Course 18.312: Algebraic Combinatorics

In-Class Exam # 1

Friday The Thirteenth, March 2009

Open books. Closed Friends and Enemies. No calculators, computers, I-pods, or Zunes. Please explain your reasoning or method, even for computational problems. You may do the problems in any order. Good Luck.

1) (10 points) Bijectively show that the number of partitions which are self-conjugate is the same as the number of partitions such that each part is odd and all of the parts are distinct.

   (10 points) Use this to show a generating function identity.

   Hint: Your identity should involve an infinite sum and an infinite product.

2) (10 points) List all permuatations π of $\{1, 2, 3, 4\}$ such that $RSK(\pi) = (P, Q)$ where $sh(P) = [2, 2]$.

3) Define a family of graphs, we call them pinwheel graphs, as follows: Let PW_T be the graph on $2T + 1$ vertices

$$u_0 \cup \{v_1, \ldots, v_T\} \cup \{w_1, \ldots, w_T\}$$

   such that there is an edge between u_0 and every other vertex and v_i is connected to w_j if and only if $i = j$.

   (20 points) a) Compute the eigenvalues of the adjacency matrix $A(PW_T)$ in terms of T.

   Hint: One way to approach this problem is to use symmetry to find a large set of linearly independent eigenvectors with first entry zero, and use combinatorial formulas to deduce the remaining eigenvalues.

   (5 points) b) Describe the set of T such that PW_T is an integral graph.

(See Back)

4) Define the following family of posets: For all integers $n \geq 1$, P_n is the poset, ordered by inclusion, consisting of subsets $\{i_1, i_2, \ldots, i_{2k}\} \subset \{1, 2, 3, \ldots, 2n\}$ satisfying

$$0 < i_1 < i_2 < \cdots < i_{2k} < 2n + 1$$

and

$$i_1, (i_2 - i_1), \ldots, (i_{2k} - i_{2k-1}), (2n+1) - i_{2k} \quad \text{are all odd.}$$

Notice that all elements are subsets with an even number of elements and that the rank of an element S is $|S|/2$. We let $\hat{0}$ denote $\emptyset$, the unique element of rank 0 and $\hat{1}$ denote $\{1, 2, \ldots, 2n\}$, the unique element of rank n.

P_n also has the property, which you do not need to show, that if $rank(S) = k$, then the interval $[\hat{0}, S]$ is isomorphic to P_k.

(5 points) a) Draw the Hasse Diagram for P_3.

(5 points) b) Compute the Möbius function $\mu(\hat{0}, \hat{1})$ for P_3.

(10 points) c) Compute the total number of elements in P_n.

(10 points) d) Show that the number of elements of rank k in P_n is $\binom{n+k}{2k}$.

Hint: One possible way to proceed with (c) and (d) is to set up a bijection between elements of P_n and domino tilings of a 2-by-$2n$ grid.

It is easy to show, you do not need to, that if S is of rank k, then the cardinality $\#\{S' : S \text{ covers } S'\}$ is a function $f(k)$ only depending on k. In other words, it is the same number, regardless of the choice of S or choice of P_n.

(5 points) e) What is $f(k)$?

(5 points) f) Using the above, deduce a formula for the number of maximal chains in P_n and prove it.

(5 points) g) Using the above or otherwise, compute $\mu(\hat{0}, \hat{1})$ for P_4.

(Bonus 5 points) Deduce a formula for $\mu(\hat{0}, \hat{1})$ in P_n and prove it.