Non-nilpotent elements in motivic homotopy theory
Plan

1. The Adams-Novikov spectral sequence
2. The stable motivic homotopy category
3. The motivic Adams-Novikov spectral sequence
4. \(\eta^{-1} \pi_{*,*}(S^{0,0})\)
5. A square of spectral sequences
6. Chromatic motivic homotopy theory
The Adams-Novikov spectral sequence

\[ H^{s,u}(BP_\ast BP) \xrightarrow{s} \pi_{u-s}(S^0) \otimes \mathbb{Z}(2). \]

\[ BP_\ast BP = \pi_\ast(BP \wedge BP), \]
\[ BP = \text{the 2-local Brown-Peterson spectrum}. \]

\[ H(BP_\ast BP) = \text{cohomology of the Hopf algebroid } BP_\ast BP \]
\[ \text{with coefficients in } BP_\ast, \text{ a } \mathbb{Z}(2)-\text{algebra}. \]

\[ \pi_\ast(S^0) \text{ is filtered}. \]
\[ F^s \pi_{u-s}(S^0)/F^{s+1} \pi_{u-s}(S^0) \text{ approximated by } H^{s,u}(BP_\ast BP). \]
The Adams-Novikov spectral sequence

Plot $H_{s,u}(BP_* BP)$ in the $(u - s, s)$-plane.

Nodes will indicate generators for $\mathbb{Z}(2)$-modules.
The Adams-Novikov spectral sequence

\[ \overline{\alpha}_n \in H^{1,2n}(BP_* BP). \]
The Adams-Novikov spectral sequence

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Generate an algebra.

Round nodes indicate \( \mathbb{Z}/2 \).

Square nodes labelled with \( n \neq \infty \) indicate \( \mathbb{Z}/2^n \).
The Adams-Novikov spectral sequence

\[ \overline{\alpha}_n \in H^{1,2n}(BP_* BP). \]
Generate an algebra.

Lines of slope 1: \( \overline{\alpha}_1 \)-multiplication.
The Adams-Novikov spectral sequence

All of $H(BP_*BP)$ in given range.
The Adams-Novikov spectral sequence

\[ H_*(BP^*BP) \text{ in the } (u-s, s)-\text{plane.} \]

Nodes will indicate generators for \( \mathbb{Z}/(2) \)-modules.

\[ \alpha_n \in H_1, 2n(BP^*BP). \]
\[ \alpha_n \in H_1, 2n(BP^*BP). \]

Generate an algebra.

Round nodes indicate \( \mathbb{Z}/2 \).

Square nodes labelled with \( n \neq \infty \) indicate \( \mathbb{Z}/2^s \).

Lines of slope 1: \( \alpha_1 \)-multiplication.

All of \( H_*(BP^*BP) \) in given range.

\[ \alpha_1 \rightarrow \eta; \]
\[ \alpha_2 \rightarrow \nu; \]
\[ \alpha_4 \rightarrow \sigma. \]

\[ \eta_4 = 0, \text{ but } \alpha_{n_1} \neq 0 \text{ for all } n. \]

Even worse: for \( k \neq 2, \alpha_n \neq 0 \text{ for all } n. \]

Family of differentials (Novikov) resolve this conflict:

\[ d_3 \alpha_4 k - 1 = \alpha_3 1 \alpha_4 k - 3, \]
\[ d_3 \alpha_4 k + 2 = \alpha_3 1 \alpha_4 k, \quad k \geq 1. \]

See \( 2 \alpha_1 = 0, \text{ so } 2 \eta = 0. \)

See \( 16 \alpha_4 = 0, \text{ so } 16 \sigma = 0. \)

\[ \alpha_5 = \langle 8 \alpha_4, 2, \alpha_1 \rangle \rightarrow \mu_9 = \langle 8 \sigma, 2, \eta \rangle. \]
The Adams-Novikov spectral sequence

\[ \alpha_n \in H^1_{2n}(BP^*BP). \]

Generate an algebra.

Nodes will indicate generators for \( \mathbb{Z}(2) \)-modules.

\[ \eta^4 = 0, \text{ but } \alpha_1^n \neq 0 \text{ for all } n. \]

\[ \alpha_1 \sim \eta. \]
The Adams-Novikov spectral sequence

\[
\bar{\alpha}_1 \sim \sim \sim \eta.
\]

\[\eta^4 = 0, \text{ but } \bar{\alpha}_1^n \neq 0 \text{ for all } n.\]

Even worse: for \( k \neq 2 \), \( \bar{\alpha}_1^n \bar{\alpha}_k \neq 0 \) for all \( n \).
The Adams-Novikov spectral sequence

Family of differentials (Novikov) resolve this conflict:

\[ d_3 \bar{\alpha}_{4k-1} = \bar{\alpha}_1^3 \bar{\alpha}_{4k-3}, \quad d_3 \bar{\alpha}_{4k+2} = \bar{\alpha}_1^3 \bar{\alpha}_{4k}, \quad k \geq 1. \]
The Adams-Novikov spectral sequence

See $2\bar{\alpha}_1 = 0$, so $2\eta = 0$.
See $16\bar{\alpha}_4 = 0$, so $16\sigma = 0$.

$\bar{\alpha}_5 = \langle 8\bar{\alpha}_4, 2, \bar{\alpha}_1 \rangle \sim \mu_9 = \langle 8\sigma, 2, \eta \rangle$. 
The (motivic) Adams-Novikov spectral sequence
The Adams-Novikov spectral sequence

\[ \eta^4 = 0 \text{ but } \bar{\alpha}_1^n \neq 0 \text{ for all } n. \]

\[ \implies \text{ the ANSS is a poor tool for computing } \pi_\ast(S^0) \otimes \mathbb{Z}(2) \text{?} \]

NO!

This arises due to the relationship between the ANSS and the motivic ANSS.
Motivic homotopy theory

The spaces of topology + the schemes of algebraic geometry
\[ \sim \text{ motivic homotopy theory.} \]

Have usual topological 1-sphere \( S^{1,0} = S^1 \)
and the algebraic 1-sphere \( S^{1,1} = \mathbb{A}^1 - 0. \)

Must specify a ground field. For us it’s \( \mathbb{C}. \)

New question: what is \( \pi_{*,*}(S^{0,0}) \otimes \mathbb{Z}(2) \)?
Second grading is called weight.
Realization functor \( X \mapsto X(\mathbb{C}) \) returns classical story.
Recall, classically
\[ \eta : S^3 \subset \mathbb{C}^2 - 0 \to \mathbb{P}^1(\mathbb{C}) = S^2 \]
so \( \eta \in \pi_1(S^0) \).

Motivically,
\[ \eta : S^{3,2} = \mathbb{A}^2 - 0 \to \mathbb{P}^1 = S^{2,1} \]
so \( \eta \in \pi_{1,1}(S^{0,0}) \).

Remarkably \( \eta^n \neq 0 \) for all \( n \) in the motivic story.
This is why \( \bar{\alpha}_1^n \neq 0 \) for all \( n \).
The motivic Adams-Novikov spectral sequence

Non-zero elements of $H_{s,u}^{s,u}(BP_* BP)$ have $u$ even and so we can assign them a weight $w = u/2$.

The motivic Adams-Novikov spectral sequence takes the form

$$H_{s,u}^{s,u}(BP_* BP)[\tau]^w \xrightarrow{s} \pi_{u-s,w}(S^{0,0}) \otimes \mathbb{Z}(2)$$

where $|\tau| = (0,0,-1)$.

Classical differentials give motivic differentials:

$$d_{2n+1}x = y \xrightarrow{\sim} d_{2n+1}x = \tau^n y.$$
The motivic Adams-Novikov spectral sequence

\[ \bar{\alpha}_4 \text{ and } \bar{\alpha}_5 \text{ detect } \]
\[ \sigma \in \pi_{7,4}(S^{0,0}) \text{ and } \mu_9 \in \pi_{9,5}(S^{0,0}). \]
\[ \eta^n \sigma \neq 0 \text{ and } \eta^n \mu_9 \neq 0 \text{ for all } n. \]
The $\eta$-local homotopy of the motivic sphere

Theorem (A., Miller)

$$\eta^{-1}\pi_{*,*}(S^{0,0}) = \mathbb{F}_2[\eta^{\pm 1}, \sigma, \mu_9]/(\eta \sigma^2),$$

where $\eta \in \pi_{1,1}(S^{0,0})$ and $\sigma \in \pi_{7,4}(S^{0,0})$ are motivic Hopf invariant one elements and $\mu_9 = \langle 8\sigma, 2, \eta \rangle \in \pi_{9,5}(S^{0,0})$ is detected by $\bar{\alpha}_5$ in the motivic Adams-Novikov spectral sequence.
Key computation: $\bar{\alpha}^{-1}_1 H(BP_\ast BP)$

Proposition (A., Miller)

$$H(BP_\ast BP) \rightarrow \bar{\alpha}^{-1}_1 H(BP_\ast BP)$$

is an isomorphism above a line of slope $1/5$ in $(u - s, s)$ coordinates and

$$\bar{\alpha}^{-1}_1 H(BP_\ast BP) = \mathbb{F}_2[\bar{\alpha}_1^{\pm 1}, \bar{\alpha}_3, \bar{\alpha}_4]/(\bar{\alpha}_1 \bar{\alpha}_4^2).$$
The Adams-Novikov $E_2$-page
My secret: a square of SSs

Classically, one has a diagram for each prime $p$:

\[
\begin{array}{c}
H(P; Q) \xrightarrow{\text{CESS}} H(A) \\
\downarrow \text{alg-Nov-SS} & \text{ASS} \\
H(BP_*BP) \xrightarrow{\text{ANSS}} \pi_*(S^0)
\end{array}
\]
The Hopf algebra $P$ and the $P$-comodule $Q$

$$P = \mathbb{F}_p[\xi_1, \xi_2, \xi_3, \ldots], \quad |\xi_n| = 2p^n - 2,$$

with Milnor diagonal:

$$\xi_n \mapsto \sum_{i+j=n} \xi_i^{p^j} \otimes \xi_j.$$

$$Q = \mathbb{F}_p[q_0, q_1, q_2, \ldots], \quad |q_n| = 2p^n - 2,$$

with coaction map similar to the Milnor diagonal:

$$q_n \mapsto \sum_{i+j=n} \xi_i^{p^j} \otimes q_j.$$
The classical chromatic story and $\mathbb{Q}$

The classical chromatic story is loaded up in $\mathbb{Q}$:

$$q_n \in \mathbb{Q} \quad \Rightarrow \quad v_n \in BP_* = \mathbb{Z}(p)[v_1, v_2, \ldots].$$

Everything one does with $BP_*$

e.g. $BP_*/(p, v_1, \ldots, v_{n-1})$, $v_n^{-1}BP_*/(p, v_1, \ldots, v_{n-1})$,
$v_n^{-1}BP_*/(p^\infty, v_1^\infty, \ldots, v_n^\infty)$

one can do with $\mathbb{Q}$ and one has appropriate algebraic Novikov SSs.
Miller’s proof of the telescope conjecture for height 1

\[ H(P; q_1^{-1}Q/(q_0)) \xrightarrow{\text{CESS}} q_1^{-1}H(A; H_*(S/p)) \]
\[ H(BP_*BP; \nu_1^{-1}BP_*/p) \xrightarrow{\text{ANSS}} \nu_1^{-1}\pi_*(S/p) \]

Miller computes this square to obtain:

**Theorem (Miller)**
\[ \nu_1^{-1}\pi_*(S/p) = \mathbb{F}_p[\nu_1^{\pm 1}] \otimes E[h] \text{ where} \]
\[ h : S^{2p-3} \xrightarrow{\nu_1} \Sigma^{-1}S/p \xrightarrow{\text{Bockstein}} S/p. \]
Uses of the square by Ravenel and myself

Ravenel: TELESCOPE CONJECTURE at height 2.
   Tried to DISPROVE by computing the corresponding square for $\nu_2^{-1}S/(p, \nu_1)$.

My thesis: understanding the square for $\nu_1^{-1}S/p^\infty$,
   the analog of Miller’s work for the SPHERE.
Theorem (A.)

For odd primes, there is an ASS for $\pi_*(v_1^{-1}S/p^\infty)$ with $E_2$-page:

We understand the differentials in this spectral sequence and above the red line it coincides with the classical ASS for the sphere.
Motivicifying the square

We localized with respect to an element $h_0 \in H(P; Q)[\tau]$ and completely described the square.

We did not only compute $\eta^{-1}\pi_{*,*}(S^{0,0})$ but also the localized motivic ASS converging to it, confirming a conjecture of Guillou-Isaksen.
A new chromatic story with $P$?

$h_0 \in H(P; Q)$ corresponds to $\xi_1 \in P$.

Classically, the chromatic story is governed by $p, v_1, v_2, \ldots \in BP_*$ and thus, $q_0, q_1, q_2, \ldots \in Q$.

Motivically, there may be other periodicity operators corresponding to $\xi_1, \xi_2, \xi_3, \ldots \in P$.

I want to call them $\eta_0, \eta_1, \eta_2, \ldots$
Mahowald already has the $\eta_j$ family so I’ll go with $w_0, w_1, w_2, \ldots$
Slopes

<table>
<thead>
<tr>
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<th>ASS</th>
<th>ANSS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_n / \nu_n$</td>
<td>$1/(2p^n - 2)$</td>
<td>0</td>
</tr>
<tr>
<td>$\xi_{n+1} / w_n$</td>
<td>$1/(2p^{n+1} - 3)$</td>
<td>$1/(2p^{n+1} - 3)$</td>
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</tbody>
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The chromatic approach to classical homotopy theory
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1. $2 : S^0 \to S^0, \quad \pi_*(2^{-1}S^0)$;
The chromatic approach to classical homotopy theory

1. \( 2 : S^0 \to S^0 \), \[ \pi_* (2^{-1} S^0) \];
2. \( \nu_1^4 : S/2 \to \Sigma^{-8} S/2 \), \[ \pi_* (\nu_1^{-1} S/2) \);
The chromatic approach to classical homotopy theory

1. \(2 : S^0 \to S^0\),
2. \(v_1^4 : S/2 \to \Sigma^{-8} S/2\), \(\pi_\ast(2^{-1} S^0); \pi_\ast(v_1^{-1} S/2)\);
3. \(v_2^{32} : S/(2, v_1^4) \to \Sigma^{-192} S/(2, v_1^4)\), \(\pi_\ast(v_2^{-1} S/(2, v_1^4))\).
The chromatic approach to motivic homotopy theory

1. $\pi_{*,*}(2^{-1} S^{0,0})$;
2. $\pi_{*,*}(v_1^{-1} S/2)$;
3. $\pi_{*,*}(v_2^{-1} S/(2, v_1^4))$. 
But is there more?

1. \( w_0 = \eta : S^{0,0} \to \Sigma^{-1,-1} S^{0,0} \).
But is there more?

1. \( w_0 = \eta : S^{0,0} \rightarrow \Sigma^{-1,-1} S^{0,0} \).
2. \( \mu_9 \in \pi_{9,5}(S^{0,0}) \) is \( v_1 \) and \( w_0 \)-periodic. It fills the gap.
But is there more?

1. \( w_0 = \eta : S^{0,0} \to \Sigma^{-1,-1} S^{0,0} \).
2. \( \mu_9 \in \pi_{9,5}(S^{0,0}) \) is \( \nu_1 \) and \( w_0 \)-periodic. It fills the gap.
3. \( w_1^4 : S/\eta \to \Sigma^{-20,-12} S/\eta \)?
But is there more? A new constellation
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But is there more? A new constellation
What powers of $w_1$ and between what?

Classically, we have non-nilpotent self maps

$$\nu_1^4 : \Sigma^8 S/2 \to S/2,$$

Motivically, we have...

**Theorem (A.)**

*There is a non-nilpotent self map*

$$w_1^4 : \Sigma^{20,12} S/\eta \to S/\eta.$$
What’s the map?

\[ \Sigma^{20,12} S/\eta \to S/\eta \]

\[ S^{20,12} \to S^{2,1} \]

\[ \eta^2 \eta_4 \]
What’s the map?

\[ \Sigma^{20,12} S/\eta \xrightarrow{w_1^4} S/\eta \]

\[ S^{20,12} \xrightarrow{\eta^2\eta_4} S^{2,1} \]
Orthogonality of the $v_n$’s and the $w_n$’s

Originally I expected a self map

$$\Sigma^{20,12}(S/2 \wedge S/\eta)/v_1 \to (S/2 \wedge S/\eta)/v_1.$$ 

I thought one would have to kill everything with higher slope than $w_1$ in the motivic ASS before defining a $w_1$-self map.

However, it is enough to kill $p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}}$ to construct a $v_n$-self map.

That we only have to kill $w_0$ to get $w_1^4$ suggests we would only have to kill $w_0^{i_0}, \ldots, w_{n-1}^{i_{n-1}}$ to get a $w_n$-self map.

The two families of self maps are orthogonal in a sense.
The nilpotence theorem

Classically the nilpotence theorem says the following.

**Theorem (Devinatz, Hopkins, Smith)**

A *non-nilpotent self map on a finite p-local spectrum induces a non-zero homomorphism on BP-homology.*

To have a nilpotence theorem motivically we’d need a spectrum $N$ such that $N_{*,*}$ is a $\mathbb{Z}_{(p)}[v_1, v_2, v_3, \ldots, w_0, w_1, w_2, \ldots]$ module and each of $2, v_1, v_2, \ldots, w_0, w_1, \ldots$ acts nilpotently on $N_{*,*}$. 
How would $N$ see our elements?

Since $2\eta = 0$ on $S^{0,0}$ but $2$ is non-nilpotent on $S/\eta$, we must have $2w_0 \cdot N_{*,*} = 0$ and $2 \cdot N_*(S/w_0) \neq 0$.

Since $\mu_9$ is non-nilpotent $\nu^4_1 w_0$ should act non-nilpotently on $N_{*,*}$.

This second observation prevents us having

$$N_{*,*} = BP_{*,*} \oplus \mathbb{Z}/2[w_0, w_1, \ldots]$$

so the two families are not completely orthogonal. Maybe

$$N_{*,*} = BP_{*,*}[w_0, w_1, \ldots]/(2w_0 \text{ and other relations})?$$

Since $S^{0,0}$ has maps inducing multiplication by $\nu_0, w_0$ and $\nu^4_1 w_0$, and $S/w_0$ has maps inducing multiplication by $\nu_0$ and $w^4_1$, even if one had a nilpotence theorem, it is hard to imagine what the thick subcategories of the stable motivic homotopy category are.
Odd primes: $\beta$

The first nontrivial element in the cokernel of $J$ is $\beta$.

$$H^{s,u}(BP_* BP) \xrightarrow{s} \pi_{u-s}(S^0) \otimes \mathbb{Z}(p)$$

$$b = \left\{ \sum_{i=1}^{p-1} \frac{(-1)^i}{i} [t_1^i | t_1^{p-i}] \right\} \xrightarrow{s} \beta$$

$\beta \in \pi_2(p^2-p-1)(S^0)$.

$\beta$ is nilpotent but $b^n \neq 0$ for all $n$. 
Odd primes: $\beta$ motivically

Can use the classical ANSS to deduce the motivic ANSS.

Motivically, $b$ detects a non-nilpotent element

$$\beta \in \pi_2(p^2-p-1), p^2-p(S^{0,0})$$

and $\tau^{p^2-2p+1} \beta^{p^2-p+1} = 0$.

**Conjecture**

$\tau^{p^2-2p} \beta^n \neq 0$ for all $n$.

$\tau^n \beta^{p^2-p} \neq 0$ for all $n$ and $p > 5$.

Moreover, this remains true in $S/(p, v_1)$.

Try to compute $\beta^{-1} \pi_{*,*}(S^{0,0})$ or $\beta^{-1} \pi_{*,*}(S/(p, v_1))$.
First step: $b^{-1} H(P; Q/(q_0, q_1))$. 
The end: thank you for listening