Course 18.312: Algebraic Combinatorics

Problem Set # 1

Due Wednesday February 11th, 2009

You may discuss the homework with other students in the class, but please write the names of your collaborators at the top of your assignment. Please be advised that you should not just obtain the solution from another source. Please explain your reasoning to receive full credit, even for computational questions.

1) Let graph G be the graph pictured in Figure 1, i.e. G is the 1-skeleton of an octahedron.

   (15 points) Compute the eigenvalues of the adjancency matrix for graph G.

   (5 points) Compute the number of closed walks of length 5 in G.

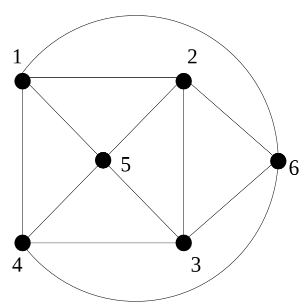

Figure 1: Graph G

2) Let K_{rs} denote the complete bipartite graph, defined on $r + s$ vertices $\{v_1, v_2, \ldots, v_r, w_1, \ldots, w_s\}$, with an edge between v_i and w_j for $1 \leq i \leq r$ and $1 \leq j \leq s$.

   (10 points) By combinatorial reasoning, give a closed formula for the number of closed walks of length k in K_{rs}.

   (10 points) Deduce from this formula the eigenvalues for the adjancency matrix of K_{rs}.

3) Recall that $\binom{n}{k}$, binomial coefficient, counts the number of subsets of $\{1, 2, \ldots, n\}$ of size k.

(10 points) Give an algebraic proof of the identity, i.e. use generating functions:

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2 . \tag{1}$$

(10 points) Give a combinatorial proof of identity (1).

4) Define a sequence of integers, $\{L_n\}$, by the initial conditions $L_0 = 2$, $L_1 = 1$, and the recurrence $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.

(10 points) Give a rational expression for the generating function

$$L(x) = \sum_{n=0}^{\infty} L_n x^n .$$

(10 points) Show that $L(x)$ has the form

$$\frac{1}{1 + \lambda_1 x} + \frac{1}{1 + \lambda_2 x} .$$

What are λ_1 and λ_2?

(5 points) Use this expression for $L(x)$ to obtain a closed formula for L_n.

(15 points) Prove that the integer sequence $\{L_n\}$ has the following combinatorial interpretation: L_n equals the number of subsets S of $\{a_1, a_2, \ldots, a_n\}$ such that

1. a_i, a_{i+1} are not both in S
2. a_1 and a_n are not both in S.

Bonus) (10 points) Define a sequence of integers, $\{P_n\}$ by the inital conditions $P_1 = 1$, $P_2 = 2$, and the recurrence $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 3$.

To what real number does the sequence

$$\Big\{ (P_1 + P_2)/P_2,\ (P_2 + P_3)/P_3,\ (P_3 + P_4)/P_4,\ \ldots \Big\}$$

converge?