binom{n+1}{2}$ , then  $8T+1 = (2n+1)^2$ .

$$8 \frac{n(n+1)}{2} + 1 = 4n^2 + 4n + 1.$$

Additionally,  $8T+1$  is never even and is monotonically increasing so there are no other possible values of  $T$  so that  $8T+1$  is a perfect square.

- 4) Define the following family of posets: For all integers  $n \geq 1$ ,  $P_n$  is the poset, ordered by inclusion, consisting of subsets  $\{i_1, i_2, \dots, i_{2k}\} \subset \{1, 2, 3, \dots, 2n\}$  satisfying

$$0 < i_1 < i_2 < \dots < i_{2k} < 2n + 1$$

and

$$i_1, (i_2 - i_1), \dots, (i_{2k} - i_{2k-1}), (2n + 1) - i_{2k} \quad \text{are all odd.}$$

Notice that all elements are subsets with an even number of elements and that the rank of an element  $S$  is  $|S|/2$ . We let  $\hat{0}$  denote  $\emptyset$ , the unique element of rank 0 and  $\hat{1}$  denote  $\{1, 2, \dots, 2n\}$ , the unique element of rank  $n$ .

$P_n$  also has the property, which you do not need to show, that if  $\text{rank}(S) = k$ , then the interval  $[\hat{0}, S]$  is isomorphic to  $P_k$ .

**Remark:** This problem appeared as exercise 52 of Chapter 3 of Stanley's Enumerative Combinatorics 1. It is due to K. Baclawski and P. Edelman.

(5 points) a) Draw the Hasse Diagram for  $P_3$ .

(5 points) b) Compute the Möbius function  $\mu(\hat{0}, \hat{1})$  for  $P_3$ .

**Solution:** An element of rank two, such as  $\{1, 2\}$ , satisfies  $\mu(\hat{0}, \{1, 2\}) = -2$  since such an element covers exactly three elements of rank one. Continuing, since there is one element of rank zero, six elements of rank one ( $\mu(\hat{0}, \{1, 2\}) = -1$ ), five elements of rank two ( $\mu(\hat{0}, \{1, 2, 3, 4\}) = 2$ ), and one element of rank three  $\mu(\hat{0}, \{1, 2, 3, 4, 5, 6\}) = \mu(\hat{0}, \hat{1}) = -5$  for  $P_3$ .

(10 points) c) Compute the total number of elements in  $P_n$ .

(10 points) d) Show that the number of elements of rank  $k$  in  $P_n$  is  $\binom{n+k}{2k}$ .

**Hint:** One possible way to proceed with (c) and (d) is to set up a bijection between elements of  $P_n$  and domino tilings of a 2-by- $2n$  grid.

**Solution:** We follow the hint. We associate a set  $\{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, 2n\}$  with a domino tiling by placing a vertical domino in precisely columns  $i_1$  through  $i_k$ . Since each  $(i_{j+1} - i_j)$  is odd, we can fill in the rest of the tiling using horizontal dominoes which each take up an even number of columns.

As we saw in the addendum to Lecture #1, the number of such domino tilings is counted by the  $(2n)$ th Fibonacci number. Thus  $|P_n| = F_{2n}$ .

A domino tiling of the 2-by- $2n$  grid can be encoded by a sequence of  $H$ 's and  $V$ 's where  $H$  indicates the presence of two horizontal dominos and  $V$

indicates a single vertical domino. There have to be an even number of vertical dominoes, let's say there are  $2k$  of them. Since they take up less space than a horizontal domino, there are precisely  $n + k$  letters in the sequence encoding the tiling, and they can come in any order. Thus the number of tiling/sets of rank  $k$  is precisely given by  $\binom{n+k}{2k}$ . This can be seen also in terms of the sets directly, but the tilings make the description more visual.

It is easy to show, you do not need to, that if  $S$  is of rank  $k$ , then the cardinality  $\#\{S' : S \text{ covers } S'\}$  is a function  $f(k)$  only depending on  $k$ . In other words, it is the same number, regardless of the choice of  $S$  or choice of  $P_n$ .

**Remark:** This follows from the fact that the interval  $[\hat{0}, S]$  is isomorphic to  $P_k$  when  $S$  is of rank  $k$ . Thus we may assume that  $S = \{1, 2, \dots, 2k\}$  in  $P_k$  and  $f(k)$  is the total number of elements of rank  $k - 1$ .

(5 points) e) What is  $f(k)$ ?

**Solution:** We may as well assume that  $k = n$  in which case  $f(n)$  equals the number of elements of rank  $n - 1$ , i.e.  $\binom{n+n-1}{2(n-1)} = 2n - 1$ .

(5 points) f) Using the above, deduce a formula for the number of maximal chains in  $P_n$  and prove it.

**Solution:** There are  $(2n - 1)$  elements of rank  $n - 1$ , which each cover  $(2n - 3)$  elements, and continuing in this way, we see that each element of rank  $k$  covers  $(2k - 1)$  elements. Choosing such a covered element at each step gives a maximal chain, and thus the number of maximal chains is

$$1 \cdot 3 \cdot 5 \cdots (2n - 1).$$

(5 points) g) Using the above, or otherwise, compute  $\mu(\hat{0}, \hat{1})$  for  $P_4$ .

**Solution:** We know that in  $P_4$ , there is one element of rank zero, with Möbius value 1,  $\binom{4+1}{2} = 10$  elements of rank one (with Möbius value  $(-1)$ ),  $\binom{4+2}{4} = 15$  elements of rank two (with Möbius value 2), and  $\binom{4+3}{6} = 7$  elements of rank three (with Möbius value  $(-5)$ ).

Thus

$$\mu(\hat{0}, \hat{1}) = -[7 \cdot (-5) + 15 \cdot 2 + 10 \cdot (-1) + 1 \cdot 1] = 14.$$

**(Bonus 5 points)** Deduce a formula for  $\mu(\hat{0}, \hat{1})$  in  $P_n$  and prove it.

**Solution:** For  $P_n$ , the formula for  $\mu(\hat{0}, \hat{1}) = C_n = \frac{1}{n+1} \binom{2n}{n}$ , the  $n$ th Catalan number.