A derivation of the Steenrod Squares

Choose $p$ prime, $n > 0$, $\pi \subseteq \Sigma_n$. Fix spaces $E\pi \xrightarrow{r} B\pi$, and basepoints $e \mapsto b$.

If $n = p = 2$ and $\pi = \mathbb{Z}_2$, we have $\tilde{H}^{nq}(B\pi_+ \wedge X) = \mathbb{Z}_2[x] \otimes \tilde{H}^*(X)$. Thus we can write $\text{SQ}(u) = \sum x^{q-i} \otimes \text{Sq}^i u$, where $\text{Sq}^i u \in \tilde{H}^{q+i}(X)$.

We are working over a field, so that homology and cohomology are dual. Thus to understand a cohomology class, it is enough to understand how it pairs with homology classes. So, we should take $s \in H_q^{q+i}(X)$ and calculate $(\text{Sq}^i u)(s)$. To do so, one takes $t_{q-i} \in H_q^{q+i} (\mathbb{R}P^\infty)$ dual to $x^{q-i}$ and calculates the pairing of $P_t(u)$ with $j^*(t_{q-i} \otimes s)$.

Thom class vs. Steenrod power

If $\xi$ is an ($R$-oriented) $\mathbb{R}^n$-bundle, get an $n$-disk bundle $D(\xi)$ and an $(n-1)$-sphere sub-bundle $S(\xi)$.

Very similarly:

$$(x^{q-i} \otimes \text{Sq}^i(u))(t_{q-i} \otimes s) = \text{Sq}^i(u)(s).$$
Construction of Steenrod Power $P_\pi$

**Lemma.** Suppose that $\tilde{H}^i(X) = 0$ for $i < q$ and that $\tilde{H}^q(X)$ is finite dimensional. Then,

$$\tilde{H}^i(D\pi X) = \begin{cases} 0; & \text{if } i < nq; \\ (\tilde{H}^q(X)^{\otimes n})_{\pi}; & \text{if } i = nq, \end{cases}$$

and $i_X^* : \tilde{H}^{nq}(D\pi X) \longrightarrow \tilde{H}^{nq}(X^{(n)})$ is the inclusion of the $\pi$-invariants $(\tilde{H}^q(X)^{\otimes n})_{\pi} \subset \tilde{H}^q(X)^{\otimes n}$.

**Proof.** We have a map (drawn with dotted arrows) of bundle-subbundle pairs:

\[
\begin{array}{cccc}
F_{n-1}X^n & \overset{\sim}{\longrightarrow} & F_{n-1}X^n \\
\downarrow & & \downarrow \\
X^n & \overset{\sim}{\longrightarrow} & X^n \\
\downarrow & & \downarrow \\
F_{n-1}X^n & = & E\pi \times_\pi F_{n-1}X^n \\
\downarrow & & \downarrow \\
X^n & \overset{\sim}{\longrightarrow} & E\pi \times_\pi X^n \\
\downarrow & & \downarrow \\
\{b\} & \longrightarrow & B\pi \\
\downarrow & & \downarrow \\
\{b\} & \longrightarrow & B\pi
\end{array}
\]

- $H^i(X^n, FW) = \tilde{H}^i(X^{(n)})$ is zero for $i < nq$, $H^*(X)^{\otimes n}$ when $i = nq$, and the identification is equivariant w.r.t. permutations.
- $E_2^{s,t} = H^s(B\pi; \{H^t(X^n, F_{n-1}X^n)\}) \Rightarrow \tilde{H}^{s+t}(D\pi X)$ is zero below row $nq$.
- Have edge homomorphism:

$$H^{nq}(E\pi \times_\pi X^n, E\pi \times_\pi FW) \xrightarrow{\text{(iso)}} E_\infty^{0,nq} \xrightarrow{\text{(iso)}} E_2^{0,nq} \xrightarrow{\text{(inv)}} E_1^{0,nq} \xrightarrow{\text{(iso)}} H^q(X^n, FW)$$

To get from $E_1^{0,nq}$ to $E_2^{0,nq}$ (after which the group is stable) one takes the $\pi$-invariants (recall that in Serre’s model, one can take $E_1^{0,nq}$ to be $C^0(B) \otimes H^{np}(X^n, F_{n-1}X^n)$, and only use cubes with corners at the basepoint).

\[\sim \] If $X = K_q = K(\mathbb{F}_p, q)$, this applies. Moreover, $K^q(K(\mathbb{F}_p, q); \mathbb{F}_p)$ is the field $\mathbb{F}_p(1_q)$.

\[\sim \] Thus $i^* : H^{nq}(D\pi K_q) \longrightarrow H^{nq}(K_q^{(n)})$ is an isomorphism. There is then a unique $P_{\pi^*}$ mapping to $i^*_q$ under $i^*$.

\[\sim \] Now suppose that $u : X \longrightarrow K_q$ represents $u \in \tilde{H}^q(X)$. Need

\[
\begin{array}{ccc}
\tilde{H}^q(X) & \xrightarrow{P_\pi} & \tilde{H}^q(D\pi X) \\
\uparrow & & \uparrow \text{(}$D\pi u$)^* \text{=} \\
\tilde{H}^q(K_q) & \xrightarrow{P_\pi} & \tilde{H}^q(D\pi K_q)
\end{array}
\]

Thus must have $P_\pi u = (D\pi u)^* P_{\pi^*}$. This map is natural, is the only way to extend $P_\mu$ from what it needs to be on $K_q$, and satisfies the formula, since it is satisfied in the universal case.
Properties

- \( \text{Sq}^i u = 0 \) for \( i < 0 \) (universality) and for \( i > q \) (no negative powers of \( x \)).
- \( \text{Sq}^q u = u^2 \).

\[
\begin{array}{c}
\xymatrix{
B\pi_+ \wedge X 
& D_\pi X \\
S^0 \wedge X 
\ar[u]_{\Delta} 
& X^{(2)} 
\ar[l]^i 
& \ar[d]_{i} 
\end{array}
\]

\[
\xymatrix{
\sum_{i=0}^q x^{q-i} \otimes \text{Sq}^i u 
\ar[u] 
& Pu 
\ar[d] 
& \ar[l]_{i} 
\end{array}
\]

\[
\xymatrix{
\tilde{H}^{2q}(-) 
& \ar[l]_{i} 
\end{array}
\]

- Cartan formula. Have map \( \delta : D_\pi (X \wedge Y) \rightarrow D_\pi (X) \wedge D_\pi (Y) \) which is \( \Delta \) on \( E_\pi \) components.

\[
\begin{array}{c}
\xymatrix{
(X \wedge Y)^{(2)} 
\ar[i] & D_\pi (X \wedge Y) \\
X^{(2)} \wedge Y^{(2)} 
\ar[i\wedge j] & D_\pi X \wedge D_\pi Y 
\ar[dl]_{j\wedge j} 
\end{array}
\]

\[
\begin{array}{c}
\xymatrix{
B\pi_+ \wedge X \wedge Y 
\ar[d]_{B\pi_+ \wedge X \wedge Y} 
& \ar[l]_{j} 
\end{array}
\]

\[
\begin{array}{c}
\xymatrix{
(t_p \wedge t_q)^{(2)} 
\ar[i] & P(t_p \wedge t_q) \\
t_p^{(2)} \wedge t_q^{(2)} 
\ar[i] & P(t_p) \wedge P(t_q) 
\ar[dl] 
\end{array}
\]

\[
\begin{array}{c}
\xymatrix{
\sum x^{p+q-k} \otimes \text{Sq}^k (u \wedge v) 
\ar[u] 
& \ar[l] \sum x^{p-i} \otimes \text{Sq}^i (u) \wedge \sum x^{q-j} \otimes \text{Sq}^j (v) 
\ar[u] 
\end{array}
\]

If \( \delta^* \) does the right thing, then we’re winners. It must, as in the universal case, \( i^* \) is injective.

- Write \( \text{Sq} u = \sum \text{Sq}^i u \). The Cartan formula says \( \text{Sq}(uv) = (\text{Sq} u)(\text{Sq} v) \).

- Exercise: show that \( \text{Sq}^0 s = s \), where \( s \) is the generator of \( \tilde{H}^1(S^1) = \mathbb{Z}_2 \).

- \( \text{Sq}^k \) is stable — it commutes with suspension. (The suspension is “smash with \( S^1 \)" and so \( \text{Sq}^k (\sigma u) = \text{Sq}^k (u \wedge e) = \text{Sq}^k u \wedge \text{Sq}^0 e = \sigma \text{Sq}^k u \).) In particular, it is additive.

- \( \text{Sq}^0 : \tilde{H}^q (X) \rightarrow \tilde{H}^q (X) \) is the identity. (The map \( \tilde{H}^q (K^q) \rightarrow \tilde{H}^q (S^q) \) induced by the fundamental class \( s^{\wedge q} \in \tilde{H}^q (S^q) \) is an isomorphism. Thus it’s enough to check this on \( s^{\wedge q} \), which follows from the exercise and Cartan.)

- \( \text{Sq}^1 \) is the Bockstein.

\[
2\tilde{H}^q (X) = \tilde{H}^q (X) \otimes \tilde{H}^1 (S^1) \xrightarrow{\Delta_{\text{e}}} \tilde{H}^{q+1} (\Sigma X)
\]
• The Adem relations are satisfied. For $a < 2b$:

\[ Sq^a Sq^b = \sum_{j=0}^{\lfloor a/2 \rfloor} \left( \frac{b-j-1}{a-2j} \right) Sq^{a+b-j} Sq^j. \]

Note that $a < 2b \implies a + b - j \geq 2j$.

• $Sq^{i_1} \cdots Sq^{i_r}$ is admissible if $i_1 \geq 2i_2$, $i_2 \geq 2i_3$, etc. The Adem relations let us write any such product as a sum of admissible products.

• Let $A$ be the free unital $\mathbb{Z}_2$-algebra on symbols $Sq^i$ for $i > 0$, mod the Adem relations.
  - $A$ acts on $\tilde{H}^*(X)$ — the free algebra obviously does, and this action descends to $A$.
  - $A$ has a basis the admissible products.
  - $A$ is generated by the $Sq^{2^i}$, which are indecomposable. (Do it by induction: use admissibility and Adem.)

• Application: Suppose that $\tilde{H}^*(X; \mathbb{Z}_2) = \mathbb{Z}_2[x]$, with $|x| = q$. Then $q = 2^i$. (For $x^2 = Sq^q x \neq 0$, but the intermediate cohomology groups between $q$ and $2q$ are zero.)

• Application: Suppose that $f : S^{2n-1} \longrightarrow S^n$ has odd Hopf invariant. Then $n = 2^i$. 

\[ \]
### Cohomology Operations

\[ H^p(K(\pi, q); G) \] is in bijection with natural transformations \( \widetilde{H}^q(-; \pi) \rightarrow \widetilde{H}^p(-; G) \).

The groups \( H^{n+q}(K(\pi, q); G) \) are stable when \( q > n \). Elements thereof are called “stable cohomology operations”. The Steenrod algebra is the algebra of stable cohomology operations \( H^*(-; \mathbb{Z}) \). To see this, we should calculate \( H^{n+q}(K(\pi, q); G) \) for large \( q \).

We’ll use the path-loop fibration \( K(\mathbb{Z}_2, q) \rightarrow * \rightarrow K(\mathbb{Z}_2, q+1) \). By induction:

\[ H^*(K(\mathbb{Z}_2, q); \mathbb{Z}_2) = \mathbb{Z}_2[\text{Sq}^I(t_q)], \] over those admissible \( I \) with excess less than \( q \).

Need Borel’s theorem:

- A graded ring \( R \) over \( \mathbb{Z}_2 \) has an order set \( x_1, x_2, \ldots \) of homogeneous elements as a simple system of generators if the monomials \( \{x_{i_1} \cdots x_{i_r} : i_1 < \cdots < i_r \} \) form a \( \mathbb{Z}_2 \)-basis, and each graded part is finite dimensional.

- Suppose \( F \rightarrow E \downarrow B \) is a fibre space with \( E \) acyclic, and that \( H^*(F; \mathbb{Z}_2) \) has a simple system \( \{x_i \} \) of transgressive generators. Then \( H^*(B; \mathbb{Z}_2) \) is a polynomial ring in the \( \{\tau(x_i)\} \).

When \( q = 1 \): have \( \mathbb{R}\mathbb{P}^\infty \), the poly alg on \( \text{Sq}^0 t_1 \).

Inductive step:

- There’s a simple system of generators \( (\text{Sq}^I t_q)^{2^r} \).
- This equals \( \text{Sq}^{2^{r-1}(q+n(I)), \ldots, 2(q+n(I)), (q+n(I))} \text{Sq}^I t_q \).
- As \( I \) runs over all admissibles with \( e(I) < q, 2^{r-1}(q+n(I)), \ldots, 2(q+n(I)), (q+n(I)) \), \( I \) runs over all with \( e(I) < q + 1 \).

What about:

\[ H^*(K(\mathbb{Z}_2^n, q); \mathbb{Z}_2) = \mathbb{Z}_2[\text{Sq}^I(t_q)], \] over those admissible \( I \) with excess less than \( q \).

Here, we swap \( \text{Sq}^I \) for \( \text{Sq}^1 \), the \( h^{th} \) Bockstein operator.

Here, the starting case is different: \( H^*(K(\mathbb{Z}_2^n, 1); \mathbb{Z}_2) = \Lambda(u_1) \otimes \mathbb{Z}_2[\text{Sq}^1(u_1)]. \)

What about:

\[ H^*(K(\mathbb{Z}, q); \mathbb{Z}_2) = \mathbb{Z}_2[\text{Sq}^I(t_q)], \] over admissible \( I, e(I) < q, I_{\text{last}} > 1 \).

Here, the starting case is different: \( H^*(K(\mathbb{Z}, 1); \mathbb{Z}_2) = \Lambda(u_1). \)

### Stable Operations

Let’s consider \( H^{n+q}(K(\mathbb{Z}_2, q); G) \) for \( q > n \). Basis corresponding to admissible sequences with degree \( n \) (and excess less than \( q \), but the excess condition is vacuous). Thus you get the whole degree \( n \) part of the Steenrod algebra.

\( H^p(K(\pi, q); \mathbb{Z}_2) \) for \( \pi \) finitely generated abelian

Using the K"unneth formula to turn \( \oplus \) to \( \otimes \), only need to know it for \( \mathbb{Z} \) and \( \mathbb{Z}_p^x \). Know when \( p = 2 \).

If \( p \) is odd, then \((q > 1)\) you can use mod \( C \) theory (where \( C \) is the class of “abelian torsion groups of finite exponent \( n \) coprime to \( p \)”).

Note that if one is interested in stable groups, the K"unneth formula turns \( \oplus \) to \( \oplus \).
Poincaré Polynomials

Let \( \theta(\pi; q) \) be the Poincaré polynomial for \( H^*(K(\pi, q), \mathbb{Z}_2) \).

The case \( \pi = \mathbb{Z}_{2^k} \)

- Have poly alg on an element of dimension \( q + n(I) \) for each admissible \( I \) with \( e(I) < q \).

\[
\theta(\mathbb{Z}_2; q) = \prod_{e(I) < q} (1 - t^{q+n(I)})^{-1}
\]

\[
= \prod_{h_1 \geq h_2 \geq \cdots \geq h_q = 0} (1 - t^{2h_1})^{-1}
\]

- To see the last equality, take a sequence \( I = (i_1, \ldots, i_r) \).
  - Let \( \alpha_1 = i_1 - 2i_2, \ldots, \alpha_r = i_r \).
  - Let \( \alpha_0 = q - 1 - \sum \alpha_i \).
  - \( n(I) = \sum (2^i - 1) \alpha_i = \sum_{i \geq 0} 2^i \alpha_i - (q - 1) \).
  - Thus generators of dimension \( N \) are in bijection with \( (\alpha_0, \ldots, \alpha_r) \) with sum \( q - 1 \) and

\[
N = 1 + 2^1 \alpha_1 + \cdots + 2^r \alpha_r.
\]

There are \( q \) summands, each a power of 2. Let \( h_1, \ldots, h_q \) be the powers, from biggest to smallest.

The case \( \pi = \mathbb{Z} \)

- Have poly alg on an element of dimension \( q + n(I) \) where \( e(I) < q \), \( I_{\text{last}} \neq 1 \) (for \( q > 1 \)).

\[
\theta(\mathbb{Z}; q) = \prod_{e(I) < q, I_{\text{last}} \neq 1} (1 - t^{q+n(I)})^{-1}
\]

\[
= \prod_{h_1 > h_2 \geq \cdots \geq h_{q-1} = 0} (1 - t^{2h_1})^{-1}
\]

- Now need \( \alpha_r > 1 \), so that \( h_1 = h_2 \).
- But then we can always swap \( 2^{h_1} + 2^{h_2} \) for \( 2^{h_1+1} \).
- Thus we get one less summand and a strict last inequality.

- \( \frac{\theta(\mathbb{Z}_2; q-1)}{\theta(\mathbb{Z}_2; q)} \) has those indexed by \( h_1 \) to \( h_{q-1} \) with \( h_1 = h_2 \).
- One can swap \( 2^{h_1} + 2^{h_2} \) for \( 2^{h_1+1} \) again, and get the product for \( \theta(\mathbb{Z}; q-1) \). Thus

\[
\theta(\mathbb{Z}; q) = \frac{\theta(\mathbb{Z}_2; q-1) \cdot \theta(\mathbb{Z}_2; q-3) \cdots}{\theta(\mathbb{Z}_2; q-2) \cdot \theta(\mathbb{Z}_2; q-4) \cdots}
\]
Convergence

- These converge for $|t| < 1$.
- They have a “dominant singularity” at $t = 1$.
  - Write $t = 1 - 2^{-x}$, so $t \to 1$ slowly as $x \to \infty$. Write:
    $$\varphi(\pi; q)(x) = \log_2(\theta(\pi; q)(t)).$$
  - Serre calculates:
    $$\varphi(\mathbb{Z}_2; q) \sim x^q/q! \quad \text{and} \quad \varphi(\mathbb{Z}; q) \sim x^{q-1}/(q-1)!$$
- As $\oplus \to \otimes$ on $H^*$, $\oplus \to \oplus$ on $\varphi$. Thus:

Theorem. Suppose $\pi$ is f.g. with $s \mathbb{Z}$ summands and $r \mathbb{Z}_2$ summands. Then:

$$\varphi(\pi; q) \sim \begin{cases} r \cdot x^q/q! & r \geq 1, \\ s \cdot x^{q-1}/(q-1)! & r = 0, s \geq 1, \\ 0 & r = s = 0. \end{cases}$$

Topological Application

Theorem. Suppose $X$ is simply connected and:

1. The $H_i(X; \mathbb{Z})$ are finitely generated;
2. The $H_i(X; \mathbb{Z}_2)$ vanish for $i \gg 0$;
3. $H_i(X; \mathbb{Z}_2)$ is nonzero for some $i \gg 0$.

Then $\pi_i(X)$ has a $\mathbb{Z}$ or $\mathbb{Z}_2$ subgroup for infinitely many $i$.

Example. In $S^n$ there are finitely many $\mathbb{Z}$ summands, and thus infinitely many $\mathbb{Z}_2$ summands.

Proof. • By Hurewicz mod $C$ the homotopy groups are finitely generated. So it’s enough to see that infinitely many $\pi_i \otimes \mathbb{Z}_2$ are nonzero.
• For a contradiction, suppose that $q$ is maximal such that $\pi_q \otimes \mathbb{Z}_2 \neq 0$.
• Let $j$ be the least positive integer such that $H_j(X; \mathbb{Z}_2) \neq 0$.
• By Hurewicz mod $C$ $j$ is the first index such that $\pi_j \otimes \mathbb{Z}_2 \neq 0$. Thus $q \geq j \geq 2$.
• Let $X_i = (X, i)$ be $X$ with the first $i-1$ homotopy groups killed, and let $A(t)$ be the Poincaré polynomial for $H^*(A; \mathbb{Z}_2)$, for any space $A$ where these are finite dimensional.
• There’s a fibration $X_{q+1} \to X_q \to K(\pi_q, q)$. $H^*(X_{q+1}; \mathbb{Z}_2)$ is trivial, by Hurewicz mod $C$.
  Thus the SSS gives $X_q(t) = \theta(\pi_q; q)$.
• Now there are fibrations:

\[ K(\pi_{q-1}, q-2) \to X_q \to X_{q-1}, \]
\[ K(\pi_{q-2}, q-3) \to X_{q-1} \to X_{q-2}, \]
\[ K(\pi_{q-3}, q-4) \to X_{q-2} \to X_{q-3}, \ldots \]
\[ K(\pi_2, 1) \to X_3 \to X_2. \]

• Whenever \( F \to E \to B \) is a fibration with \( B \) simply connected, we have, on every co-efficient, \( E(t) \leq F(t) \cdot B(t) \) (if all the terms make sense). The \( E_2 \) page has Poincaré poly \( F(t) \cdot B(t) \), but the differentials cut down dimensions.

• Thus, on coefficients:

\[
\theta(\pi_q; q) = X_q(t) \\
\leq X_{q-1}(t) \cdot \theta(\pi_{q-1}; q-2) \leq \cdots \\
\leq X(t) \cdot \prod_{1<i<q} \theta(\pi_i; i-1).
\]

• \( X(t) \) is a polynomial, so its values on \([0, 1]\) are bounded by \( h \). Taking log₂:

\[
\varphi(\pi_q; q)(x) \leq \log_2 h + \sum_{i=2}^{q-1} \varphi(\pi_i; i-1)(x).
\]

This is a contradiction — the terms on the right grow at most as fast as \( x^{q-2} \) as \( x \to \infty \), but those on the right grow as either \( x^q \) or \( x^{q-1} \).

\[ \square \]
Jeremy’s Talk on Thom’s paper

We’d really like to calculate the homotopy groups

\[ \pi_*(MBO(r)), \quad \pi_*(MBSO(r)). \]

We’ll do the first, which is easier, and that’s all we have time for.

Let’s just start trying to understand the cohomology thereof and the Steenrod operations thereupon (the cohomology should be the most accessible topological invariant). We know:

\[ H^{r+i}(MBO(r)) \cong H^i(BO(r)) = \mathbb{Z}_2[\text{Stiefel-Whitney classes}] \]

Although we know the cohomology ring, we aren’t completely sure yet as the action of the steenrod algebra. Anyway, we know that

\[ H^i(BO(r)) \cong H^i(BO(s)) \text{ where } i \leq r, s \]

so \( H^{r+i}(MBO(r)) \) is independent of \( r \) for \( i \leq r \), and we define \( H^i(MBO) \) to be this stable value. This connotes that we’re taking a spectrum, but we’re not so sophisticated. We can define \( H^i(MBO) = \bigoplus H^i(MBO) \). For example, \( H^0(MBO) = H^r(MBO(r)) \) for large \( r \), and this is just a one-dimensional \( \mathbb{F}_2 \)-vector space generated by the Thom class \( U \).

Digression on the Steenrod algebra

Let \( A \) be the Steenrod algebra. For a space \( X \), \( H^*(X) \) is as graded \( A \)-module. A nice space \( X \) to use here is \( X = \mathbb{R}P^\infty \times \cdots \times \mathbb{R}P^\infty \). We have a map

\[ \omega : A \longrightarrow H^*(X) \]

\[ a \longmapsto a(u_1 \otimes \cdots \otimes u_n) \]

\( \omega \) is injective in degree \( \leq n \), and this is how we see that the Adem relations are the only relations. This is the simplest case where we see the linear independence of a bunch of the admissible squares.

**Proposition.** Have \( \psi : A \longrightarrow A \otimes A \) the unique algebra homomorphism\(^7\) taking \( Sq^i \) to the Whitney sum \( \sum Sq^i \otimes Sq^j \) — the comultiplication map.

**Proof.** Let \( A \) be the free associative algebra on the \( Sq^i \). There’s a projection \( \pi : A \longrightarrow A \). The definition of \( \psi \) definitely extends to a map \( A \longrightarrow A \otimes A \). We need to check that \( \ker \pi \) is contained in \( \ker \psi \). But:

\[ A \xrightarrow{\psi} A \otimes A \xrightarrow{\omega \otimes \omega} H^*(X) \otimes H^*(X) \xrightarrow{\sim} H^*(X \times X) \]

And \( \omega \otimes \omega \) is injective in a range which depends on how many factors we choose in \( X \). \( \square \)

\(^7\)To check it we need to use a bit of knowledge about the Steenrod.

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Also have a counit map $\epsilon : A \longrightarrow \mathbb{Z}_2$ which takes $\text{Sq}^0 \mapsto 1$ and $\text{Sq}^i \mapsto 0$ for $i > 0$. These satisfy coherence conditions:

\[
\begin{array}{ccc}
A & \longrightarrow & A \otimes A \\
\downarrow & & \downarrow \\
A \otimes A & \longrightarrow & A \otimes A \otimes A
\end{array}
\]

This is enough to make $A$ a bialgebra. As $\epsilon$ is an isomorphism between the degree 0 part and the field:

**Fact.** $A$ is a connected associative, coassociative and cocommutative Hopf algebra.

**Remark.** Haynes said some things which I only half remember. Something like: “If a bialgebra is connected, there’s exactly one antipode making it a group object in coalgebras. If the bialgebra is either commutative or cocommutative, then the antipode is an involution.”

**Back to $MBO(r)$**

We should note that there’s a Whitney sum map

\[
BO(r) \times BO(s) \longrightarrow BO(r + s)
\]

giving

\[
MBO(r) \wedge MBO(s) \longrightarrow MBO(r + s)
\]

and on cohomology

\[
\psi : \tilde{H}^*(MBO(s + r)) \longrightarrow \tilde{H}^*(MBO(s)) \otimes \tilde{H}^*(MBO(r))
\]

\[
\epsilon : H^*(MBO) \longrightarrow \mathbb{F}_2 \quad 1 \longrightarrow 1
\]

This structure makes $H^*(MBO)$ a coalgebra and a left $A$-module. $a \in A$ acts on $H^{n+r}(MBO(r))$ (for $r$ very large) via the natural action. This should be compatible with the stabilisation process. There’s one sticky point — cupping with the Thom class is not natural, but we’ll apply this twice, so what we end up with is correct.

**Theorem.** Let $A$ be a connected Hopf algebra, and let $M$ be a connected co-algebra which is also an $A$-module. Suppose also that $\psi : M \longrightarrow M \otimes M$ is an $A$-module homomorphism. Let $\nu : A \longrightarrow M$ be defined by $\nu(a) = a \cdot 1$, where $1 \in M^0$ is the unit (well defined by connectivity). Suppose that $\nu$ is injective. Then $M$ is a free $A$-module.

**Sketch of proof.** Do the dumbest thing in the world. Define $N = M/A_+ M$, where $A_+$ is the positive degree part of $A$. Choose any vector space splitting $f : N \longrightarrow M$. Define $C : A \otimes N \longrightarrow M$ by $a \otimes n \mapsto af(n)$. Once you see that this is an isomorphism, you’re done, and you can show this by induction on degree.

To apply this to our example, we should check:

\footnote{Should be associative because Whitney sums are!}
Proposition. $\nu: \mathcal{A}_2 \rightarrow H^*(MBO)$ sending $a \mapsto a(U)$ is injective.

Proof. Assume that $a \in \mathcal{A}_2^n$ (the $n^{th}$ degree part) has $\nu(a) = 0$. Think of this as happening in:

$$H^r(MBO(r)) \rightarrow H^{r+n}(MBO(r)) \quad U \mapsto a(U).$$

As this is the universal case, it’s enough to pull all this back, and examine

$$H^r(MBO(1) \wedge \cdots \wedge MBO(1)) \rightarrow H^{r+n}(MBO(1) \times \cdots \times MBO(1))$$

This is a general thing — the splitting principle — these constructions don’t see the difference between the split and nonsplit cases. $U = x_1 \cdots x_r$ where $x_i$ is the Thom class of $MBO(1)$. Note that $MBO(1) = \mathbb{RP}^\infty$. Now write $a = \sumSq^I$ in the admissible sequence basis. Now how does $\Sq^I$ hit $U$? Inductively, we see

$$a(U) = \sum x_1^{v_1} \cdots x_r^{v_r}$$

where $v_i = i - 2i_{i+1}$. This implies that $\Sq^I$ is injective. \(\square\)

Now note that $H^*(MBO)$ is a free $\mathcal{A}_2$-module, due to the proposition and theorem above.

Some consequences

Lets derive some consequences. When we destabilise, this says:

$$H^{n+r}(MBO(r)) \text{ is a free } \mathcal{A}_2\text{-module for } r \geq n.$$ 

Moreover, (as some stuff is zero):

$$H^i(MBO(r)) \text{ is a free } \mathcal{A}_2\text{-module for } i \leq 2r.$$ 

Let’s choose a basis $x_1, \ldots, x_m$ for $H^*(MBO(r))$ in the range $* \leq 2r$. Suppose $x_i$ lives in dimension $n_i$, $r \leq n_i \leq 2r$. Each $x_i$ induces a map $\varphi_i: MBO(r) \rightarrow K(\mathbb{Z}_2, n_i)$, these maps can be glued together to get

$$\varphi: MBO(r) \rightarrow \prod_{i=1}^n K(\mathbb{Z}_2, n_i) = K.$$ 

Consider

$$H^j(K) = \bigoplus_{\sum j_i = j} H^{j_i}(K(\mathbb{Z}_2, n_i)) \otimes \cdots \otimes H^{j_m}(K(\mathbb{Z}_2, n_m)).$$

For $j \leq 2r$, the only way we can get anything here is when all but one of the $j_i$ are zero, so

$$H^j(K) \cong \bigoplus_{i=1}^m H^{j_i}(K(\mathbb{Z}_2, n_i)) \cong \bigoplus_{i=1}^m \mathcal{A}_2^{j_i-n_i}.$$ 

Now $\varphi^*: H^j(K) \rightarrow H^j(MBO(r))$ sends

$$(a_1, \ldots, a_m) \mapsto a_1x_1 + a_2x_2 + \cdots + a_mx_m$$

and because this stuff is free, in terms of a bases, this is an isomorphism. That is, $\varphi$ is a $\mathbb{Z}_2$-cohomology isomorphism in the range $j \leq 2r$. 

11
Now the cohomology of $MBO(r)$ is universally zero in mod $p$ coefficients. Thus we have an iso with mod $p$ coefficients (of zero objects). Thus $\varphi$ is a homology isomorphism in this range. As everything is simply connected, it’s a $\pi_\ast$-equivalence in this range! In particular,

$$\mathfrak{M}_n = \lim_{\to} \pi_{n+r}(MBO(r))$$

And it’s going to follow, just by counting, that:

**Corollary.** For large $r$, $\dim \pi_{n+r}(MBO(r)) = p_{nd}(n)$, the number of partitions of $n$ that include no numbers one less than a power of 2.

**Applications**

We have seen that the Hurewicz homomorphism from $\pi_\ast(MBO(r)) \to H^\ast(MBO(r))$ is injective for $\ast < 2r$. So (using the duality of homology and cohomology)

$$\lim_{\to} \pi_{n+r}(MBO(r)) \to H^{n+r}(MBO(r))$$

is an injection from the cobordism groups of an $n$-manifold. This tells us that the Stiefel-Whitney numbers of a manifold determine its cobordism class.

Now $p_{nd}(3) = 0$, so that every closed 3-manifold is null-cobordant. This was once considered very hard.

---

9c.f. Milnor-Stasheff. One can use Pontrjagin classes and Steenrod powers.
The Signature Theorem

Remarks on oriented $4k$-dimensional manifolds:

1. There is a nondegenerate symmetric bilinear form on $H^{2k}(M^{4k})$.
2. The signature is called the signature $\sigma(M^{4k})$ of $M^{4k}$.
3. The signature is a genus — a ring homomorphism from $\Omega_* \to \mathbb{Q}$.
4. $\sigma$ must factor through $\Omega_* \to \Omega_* \otimes \mathbb{Q}$. We’ll study $\Omega_* \otimes \mathbb{Q}$ using Pontrjagin numbers.

Chern and Pontrjagin classes

⇝ An $n$-dimensional complex vector bundle $\omega \downarrow X$ has Chern classes: characteristic classes
$c_i(\omega) \in H^{2i}(X), \ i = 0, \ldots, n$.
⇝ These satisfy a Cartan formula, $c_0 = 1$, and a normalisation, and are pulled back from:

$$H^*(BU(n); \mathbb{Z}) = \mathbb{Z}[c_1(\gamma^n), c_2(\gamma^n), \ldots, c_n(\gamma^n)] \quad \text{with} \quad |c_i| = 2i.$$ 

⇝ Define the Pontrjagin classes of an $n$-dimensional real vector bundle $\xi \downarrow X$ to be the even Chern classes of the complexification $\xi \otimes \mathbb{C}$, with a sign:

$$p_i(\xi) = (-1)^i c_{2i}(\xi \otimes \mathbb{C}) \in H^{4i}(X; \mathbb{Z}).$$

We could abbreviate $p(\xi) := c_{\pm \text{ev}}(\xi \otimes \mathbb{C})$.
⇝ Ignore the odd Chern classes — they have order 2, as a complexification is isomorphic to its conjugate.
⇝ The total Pontrjagin class inherits a Cartan formula which holds modulo elements of order 2.
⇝ If $M^{4k}$ is an oriented $4k$-manifold, for each partition $I = (i_1, \ldots, i_r) \in P(k)$, we have:

$$p_I[M] = p_{i_1} \cdots p_{i_r}[M] = \langle p_{i_1}(\tau_M) \cdots p_{i_r}(\tau_M), [M] \rangle \quad \leftarrow \text{Pontrjagin number}$$

⇝ Pontrjagin numbers of $\mathbb{CP}^{2n}$: Write $H^*(\mathbb{CP}^{2n}) = \mathbb{Z}[a]/a^{2n+1}$.

We know: $c(\tau) = (1 + a)^{2n+1}$ and $c(\overline{\tau}) = (1 - a)^{2n+1}$, so that $c(\tau \oplus \overline{\tau}) = (1 - a^2)^{2n+1}$.

$$p(\tau) := c_{\pm \text{ev}}(\tau \otimes \mathbb{C}) = c_{\pm \text{ev}}(\tau \oplus \overline{\tau}) = (1 + a^2)^{2n+1}.$$ 

Unravelling this, we obtain: $p_I[\mathbb{CP}^{2n}] = \left(\begin{array}{c} 2n+1 \\ i_1 \end{array}\right) \cdots \left(\begin{array}{c} 2n+1 \\ i_r \end{array}\right)$.
⇝ We have that:

$$H^*(B(SO(k)); \mathbb{Z}[1/2]) = \mathbb{Z}[1/2][p_1, p_2, \ldots, p_{k/2}, e]/\begin{cases} e = 0; & k \text{ odd; } \\ e^2 = p_{k/2}; & k \text{ even,} \end{cases}$$

where $p_i$ has dimension $4i$, and $e$ has dimension $k$. Thus, this group has rank $p(n/4)$ in dimension $n$, for any $n < k$. 

Michael Donovan
Symmetric functions

\[ \prod (1 + t_i) = 1 + \sum \sigma_i. \]

For each partition \( I = (i_1, \ldots, i_r) \in P(k) \) with length \( r \leq n \), can form

\[ \sum t_{i_1} \cdots t_{i_r} =: s_I(\sigma_1, \ldots, \sigma_n) \]

so that \( s_I \) is a polynomial in \( n \) variables of degree \( k \) if its \( i^{th} \) input has degree \( i \).

Note that \( s_{(n)}(\sigma_1, \ldots, \sigma_n) = \sigma_n \). By convention, \( s_{(1)}(\sigma_1, \ldots, \sigma_n) = 1 \).

If \( \xi \downarrow X \) is a real vector bundle and \( I \in P(k) \), write

\[ s_I(p(\xi)) := s_I(p_1(\xi), \ldots, p_r(\xi)) \in H^{4k}(X). \]

For \( M \) a 4\( n \)-manifold and \( I \in P(n) \), write

\[ s_I(p)[M] := (s_I(p(\tau_M)), [M]). \]

\( s_{(n)}(p)[\mathbb{CP}^{2n}] = 2n + 1 \). As \( p(\tau) = (1 + a^2)^{2n+1} \), each of \( t_1, \ldots, t_{2n+1} \) represents \( a^2 \).

In particular, \( s_{(n)}(\sigma_1, \ldots, \sigma_{2n+1}) := \sum t_i^{n} \) should then take the value \( (2n+1)(a^2)^n \).

Why is \( \sigma \) a genus?

That is is additive is easy. Suppose given \( V^n \) and \( W^m \). If \( m + n \) is not divisible by 4, we’re clearly done, so write \( 4k = m + n \). If \( m \) and \( n \) not both divisible by four, need to show \( \sigma(V \times M) = 0 \). Otherwise, need \( \sigma(V)\sigma(W) \). Always use real coefficients. Define

\[ B = \bigoplus_{s=0}^{2k} H^s(V) \otimes H^{2k-s}(W). \]

Give a basis \( \{ v_i^s \} \) of the modules \( H^s(V) \) that appear in this sum, so that \( \langle v_i^s \cup v_j^{n-s}, V \rangle = \delta_{ij}. \) Give a similar basis \( \{ w_i^s \} \) of the modules \( H^s(W) \) appearing. Then \( B \) has a basis \( \{ v_i^s \otimes w_j^{2k-s} \} \), and the only nontrivial pairings between these basis elements are:

\[ (v_i^s \otimes w_j^{2k-s}) \cup (v_i^{n-s} \otimes w_j^{2k-n+s}) = (-1)^n = (v_i^{n-s} \otimes w_j^{2k-n+s}) \cup (v_i^s \otimes w_j^{2k-s}). \]

So we get antidiagonal \( 2 \times 2 \) matrices down the diagonal, and can thus disregard \( B \).

We are left to deal with \( A = H^{m/2}(V) \otimes H^{n/2}(W) \). If \( m \) and \( n \) are odd, we’re done. If \( m \) and \( n \) are multiples of four, the signatures multiply, and we’re done (note that everything is in even dimensions). If \( m \) and \( n \) are congruent to 2 \((\mod 4)\), we have the tensor product of skew-symmetric forms, which have zeros on the diagonal.

To see that \( \sigma \) vanishes on boundaries, we have, by the same argument as Michael gave on Wednesday, if \( j : M^{4k} \to V \) is the inclusion of the boundary, an equation

\[ \dim[\text{im } j^*] = 1/2 \cdot \dim[H^{2k}(M)], \text{ where } j^* : H^{2k}(V) \to H^{2k}(M) \]

However, \( \langle j^*(x) \cup j^*(y), [M] \rangle = \langle x \cup y, j_*[M] \rangle = 0 \) as \( M \) bounds, so we have a totally isotropic subspace of half the total dimension. Apparently this only happens when the signature is zero.
A lower bound on the rank of $\Omega_*$

**Theorem (16.8).** Suppose that $M^4, \ldots, M^{4n}$ are oriented manifolds with $s_{(k)}(p)[M^{4k}] \neq 0$. Define $M^J = M^{4j_1} \times \cdots \times M^{4j_s}$ for $J \in \mathcal{P}(n)$. Then the $p(n) \times p(n)$ matrix

$$[p_I[M^J]]_{I,J}$$

of Pontrjagin numbers is non-singular.

**Remark.** Pontrjagin numbers are oriented cobordism invariant.

*Proof of Remark.* Suppose that $M^{4k} = \partial V$, with inclusion $j : M \rightarrow V$. Then $j^*\tau_V = 1 \oplus \tau_M$, so that $p(\tau_M) = j^*p(\tau_V)$. But $\langle j^*(p(\tau_V)), [M] \rangle = \langle p(\tau_V), j_*[M] \rangle = 0$, as $[M]$ is a boundary. \hfill \Box

As it is readily calculated that $s_{(k)}[\mathbb{CP}^{2k}] = 2k + 1$, we obtain:

**Corollary.** The manifolds $\mathbb{CP}^2, \mathbb{CP}^4, \mathbb{CP}^6, \ldots$ are algebraically independent in $\Omega_*$. In particular, they generate a polynomial subalgebra whose rank in dimension $n$ is $p(n/4)$.

*Proof of corollary.* A homogeneous polynomial relation of dimension $4n$ is a linear relation between the manifolds $M^J$ for $J \in \mathcal{P}(n)$, using $M^{4i} = \mathbb{CP}^{2i}$. This would give a linear relation between their vectors of Pontrjagin numbers, which is impossible. \hfill \Box

*Proof of Theorem (16.8).* It follows from readily verifiable (but time-costly) properties of the $s_I$ and the Cartan formula that, if $J = (j_1, \ldots, j_q) \in \mathcal{P}(n)$:

$$s_I(p)[M^J] = \sum_{I_1 \cdots I_q = I} \prod_{k=1}^q s_{I_k}(p)[M^{4j_k}].$$

Thus, if $I$ does not refine $J$, this quantity is zero, and if $I = J$, this quantity is nonzero, being the product of certain of the $s_{(k)}(p)[M^{4k}]$. Thus the matrix is triangular with nonzero diagonal, w.r.t. refinement. \hfill \Box
Some preliminaries before proving the converse

**Theorem** (18.3). Let $X$ be a finite $(k - 1)$-connected complex ($k \geq 2$). Then the Hurewicz homomorphism is a $C$-isomorphism up to dimension $2k - 2$.

**Proof.** Let $\mathcal{K}$ be the collection of $(k - 1)$-connected complexes for which the theorem holds.

$\Rightarrow$ $S^n \in \mathcal{K}$, by a calculation of Serre.

$\Rightarrow$ If $X, Y \in \mathcal{K}$, then $X \times Y$ and $X \vee Y$ are both in $\mathcal{K}$. To see this, note that $X \wedge Y$ is $(2k-1)$-connected, and use the relative Hurewicz theorem to see that the groups $\pi_i(X \times Y, X \vee Y)$ are zero for $i < 2k$. Now examine the homotopy-homology ladder (with $i \leq 2k - 2$), in light of the Künneth theorem:

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
\pi_i(X \vee Y) & \rightarrow & H_i(X \vee Y) \\
\downarrow & & \downarrow \\
\pi_i(X \times Y) & \rightarrow & H_i(X \times Y) \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0 \\
\pi_i(X) \oplus \pi_i(Y) & \overset{\text{C-iso}}{\rightarrow} & H_i(X) \oplus H_i(Y) \\
\end{array}
\]

$\Rightarrow$ Suppose that $X$ is an arbitrary finite $(k - 1)$-connected complex. Choose a homogeneous basis $f_i : S^{r_i} \rightarrow X$ for $\bigoplus_{i=k}^{2k-1} \pi_*(X)$, and let $f : \vee S^{r_i} \rightarrow X$ be the resulting map.

\[
\begin{array}{ccc}
\pi_*(\vee S^{r_i}) & \overset{\pi_*f}{\rightarrow} & \pi_*(X) \\
\downarrow h & & \downarrow h' \\
\pi_*(\vee S^{r_i}) & \overset{H_*f}{\rightarrow} & H_*(X) \\
\end{array}
\]

As $h$ is a $C$-iso to dimension $2k - 2$, we see that $\pi_*f$ is a $C$-iso to dimension $2k - 2$, and a $C$-epi in dimension $2k - 1$. By Whitehead’s theorem mod $\mathcal{C}$, $H_*f$ is a $C$-iso to dimension $2k - 2$. Thus $h'$ must be a $C$-iso to dimension $2k - 2$, and $X \in \mathcal{K}$. \[\square\]

---

10 Suppose that $(X, A)$ is an $(n - 1)$-connected pair of path-connected spaces with $n \geq 2$ and $A \neq \emptyset$. Then $H_i(X, A) = 0$ for $i < n$ and $h : \pi_n(X, A) \rightarrow H_n(X, A)$ is an isomorphism.

11Whitehead’s theorem mod $\mathcal{C}$: Suppose that $f : A \rightarrow X$ is a map of simply connected spaces that induces an isomorphism on $\pi_2$. Then TFAE:

1. $f_* : H_i(X) \rightarrow H_i(Y)$ is $\mathcal{C}$-iso for $i < n$ and $\mathcal{C}$-epi for $i = n$; and

2. $f_# : \pi_i(X) \rightarrow \pi_i(Y)$ is $\mathcal{C}$-iso for $i < n$ and $\mathcal{C}$-epi for $i = n$.

16
An upper bound on the rank of $\Omega_n$

Fix $n$. Seek to bound the rank of $\Omega_n$.

$\Rightarrow$ Choose $k, p \geq n + 2$.

$\Rightarrow$ Let $\tilde{\gamma}$ be the universal oriented $k$-bundle over $\tilde{G} = B(SO(k)) = G_k(\mathbb{R}^\infty)$.

$\Rightarrow$ Let $\tilde{\gamma}'$ be the tautologous $k$-bundle over $\tilde{G}' = \tilde{G}_k(\mathbb{R}^{k+p})$.

$\Rightarrow$ $\tilde{G}'$ is a subcomplex of $\tilde{G}$ containing the $(n+1)$-skeleton.

$\Rightarrow$ We thus have maps:

$$\Omega_n \hookrightarrow \pi_{n+k}(T(\tilde{\gamma}')) \xrightarrow{C} H_{n+k}(T(\tilde{\gamma}')) \cong H_n(\tilde{G}') \cong H_n(\tilde{G})$$

$\Rightarrow$ By the universal coefficient theorem:

$$0 \to \text{Ext}(H_{n-1}(\tilde{G}), \mathbb{Z}[1/2]) \to H^n(\tilde{G}; \mathbb{Z}[1/2]) \to \text{Hom}(H_n(\tilde{G}), \mathbb{Z}[1/2]) \to 0.$$ 

Now the Ext group is finite as $H_{n-1}(\tilde{G})$ is finitely generated, so:

$$\text{rank}_\mathbb{Z}[1/2] H_n(\tilde{G}) = \text{rank}_\mathbb{Z}[1/2] \left[ \text{Hom}(H_n(\tilde{G}), \mathbb{Z}[1/2]) \right] \quad \text{(as } \mathbb{Z}[1/2] \text{ is torsion free)}$$

$$= \text{rank}_\mathbb{Z}[1/2] \left[ H^n(\tilde{G}, \mathbb{Z}[1/2]) \right] \quad \text{(as the Ext group is finite)}$$

$$= p(n/4).$$

So, for all $n$, $\Omega_n$ is finitely generated, with rank at most $p(n/4)$. We conclude:

**Theorem.** $\Omega_* \otimes \mathbb{Q}$ is polynomial with generators the even complex projective spaces.

Multiplicative sequences

$\Rightarrow$ Let $x_1, x_2, \ldots$ be a sequence of indeterminates with $|x_i| = i$.

$\Rightarrow$ Consider a sequence $K_*$ of polynomials (with $\mathbb{Q}$ coefficients):

$$K_1(x_1), K_2(x_1, x_2), \ldots$$

such that $|K_i| = i$.

$\Rightarrow$ Suppose that $A^*$ is a unital graded $\mathbb{Q}$-algebra. Write $A^\times$ for the group (under multiplication) of formal sequences $1 + a_1 + a_2 + \cdots$, with leading term 1.

$\Rightarrow$ Then $K_*$ defines a function $K : A^\times \to A^\times$ which is ‘polynomial’:

$$a = 1 + a_1 + a_2 + \cdots \xrightarrow{K} 1 + K_1(a_1) + K_2(a_1, a_2) + \cdots$$

$\Rightarrow$ $K_*$ is called a multiplicative sequence if $K : A^\times \to A^\times$ is a group automorphism for all unital graded $\mathbb{Q}$-algebras $A^*$.

---

\[12\] For the number of $n$-cells in $G_k(\mathbb{R}^{k+p})$ equals the number of partitions of $n$ into at most $k$ integers, each of which is at most $p$.

\[13\] In fact, $\text{Ext}(\mathbb{Z}/(2^r(2s+1))\mathbb{Z}, \mathbb{Z}[1/2]) \cong \mathbb{Z}/(2s+1)\mathbb{Z}$, and of course $\text{Ext}(\mathbb{Z}, \mathbb{Z}[1/2]) = 0$. 

17
The $K$-genus

We can obtain a ring homomorphism $K : \Omega_* \rightarrow \mathbb{Q}$ as follows.

$\leadsto$ Choose $M^m$ a representative of a cobordism class.

$\leadsto$ If $m$ is not divisible by 4, set $K(M^m) = 0$.

$\leadsto$ If $m = 4n$, write $K[M] = \langle K_n(p_1, \ldots, p_n), M^{4n} \rangle$.

That this function is additive is obvious. That it vanishes on oriented boundaries follows since Pontrjagin numbers do. That it is multiplicative follows from the Cartan formula which holds (mod order two elements), as $K$ is multiplicative.

Classification of multiplicative sequences

Lemma (Hirzebruch).

$\leadsto$ Multiplicative sequences are in bijective correspondence with elements of $\mathbb{Q}[[t]]$.

$\leadsto$ To obtain a power series from its multiplicative sequence, one considers the $\mathbb{Q}$-algebra $\mathbb{Q}[t]$, and evaluates $K(1 + t)$.

$\leadsto$ If $K$ belongs to $f(t)$, and $a \in A^1$, we always have $K(1 + a_1) = f(a_1)$.

Note that in our application, $A^1$ will be $H^4(M)$, where $M$ is some $4k$-manifold.

The signature theorem

Theorem (Hirzebruch signature theorem). Let $L_*$ be the multiplicative sequence corresponding to the power series

$$\sqrt{t}/\tanh \sqrt{t} = \sum_{k \geq 0} \frac{(-1)^{k-1}2^k B_k}{(2k)!} t^k,$$

so that, for example,

$$L_4 = \frac{1}{14175} \left(381\sigma_4 - 71\sigma_3\sigma_1 - 19\sigma_2^2 + 22\sigma_2\sigma_1^2 - 3\sigma_1^4\right).$$

Then the signature $\sigma(M^{4k})$ equals the $L$-genus $L[M^{4k}]$.

(For example, for a smooth compact oriented 16-manifold, its signature is the above $\mathbb{Q}$-linear combination of Pontrjagin numbers.)

Proof.

$\leadsto$ It’s enough to check the generators $\mathbb{CP}^{2k}$ of $\Omega \otimes \mathbb{Q}$, whose signature is obviously one (after getting the right orientation convention for $\mathbb{CP}^{2k}$).

$\leadsto$ Let $p = p(\tau) = (1 + a^2)^{2k+1}$ be the total Pontrjagin class of $\mathbb{CP}^{2k}$.

$\leadsto$ $L(p) = L(1 + a^2)^{2k+1} = (a/ \tanh a)^{2k+1}$.

$\leadsto$ Thus we need only check that the coefficient of $a^{2k}$ in $(a/ \tanh a)^{2k+1}$ is one — an exercise in the theory of residues.

\[14\text{We have } p(M \times M) \equiv p(M) \times p(M') \text{ (mod elements of order 2), and so } K(p \times p') = K(p) \times K(p') \text{ (operating in the algebra } H^*(M \times M')) \text{. Then } K[M \times M'] = \langle K(p \times p'), \mu \times \mu' \rangle = (-1)^{m'm'} \langle K(p), \mu \rangle \langle K(p'), \mu' \rangle = K[M]K[M']\]
Michael Andrews’ Talk on Kervaire’s paper

The plan:

1. Construct $\Phi$ a (homotopy type) invariant of closed 4-connected 10-manifolds.
2. If $M$ is smooth, then $\Phi(M) = 0$.
3. Construct $M_0$ such that $\Phi(M_0) = 1$.

Question. Is this the standard Kervaire invariant?

The normal Kervaire invariant is defined for $(4k - 2)$-dim framed manifolds. Define a quadratic refinement $q : H^{2k+1}(M; \mathbb{Z}) \to \mathbb{Z}_2$ of the cup product pairing (i.e. a function satisfying $q(x + y) = q(x) + q(y) + (x \smile y)[M]$). Define $\Phi = \text{Arf}(q)$.

To define $q$, take $M^n \subset \mathbb{R}^{n+N}$ giving $S^{n+N} \to \text{Th}(\nu) = \Sigma M_+$

$$
(q : M_+ \to K) \mapsto (S^{n+N} \to \Sigma M_+ \to \Sigma K)
$$

this gives an element of $\pi_n^s(K) = \mathbb{Z}_2$. ($K = K(\mathbb{Z}_2, 2k+1)$)

Defining $\Phi$

Define $\varphi_0 : H^5(M) \to \mathbb{Z}_2$ by letting $e_1, e_2$ be the generators of $H^5(\Omega S^6), H^{10}(\Omega S^6)$, and $u_2$ be the generator $H^{10}(\Omega S^6; \mathbb{Z}_2)$. For $X \in H^5(M)$:

1. Find $f_X : M \to \Omega$ such that $f_X^*(e_1) = X$.
2. Define $\varphi_0(M) := f_X^*(u_2)[M]$.

Can show that $\varphi_0$ descends (nothing tricky here, by Poincaré duality...) to $\varphi : H^5(M; \mathbb{Z}_2) \to \mathbb{Z}_2$.

Let $\Phi$ be $\text{Arf}(\varphi)$, i.e. $\sum \varphi(x_i)\varphi(y_i)$, where the $x_i, y_i$ are a symplectic basis.

Constructing $M_0$

Start with $S^5$, and let $\tau$ be its tangent bundle. Pull back in two ways, with the factors $D^5$ playing opposite roles:

$$
\begin{array}{ccc}
D^5 \times D^5 & \longrightarrow & D(\tau)_1 \\
\pi_1 & & \pi_2 \\
D^5 & \text{top hemi} & D^5 \\
\longrightarrow & S^5 & \text{top hemi} \\
& \longrightarrow & S^5 \\
\end{array}
$$

Push out:

$$
\begin{array}{ccc}
D^5 \times D^5 & \longrightarrow & D(\tau)_1 \\
& & \downarrow \\
& & D(\tau)_2 \\
& & \longrightarrow W \\
\end{array}
$$

15Simply connected implies oriented!
16Arf invariant returns which value is taken on more often by $q$. Michael Donovan
That is, $W$ is obtained by plumbing two copies of the disc bundle. Need to do some smoothing of the boundary $\partial W$, (unless you’re Milnor, who’s really smart somehow).

By Morse theory, $\partial W = S^9$ (Milnor writes down a Morse function with only two critical points, and applies a theorem of Milnor)\footnote{To construct it, you note that the tangent bundle of $S^5$ has a non-vanishing section, so we can reduce the structure group to $SO(4)$. This comes in handy somehow.} Let $M_0$ be the cofibre of $S^9 \rightarrow W$. Our proof will then also show that $\partial W$ must be exotic.

**Showing that $\Phi(M_0) = 1$:**

Now to $\Phi(M_0) = 1$. The 0-sections of the two disk bundles meet transversely at a point. We have $S^5 \vee S^5 \rightarrow W$, which is a homotopy equivalence. So adding a 10-cell, $M_0$ has a CW-structure with one 0-cell, two 5-cells and one 10-cell (which shows that it’s 4-connected). Haynes: “so it looks a lot like $S^5 \times S^5$, a whole lot! I think then you’re plumbing by *not* swapping the roles of the disks.”

The copies of $S^5$ give generators of $H_5(M_0) = \mathbb{Z} \oplus \mathbb{Z}$. Let $X$ and $Y$ be the Poincaré dual generators. Let $x$ and $y$ be their reductions mod 2, and let $K = \text{Th}(\tau)$. Now $x \cdot y = 1$ (using the intersection pairing), so that $\{x, y\}$ is a symplectic basis. We need then to show that $\varphi(x)\varphi(y) = 1$.

Let’s take $f_X : M_0 \rightarrow K$ which collapses everything outside $D(\tau)_X$ (the copy of $D(\tau)$ corresponding to $X$) to the basepoint and takes $D(\tau)_X \rightarrow K = D(\tau)/S(\tau)$. This $D(\tau)_X$ is a tubular neighbourhood of the $S^5$ corresponding to $X$, so that $f_X^*(t) = X$, where $t$ is the Thom class. By a local degree argument, $f_X^*$ is an iso on $H^{10}$. We do the same to construct $f_Y$.

If $K$ were equal to $\Omega S^6$, we’d be done. But it’s not! Now $K$ has the form $S^5 \cup_f e^{10}$. Here, $f = [x_5, x_5]$, the Whitehead square. Now $\Omega S^6$ has a cell structure (Michael likes to think about this using Morse theory) with cells of dimension $5k$ for $k \in \mathbb{N}$, so that the 10-skeleton is $S^5 \cup e^{10}$ too, and it turns out that you also get the Whitehead square. Thus $K \subset \Omega S^6$ is the 14-skeleton, and you’re a winner.

By the way, we’re using $\Omega S^6$ because it’s an H-space, which will be useful when one wants to prove the quadratic refinement formula. We’ll form $f_{X+Y}$ as the composite:

$$M \xrightarrow{M} M \times M \xrightarrow{f_X \times f_Y} \Omega \times \Omega \xrightarrow{\Omega} \Omega.$$  

**Is $\Phi$ well defined?**

We’ve gotta show that $f_X$ does exist. We do obstruction theory:

If $(X, A)$ is a relative CW-complex, for each $(n + 1)$-cell $\sigma$, let $g_{\sigma} : S^n \rightarrow X$ be the attaching map. Given $f : X^n \rightarrow Y$, one would like to extend this to the $(n + 1)$-skeleton. To make the following make sense, one needs $\pi_1(Y)$’s action on $\pi_n(Y)$ to be trivial, in order to disregard basepoints. Let’s also have $Y$ connected.

All we really want is for the compositions of $f$ with the $g_{\sigma}$ to be null. We can define:

$$c(f) \in C^{n+1}_{CW}(X, A; \pi_n(Y)) \text{ given by } c(f)(\sigma) = [f \circ g_{\sigma}]$$

This last thing is supposed to be an element of $\pi_n(Y)$, but the basepoints are mucked up. Who cares — $Y$ is connected and $\pi_1$ acts trivially.

**Theorem.** \(\Rightarrow\) $c(f) = 0 \iff f$ can be extended to $X^{n+1}$.

\(\Rightarrow\) $c(f)$ is a cocycle, and $[c(f)] = 0$ iff $f|_{X^{n-1}}$ can be extended to $X^{n+1}$.\footnotemark
There’s a bit left to do for well-defined-ness. We’ll talk about why it descends to mod 2 cohomology. How come? Well, the relation

$$\varphi_0(X + Y) = \varphi_0(X) + \varphi_0(Y) + x \cdot y$$

is proven by considering a composite

$$M \xrightarrow{f} M \times M \xrightarrow{f_X \times f_Y} \Omega \times \Omega \xrightarrow{\Omega} \Omega$$

and exploiting our knowledge of the Pontrjagin coproduct on the cohomology of $\Omega$.

**$M$ smooth implies $\Phi(M) = 0$**

Just use Milnor’s paper repeatedly. Milnor has a tool for killing homotopy groups of smooth manifolds. Suppose we’re given an $N$-manifold $M^N$ and $[f] \in \pi_n(M)$. Have $S^n \to M$, change it to get an embedding $S^n \times D^{N-n} \hookrightarrow M$ (by the tubular neighbourhood theorem, to do this is to embed $S^n$ in $M$ with trivial normal bundle). Then apply surgery, to get

$$(M - (S^n \times D^{N-n})) \cup (D^{n+1} \times S^{N-n-1})$$

Two problems, you create a new hole, so can only do half the dimensions, i.e. you want $n < N/2$ or so. Also, you need a $\pi$-manifold, in order to choose the embedding (still only get certain dimensions — if you can embed $M$ into a Euclidean space, with trivial normal bundle, then you can embed $S$ inside with trivial normal bundle how?).

Using differential topology, we can make $S^n \to M$ an embedding. Moreover, we can embed $M \subset \mathbb{R}^N$ with trivial normal bundle, by as $M$ is a $\pi$-manifold. Then the normal bundle of $S^n$ in $\mathbb{R}^N$ is trivial, as the embedding of $S^n$ in $\mathbb{R}^N$ is isotopic to the standard embedding ($N$ being large). Thus the normal bundle of $S^n$ in $M$ is stably trivial. However, by the proof of the theorem soon to follow on $\pi$-manifolds, as the dimension of $S^n$ is less than the dimension of its normal bundle in $M$, its normal bundle must be trivial.

The killing procedure preserves $\pi$-manifold-ness. Milnor does all this very well.

**Definition.** A $\pi$-manifold is one whose tangent bundle $\tau$ has $\tau \oplus \epsilon$ trivial.

This obviously implies that $\tau$ is stably trivial. It also shows that $\nu_M$ is stably trivial. In fact all these three are equivalent.

**Theorem.** If $M$ is a smooth $n$-manifold with tangent bundle $\tau$, TFAE:

(a) $M$ is a $\pi$-manifold.
(b) $\tau$ is stably trivial.
(c) $\tau \oplus \epsilon$ is trivial.
(d) The stable normal bundle of $M$ is trivial.

Moreover, if $M$ is connected with non-empty connected boundary, these are all equivalent to:
(e) $\tau$ is trivial.
**Proof.** Consider the following diagram, in which the ‘Γ’ shapes are all fibrations. Let \( M \to BO(n) \) classify the tangent bundle.

\[
\begin{array}{c}
M \to BO(n) \leftarrow S^n \\
\downarrow \\
BO(n + 1) \leftarrow S^{n+1} \\
\downarrow \\
BO(N - 1) \leftarrow S^{N-1} \\
\downarrow \\
BO(N) \\
\end{array}
\]

We only need to prove that (b) implies (c) (and that (b) implies (e) under the extra hypothesis). Assuming (b), we see that the composite \( M \to BO(N) \) is trivial. Thus, it factors through \( S^{N-1} \to BO(N) \). However, by connectivity, the map to \( S^{N-1} \) must be null, so that in fact, \( M \to BO(N - 1) \) is null. This argument works all the way up the ladder, to show that \( M \to BO(n + 1) \) is null, giving (c). If \( M \) has a nonempty connected boundary, it is homotopy equivalent to a CW-complex of lower dimension (using a handlebody decomposition argument), so that we can take this argument one step further. Identify the spectral sequence here... 

**There are three big things left to do:**

1. A smooth 4-connected 10-manifold is \( \pi \) (obstruction theory, involving the injectivity of the \( J \) homomorphism on \( \pi_9 \)). This is the only place we use smoothness, in order to apply Milnor’s procedure.

2. Given \( (M^{10}, f_n) \), get element \( \alpha(M, f_n) \in \pi_{n+10}(S^n) \), and if this element is zero, the the Kervaire invariant \( \Phi(M) \) is zero.

**Proof.** The first thing you do is note that Thom shows that there’s a \( (V^{11}, F) \) framed with \( \partial V = M \). Can assume \( V \) is connected (as \( M \) is). Then the tangent bundle of \( V \) is trivial, in particular, it’s a \( \pi \)-manifold.

By applying Milnor’s killing process, can assume \( V \) is 4-connected. Then Poincaré duality shows that \( V \) only has nonzero cohomology in dimensions 0, 5, 6, 11. We have the following beautiful diagram from Poincaré duality, with \( \mathbb{Z} \) coefficients:

\[
\begin{array}{cccccccc}
H^5(V) & \xrightarrow{i^*} & H^5(M) & \xrightarrow{\delta} & H^6(V, M) \\
\sim & & \sim & & \sim \\
H_6(V, M) & \xrightarrow{\partial} & H_5(M) & \xrightarrow{i_*} & H_5(V)
\end{array}
\]

(There’s a sign in this diagram with \( \mathbb{Z} \) coefficients, depending on your sign conventions!) Now

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we have the following equalities:

\[
\dim \ker(\delta) = \dim \text{im}(i^*) \quad \text{(exactness)}
\]

\[
= \dim \text{im}(i_*) \quad \text{(as } i_* \text{ and } i^* \text{ are dual)}
\]

\[
= \dim \text{im}(\delta) \quad \text{(by commutativity)}
\]

\[
= \dim H^5(M) - \dim \ker(\delta) \quad \text{(rank-nullity)}
\]

Thus, \( \ker(\delta) \) is a subspace of \( H^5(M) \), of half the dimension. Moreover, suppose that \( x, y \in \ker(\delta) = \text{im}(i^*) \). Then \( x \cup y \in \text{im}(i^*) \), and as \( H^{10}(V) = 0 \) (by Poincaré duality), this shows that \( x \cup y = 0 \). Thus, \( \ker(\delta) \) is a “Lagrangian subspace”, and so can be extended to a symplectic basis. In particular, it’ll be enough to show that \( \varphi(x) = 0 \) for all \( x \in \ker(\delta) \).

Haynes: “This is an important general idea: that if \( M \) is the boundary of \( V \), the the kernel of \( \delta \) gives a Lagrangian subspace.”

So, choose \( x \in \ker(\delta) \), and lift it to an integral class \( X \in H^5(M) \). Choose \( f_X : M \to \Omega \) such that \( f_X^*(e_1) = X \). We need only show that \( f_X^*(u_2) = 0 \).

Now if we could form a diagram

\[
\begin{array}{ccc}
M & \to & \Omega \\
\downarrow & & \downarrow \\
V & \to & \Omega^*
\end{array}
\]

such that \( H^{10}(\Omega; \mathbb{Z}_2) \to H^{10}(\Omega^*; \mathbb{Z}_2) \) is an isomorphism, we’d be happy, since \( H^{10}(V; \mathbb{Z}_2) = 0 \). (Note that it’s too hard to do this when \( \Omega^* = \Omega \).) If we had a choice of \( \Omega^* \), we still need to fill in the map \( V \to \Omega^* \), and we approach this problem with obstruction theory. We have a map \( M \to \Omega^* \) which we need to extend to all of \( V \). Note that \( (V, M) \) can be triangulated into a relative CW-complex. Now \( \Omega \) has a cell structure with one cell in each dimension a multiple of 5. We’ll form \( \Omega^* \) by attaching a 6-cell a map of degree 2 to the 5-cell. The map \( \Omega \to \Omega^* \) is obviously an isomorphism on \( H^{10} \). Somehow, he manages to pin down all the obstructions up to the one in dimension 10, and then brings in a lemma to show that the map can be extended to the (unique) 11-cell of \( V \) (This 11-cell can be unique by the dualisability of handlebody decompositions — Morse theory).

3. The lemma taking two pages to prove, with \( \beta \circ \alpha \) in the statement.
Markus’s talk on \( K \)-theory

\( K \)-theory

Let \( X \) be a compact topological space. Define \( \text{Vect}(X) \) to be the set of isomorphism classes of finite dimensional complex vector bundles on \( X \). \( \text{Vect}(X) \) is a commutative monoid under Whitney sum.

Define \( K(X) := \text{GrthGp}(\text{Vect}(X)) \). For \( X \) pointed, we define \( \widetilde{K}(X) := \ker \{ K(X) \to K(*) \} \). This map is actually split by \( * \to X \to * \), so that \( K(X) \cong K(*) \oplus \widetilde{K}(X) \). (You also thus have \( \widetilde{K}(X) = \text{coker}\{ K(*) \to K(X) \} \).)

Assume \( X \) connected. Then (the colimit is via stabilisation by adding trivial bundles)

\[
\widetilde{K}(X) \cong \text{colim}_n \text{Vect}_n(X) \cong \text{colim} [X, BU(n)] \cong [X, BU] \cong [X, BU]_*.
\]

Here the second last isomorphism holds as \( X \) is compact, and the last isomorphism holds since \( BU \) is simply connected. So for compact connected \( X \), it’s representable. For \( X \) not necessarily connected, but still compact, we get that \( \widetilde{K}(X) \cong [X, \mathbb{Z} \times BU]_* \) and \( K(X) \cong [X, \mathbb{Z} \times BU] \).

We now define \( \widetilde{K}(X) := [X, \mathbb{Z} \times BU]_* \), and \( K(X) := [X, \mathbb{Z} \times BU] \) for arbitrary \( X \). Note that for general \( X \), the vector bundle construction is not the same as this.

**Example.** The tautologous bundle on \( \mathbb{CP}^\infty \) doesn’t have an inverse.

[Example: the thing represented by the identity map of \( BU \).]

**Multiplication**

If \( X, Y \) are compact, then the tensor product induces pairings \( K(X) \otimes K(Y) \to K(X \times Y) \), and \( \widetilde{K}(X) \otimes \widetilde{K}(Y) \to \widetilde{K}(X \wedge Y) \). If \( X = Y \) we can pull this back to an internal product.

We’ll get a multiplication on \( \mathbb{Z} \times BU \) (more twisted than you might think). Suppose \( X \) is compact and connected and have \( X \to BU \), can think of this as a map \( X \to n \times BU \) for any \( n \). This factors via \( X \to BU(n) \), giving a bundle \( V \) of dimension \( n \). If \( w \) of dim \( m \), ummm should pull back to 0-dim stuff.

\[
([V] - [e^n])([W] - [e^m]) = [V \otimes W \oplus e^{mn}] - [W^n \oplus V^m].
\]

So get \( m \times BU \times n \times BU \to mn \times BU \) which doesn’t agree with \( BU \times BU \to BU \) under the obvious identification.

[Currently unsure as to how to extend this to arbitrary \( X \) and \( Y \).]

Now we have a representable functor, and a representing object gives half a cohomology theory. We just need to make it into an infinite loop space. In fact, it already is:

\[
\Omega U \simeq \mathbb{Z} \times BU, \text{ so that } \Omega^2(\mathbb{Z} \times BU) \cong \mathbb{Z} \times BU.
\]

This implies that \( \widetilde{K}(X) \cong \widetilde{K}(S^2 \wedge X) \), where the isomorphism is given by smashing with the Bott class \([H] - 1 \in \widetilde{K}(S^2)\), where \( H \) is the taut bundle on \( \mathbb{CP}^1 \). Moreover,\(^{18}\) \( K(S^2) = \mathbb{Z}[H]/(H - 1)^2 \).

It follows that \( \widetilde{K} \) is the 0-th part of a 2-periodic reduced cohomology theory: \( \widetilde{K}^{-1}(X) := [X, U]_* \).

\(^{18}\)To see why \((H - 1)^2 = 0\), one could note that it is a statement about 2-dimensional vector bundles on \( S^2 \), and there are only trivial ones. Alternatively, note that this is a reduced class, and any product of two reduced classes on a suspension is zero.

24 Michael Donovan
The difficulty with forming operations on $K$-theory

**Problem:** The representing object $Z \times BU$ is not compact. This makes it harder to construct operations on $K$-theory — we don’t actually have vector bundles at the representing object.

In order to circumvent this, we’ll show\(^{[19]}\) that

**Theorem.** $K(Z \times BU) = \varprojlim K(G_n)$, where $G_n := \{-n, \ldots, n\} \times \text{Gr}_n(\mathbb{C}^{2n})$.

**Proof.** We use the fact that $Z \times BU = \text{colim} G_n$. We’ll show that the $K$-theory of $Z \times BU$ is the inverse limit of that of the $G_n$. There’s a Milnor exact sequence:

$$0 \rightarrow \varprojlim \lambda^i(G_n) \rightarrow K^0(Z \times BU) \rightarrow \varprojlim K^0(G_n) \rightarrow 0$$

We do this using the AHSS, $H^*(G_n, K^{d(*)}) \rightarrow K^{s+t}(G_n)$, as this collapses, due to a checker-board pattern\(^{[20]}\), and $G_n$ has no odd $k$-theory. Thus the $\varprojlim^1$ vanishes. \(\square\)

Thus, given a transformation defined on the $K$-theory of compact spaces, we can extend to a transformation on all $K$-theory, by taking the universal class in $K^0(Z \times BU)$, representing it by a sequence in the above inverse limit, and applying the transformation to the terms individually.

Similarly, $K(Z \times BU \times Z \times BU) \cong \varprojlim K(G_n \times G_n)$, so can extend pairings on compact spaces to pairings overall. I.e. we can use this to construct a multiplication on $Z \times BU$ which classifies a multiplication of the $K$-theory of compact spaces.

**Splitting principle**

If $X$ is a space and $V$ a vector bundle on $X$, then there is a space $Y$ and a map $f : Y \rightarrow X$ such that $f^* V$ is a sum of line bundles, and $f^* : K(X) \rightarrow K(Y)$ is injective. [The standard construction using the projectivisation, splitting off one bundle at a time, works in $K$-theory, too.]

**$K$-theory operations**

For $X$ compact, $V$ a vector bundle over $X$, let $\lambda^n(V)$ be the $n^{th}$ exterior power of $V$. This is *not* additive. Let $\lambda(V)$ be the formal sum $\sum_{n=0}^{\infty} \lambda^n(V)t^n \in K(X)[[t]]$. This lands in the multiplicative group of elements with leading coefficient 1. This $\lambda$ is a monoid homomorphism from $\text{Vect}(X)$ to this multiplicative group, which thus extends to $K(X)$. This follows from the equation $\lambda^n(V \oplus W) = \bigoplus_{i+j=n} \lambda^iV \otimes \lambda^jW$.

Now let $\lambda^i$ be the $i^{th}$ component of $\lambda$, thus $\lambda^i$ is a natural transformation $K(X) \rightarrow K(X)$. These don’t have great properties [components are neither additive nor multiplicative].

**Adams operations**

We define, for $x \in K(Y)$, $\psi(x) = \text{rank}(x) - t \frac{d}{dt} \log \lambda_{-t}(x) \in K(X)[[t]]$, expanding the log via $\log(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(t-1)^n$. Call this the total Adams operation. Let $\psi^i(x)$ be the $i^{th}$ component. The Adams operations have the following properties:

1. The $\psi^i$ are additive.

---

\(^{[19]}\)One can use the AHSS directly to show that $K^{-1}(Z \times BU) = 0$.

\(^{[20]}\)Note that $K^{-1}(*) = [*, U] = 0$, and $G_n$ only has cells in even dimensions.
2. \( \psi^i(\mathcal{L}) = \mathcal{L}^{\otimes i} \) for line bundles \( \mathcal{L} \).
3. The \( \psi^i \) are multiplicative (in particular, they are ring homomorphisms).
4. \( \psi^i \circ \psi^j = \psi^{ij} \).
5. \( \psi^i(u) = i^n u \) for all \( u \in \tilde{K}(S^{2n}) \) (in particular, they are unstable).

**Proof.** By the splitting principle, the first two properties characterise.

1. This holds by construction.
2. We compute. Let \( V \) be a line bundle. Then \( \lambda([V]) = 1 + Vt \), so
   \[
   \log \lambda_{-t}[V] = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (-Vt)^n,
   \]
   and we get \( \psi([V]) = \sum [V]^n t^n \).
3. We can check this on line bundles by 2. But then it also holds for sums of these, so we can apply the splitting principle.
4. This is true on line bundles, and both sides are additive, so they must agree everywhere, as the first two properties characterise.
5. Starting with \( n = 1 \), we know that \( x = [H] - 1 \) is a generator of \( \tilde{K}(S^2) \), and \( \psi^i(x) = \psi^i((x + 1) - 1) = (x + 1)^i - 1 = ix \). In general, to increase \( n \), we have \( \psi^i \) multiplicative in terms of the inner operation, but this implies that it’s multiplicative in terms of the smash operation. \( \square \)
Guozhen’s Talk on “Equivariant $K$-theory and completion”

Our main theorem will be that $K(BG) = \hat{R}(G)$, for a compact Lie group $G$.

**Definition.** Let $G$ be a compact Lie group, and $X$ a compact $G$-space. Define the monoid of $G$-vector bundles over $X$ to be the set of (iso classes) of vector bundles on $X$ so that $G$ acts on the total space. Taking the group completion under Whitney sum, we obtain $K_G(X)$. The tensor product makes this a ring. We can define compactly supported $K$-theory, denoted $K_G^c(X)$, to be the $K$-theory of the one point compactification of $X$, and define relative equivariant $K$-theory as usual.

1. $K_G(* \equiv R(G)$.
2. If $G$ acts trivially on $X$, we have $K_G(X) = K(X) \otimes R(G)$ (by Schur’s lemma).
3. If $G$ acts freely on $X$, $K_G(X) = K(X/G)$. More generally, if $N$ is a normal subgroup of $G$, and $N$ acts freely on $X$, $K_G(X) = K_{G/N}(X/N)$.
4. Given $H \rightarrow G$, get a map $K_G \rightarrow K_H$.
5. If $H \subset G$, then $K_H(X) \simeq K_G(X \times_H G)$, so that we can always recover the $K$-theory of a smaller group by studying a larger group.

Let $V$ be an $n$-dimensional $G$-vector space, and $X$ a compact $G$-space. Define a class $\lambda[V] \in K_G^0(V)$ (compactly supported — i.e. $K$-theory of a sphere). We construct a complex on $V$:

$$0 \rightarrow \lambda^0(V) \xrightarrow{\land v} \Lambda^1(V) \rightarrow \cdots \rightarrow \lambda^n(V) \rightarrow 0$$

Here, $\Lambda^i(V)$ is the $i$\textsuperscript{th} exterior algebra. This above is a complex on $V$, and $\land v$ is the operation which sends $w \mapsto w \land v$.

This complex is exact when $v \neq 0$, as can be checked on a basis. Thus, this complex defines an element $\lambda(V)$ of $K_G(V, V - 0) = K_G^c(V)$. (For this, we take the alternating sum, $\sum (-1)^i \lambda^i(V)$ as elements of $K$-theory, and note that this is trivial wherever the complex was exact.) Let $\lambda^*(V)$ be the complex conjugate of $\lambda(V)$.

**Theorem** (Bott periodicity). $K_G(X) \rightarrow K_G^c(X \times V)$ is an isomorphism, where $W \mapsto W \otimes \lambda^*(V)$. Moreover, this is a $K_G(X)$-module isomorphism.

Note here that if $X$ is compact, the compactly supported $K$-theory of $X \times V$ is the reduced $K$-theory of the Thom space. This holds by definition of the compactly supported $K$-theory. Note that as in the standard Thom isomorphism, we have $\lambda^*(V \oplus W) = \lambda^*(V) \cdot \lambda^*(W)$, where the dot is the external smash product.

**Corollary** (Thom isomorphism). If $V$ is a $G$-vector bundle over $X$, then $K_G(X) \rightarrow K_G(V)$ is an isomorphism, also sending $w \mapsto w \otimes \lambda^*[V]$.

**Proof.** Broken.

**The main theorem**

If $G$ is compact Lie, $R(G)$ is noetherian, containing the augmentation ideal $I_G$, the kernel of the dimension map $R(G) \rightarrow \mathbb{Z}$. If $X$ is a finite $G$-CW-complex (built out of cells $G/H \times D^n$), $K_G(X)$ is a finite module, as $K_G(G/H) = R(H)$ is a finite $R(G)$-module.
In general, $K_G(X)$ is a finite module over $R(G)$ when $X$ is a finite complex. Let $I_G \subset R(G)$ be the augmentation ideal, the kernel of the dimension map $R(G) \rightarrow \mathbb{Z}$.

We have a map $K_G(X) \rightarrow K_G(X \times EG) = K(X \times_G EG)$ induced by the projection.

**Theorem.** If $X$ is a finite $G$-CW-complex, then $K(X \times_G EG)$ is the $I_G$-adic completion of $K_G(X)$. That is, $K(X \times_G EG) = \lim_{\rightarrow} K_G(X)/I^n_G K_G(X)$.

**Proof.** We have a map $K_G(X) \rightarrow K_G(X \times EG) = K(X \times_G EG)$ induced by the projection. We’ll prove that $K_G(X) \rightarrow K_G(X \times EG)$ is the completion, in the following four steps.

1. When $G = T^1$.

   **Lemma.** Let $G \rightarrow T$ be a homomorphism, let $X$ be a $T$-space, so that $G$ acts on $X$ through this homomorphism. Then there’s an isomorphism $K_G(X \times ET) \simeq K_G(X)_{I_T^1}$.

   **Proof of lemma.** Have $ET = S^\infty = \lim S^{2n-1}_2 \subset C^n$. $T = U(1)$ acts as usual. We’ve got a pair $(D^{2n}, S^{2n-1})$ acted on by $G$. Use the Thom iso, so get

   $$K_G(X \times (D^{2n}, S^{2n-1})) \cong K_G(X) \quad \text{Thom class} \hookrightarrow 1$$

   Get a LES

   $$K_G(X \times pr) \rightarrow K_G(X \times D) \rightarrow K_G(X \times S) \rightarrow K^{-1}_G(X \times pr) \rightarrow K^{-1}_G(X \times D)$$

   Identifying the first and second terms with $K_G(X)$, the map between them is multiplication by the Thom class. Now the Thom class is multiplicative, so it’s the $N$th power of the Thom class of $\mathbb{C}$, which is $1 - \rho$ (as in the traditional case). Now $\lambda^*(C^n) = (\lambda^*(C))^n = (1 - \rho)^n = \xi^n$. Have SES

   $$0 \rightarrow K_G(X)/\xi^n \rightarrow K_G(X \times S^{2n-1}) \rightarrow \xi^n K_G^{-1} \rightarrow 0$$

   where the last term is the kernel of multiplication by $\xi^n$.

   In the special case $\lim K_G(X \times S^{2n-1})$ vanishes and the $\lim K_G(X \times S^{2n-1})$. But the inverse limit of the third terms vanishes. [Get ML because of finiteness]

2. When $G = T^n$, $n > 1$.

   **Lemma.** Let $G \rightarrow T^n$ be a homomorphism, let $X$ be a $T^n$-space, so that $G$ acts on $X$ through this homomorphism. Then there’s an isomorphism $K_G(X \times ET^n) \simeq K_G(X)_{I_{T^n}}$.

   **Proof of lemma.** We do the rest by induction:

   $$K_G(X \times ET^n) = K_G(X \times ET^{n-1} \times ET) = K_G(X \times ET^{n-1})_{I_T} = (K_G(X)_{I_{T^n-1}})_{I_T}$$

   We know that $R(T^n) = \mathbb{Z}[X_1^\pm, \ldots, X_n^\pm]$, so that this is $K_G(X)_{I_{T^n}}$ as desired. \[21\]

21In the paper, we only need $K_G(X)$ to be a finite $R(G)$-module.
3. $G = U(n)$: we have $T^n \subset U(n)$ the diagonal maximal torus, and the following commuting diagram, in which the vertical maps are the restrictions, and the bottom map is known to be the completion, by part 2. We take $EU(n)$ and $ET^n$ to be the same topological space.

$$
\begin{array}{ccc}
K_{U(n)}(X) & \longrightarrow & K_{U(n)}(X \times EU(n)) \\
\downarrow \pi^* & & \downarrow \\
K_{T^n}(X) & \longrightarrow & K_{T^n}(X \times ET^n)
\end{array}
$$

We’ll show that the second row is a direct summand of the first. This will suffice, as the $I_{U^n}$ and the $I_{T^n}$ topologies coincide, as $R(U(n)) \longrightarrow R(T^n)$ is a finite extension — note that $R(U(n))$ is the invariants of the Laurent series ring $R(T^n)$ under the action of the symmetric group.

Now we’ll prove $K_{U(n)}(X)$ is a direct summand of $K_{T^n}(X)$. We also have

$$
K_{U(n)}(X) \longrightarrow K_{U(n)}(X \times T^n \ U(n)) = K_{T^n}(X).
$$

We have a fibration

$$
U(n)/T^n \longrightarrow X \times T^n U(n) \longrightarrow X
$$

Now $U(n)/T^n = \text{GL}(N, \mathbb{C})/H$ with $H$ is the upper triangular matrices — a flag variety, so that $U(n)/T^n$ is a complex manifold, and moreover a rational variety (as a flag variety has a CW-decomposition and the top cell gives a rational map to complex projective space).

Now for some index theory. On a complex manifold, we have the Dolbeault complex:

$$
0 \longrightarrow \Omega^{0,0} \overset{\partial}{\longrightarrow} \Omega^{0,1} \overset{\bar{\partial}}{\longrightarrow} \Omega^{0,2} \overset{\bar{\partial}}{\longrightarrow} \Omega^{0,3} \overset{\bar{\partial}}{\longrightarrow} \ldots
$$

Now for any vector bundle $W$ over $X \times T^n U(n)$, we define

$$
d_*(W) = \text{index}\{W \otimes \text{the Dolbeault complex of the fibre}\}
$$

Take $D = \bar{\partial} + \bar{\partial}^* : W \otimes \Omega^{0,\text{even}} \longrightarrow W \otimes \Omega^{0,\text{odd}}$. We can take the index of $D$:

$$
d(W) = \ker (D) - \text{coker}(D)
$$

Not that these aren’t actually vector bundles, but somehow their difference is, and in fact gives a well defined element of $K$-theory. Thus we obtain $d : K_G(X \times T^n U(n)) \longrightarrow K_G(X)$.

Now $d$ preserves the module structure, so if we prove that $D(\pi^*(1))$, we’ll see that $d \pi^* = \text{id}$. Now $D(\pi^*(1))$ is the index of the Dolbeault complex $\sum (-1)^i H^i(U/T, \mathcal{O})$, but the something numbers are birational invariants, so get $h_i = \delta_{i0}$. This implies that $d(\pi^*(1)) = 1$.

4. The general case: we embed $G \hookrightarrow U(n)$ for some $n$. Then we have $K_G(X) = K_{U(n)}(X \times G U(n))$. Moreover

$$
K(X \times G EG) = K(X \times G EU(n)) = K(X \times G U(n) \times U(n) \ EU(n))
$$

Using again the fact that $R(G)$ is finite over $R(U(n))$ to see that the topologies are right, we conclude that the theorem holds for general $G$. 

$\square$
Comments and Questions after the lecture

If $G = U(n)$, $R(G) = \mathbb{Z}[\lambda_1, \ldots, \lambda_n, \lambda_n^{-1}]$ ($\lambda_n$ is the determinant, which is invertible). Then $K(BU(n)) = \mathbb{Z}[[u_1, \ldots, u_n]]$, and the $\lim_{\leftarrow}$ is zero, so $K(BU)$ is the inverse limit of all these.
Markus’s talk on the Hopf Invariant One problem

Given $f : S^{2n-1}(S^n)$, for $n > 1$, the cone $C(f)$ of $f$ has a CW-structure with one cell in dimensions 0, $n$ and $2n$. Its cohomology ring has a copy of $\mathbb{Z}$ in each of these dimensions. Let $b \in H^{2n}(C(f); \mathbb{Z})$ be the generator corresponding to a fixed orientation of $S^{2n}$. Then if $a$ is any generator of $H^n(C(f); \mathbb{Z})$, we have $a \cup a = h(f) \cdot b$ for some unique $h(f) \in \mathbb{Z}$. This is the Hopf invariant.

This descends to a map $h : \pi_{2n-1}(S^n) \to \mathbb{Z}$, which necessarily zero unless $n$ is even.

**Lemma.** $h$ is a group homomorphism $h : \pi_{2n-1}(S^n) \to \mathbb{Z}$.

**Proof.** All the maps in the diagram on the left are isomorphisms on $H^n$. Their action on $H^{2n}$ is recorded on the right:

\[
\begin{array}{ccc}
C(f + g) & \longrightarrow & C(f \vee g) \\
\downarrow & & \downarrow \\
C(g) & \longrightarrow & \mathbb{Z} \\
\end{array}
\quad
\begin{array}{ccc}
\mathbb{Z} & \longrightarrow & \mathbb{Z} \\
\downarrow & & \downarrow \\
\mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} \\
\end{array}
\]

$\square$

**Proposition.** If $n$ is even, then $2 \in \text{im}(h)$.

**Proposition.** If $S^{n-1}$ is parallelisable, then $\pi_{2n-1}(S^n)$ contains an element of Hopf invariant one.

**Theorem.** If $\pi_{2n-1}(S^n)$ contains an element of Hopf invariant one, then $n = 2, 4, 8$.

(Note we can construct these parallelisations via multiplications from division algebras.)

Our main tool for constructing maps of non-trivial Hopf invariant is the ‘Hopf construction’. Suppose given a map $\varphi : S^{n-1} \times S^{n-1} \to S^{n-1}$. Fixing a basepoint in $S^{n-1}$, we can embed

\[ S^{n-1} \vee S^{n-1} \to S^{n-1} \times S^{n-1} \xrightarrow{\varphi} S^{n-1} \]

Call this composite $\varphi_1 \vee \varphi_2$. Let $\alpha, \beta$ be the degrees of these maps.

We use the Hopf construction to get a map $S^{2n-1} \to S^n$, by inducing a morphism of pushout diagrams:

\[
\begin{array}{ccc}
S^{n-1} \times S^{n-1} & \longrightarrow & S^{n-1} \times CS^{n-1} \\
\downarrow & & \downarrow \\
CS^{n-1} \times S^{n-1} & \longrightarrow & S^{n-1} \to CS^{n-1} \\
\end{array}
\]

This induces a map $f : S^{2n-1} \to S^n$ (note that the first pushout is a boundary decomposition of $D^n \times D^n$). We will not prove the following:

**Proposition.** The Hopf invariant of $f$ is equal to $\pm \alpha \beta$. 

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Proof of first proposition. The clutching function of the tangent bundle of $S^n$ is the adjoint of

$$\bar{\varphi} : S^{n-1} \rightarrow \text{Map}(S^{n-1}, S^{n-1})$$

which sends $x$ to “reflection along the hyperplane orthogonal to $x$”. Let $\varphi : S^{n-1} \times S^{n-1} \rightarrow S^{n-1}$ be the adjoint of this.

Fixing one component, we’re just measuring the degree of a reflection on $S^{n-1}$, which is $-1$ (as $n$ is even).

The other map is “$x \mapsto \bar{\varphi}(x)(\text{bpt})$”. Well, any point has two preimages, and they have the same local degree, as the antipode has degree $(-1)^n = 1$, so we have total degree either $\pm 2$. □

Proof of second proposition. Let $S^{n-1}$ be parallelisable. This means that there is a section $s$ of (the fibre bundle) $O(n) \rightarrow S^{n-1}$(using that the normal bundle of $S^{n-1}$ in $\mathbb{R}^n$ is trivial).

$$\begin{align*}
S^{n-1} \times S^{n-1} &\xrightarrow{\bar{\varphi}} S^{n-1} \\
\quad &\downarrow s \times 1 \\
O(n) \times S^{n-1} &\xrightarrow{\text{act}} S^{n-1}
\end{align*}$$

Then $e_1$ is a right unit of this ‘multiplication’ $\varphi$. Replacing $s$ by $s(e_1)^{-1} s$, we still have a section, and $e_1$ is now an identity for $\varphi$. But now, $\varphi_1, \varphi_2$ are the identity, and so have degree 1, so that the resulting Hopf construction has Hopf invariant one. □

This, taken with the theorem, also shows that the only spheres which are $H$-spaces are $S^1, S^3$ and $S^7$, putting the required bound on the number of real division algebras that can be!

Proof of theorem. Assume $n$ is even, as we may. The Hopf invariant, having been defined in singular cohomology, needs to be translated to $K$-theory. We use the Atiyah-Hirzebruch spectral sequence:

$$\tilde{H}^s(C(f), K^t(*)) \Rightarrow \tilde{K}^{s+t}(C(f)).$$

As $n$ is even, there can be no non-trivial differentials by $E_2$. So

$$\tilde{K}(C(f)) \cong \tilde{H}^{2n}(C(f); \mathbb{Z}) \oplus \tilde{H}^{n}(C(f); \mathbb{Z}).$$

In fact this holds as a ring where we take an associated graded object on the left hand side. However, one notes that the error terms lie in lower gradings, which are zero, so we actually get a ring isomorphism. So, squaring in $K$-theory corresponds to squaring in singular cohomology. That is, if $a, b$ are the relevant generators in $K$-theory, we still have $a^2 = h(f) \cdot b$.

Recall the Adams operations from the previous lecture. One thing that wasn’t mentioned:

**Lemma.** For a prime $p$, $\varphi^p(x) \equiv x^p$ modulo multiples of $p$.

**Proof.** This holds for line bundles with equality. Then use the fact that

$$(x_1 + x_2 + \cdots + x_n)^p \equiv x_1^p + \cdots + x_n^p + p \cdot f(\lambda^1(x), \ldots, \lambda^p(x)),$$

(which follows by considering symmetric polynomials...) and apply the splitting principle. □

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22using the multiplication on $K$-theory, and that the spectral sequence is multiplicative
We want to show that \( a^2 \cong 0 \pmod{2} \) if \( n \neq 2, 4, 8 \). This is equivalent to showing that \( \varphi^2(a) \cong 0 \pmod{2} \). Let \( n = 2m \). Now
\[
\psi^2(a) = 2^m a + \mu b
\]
by the naturality of the AHSS. We also note that
\[
\psi^3(a) = 3^m a + \nu b.
\]
As \( \psi^3 \) and \( \psi^2 \) commute (and \( b \) actually comes from a spherical class), have
\[
3^m (2^m a + \mu b) + \nu 2^{2m} b = 2^n (3^m a + \nu b) + \mu 2^{2m} b
\]
So \( 2^m(2^m - 1)\nu = 3^m(3^m - 1)\mu \). If \( \mu \) is odd then \( 2^m | (3^m - 1) \), but this only happens when \( m = 1, 2, 4 \), by elementary number theory.
Krishanu’s talk on Stable homotopy and generalised homology

A spectrum $E$ is a sequence of pointed spaces $E_n$ with maps $SE_n \to E_{n+1}$. A CW-spectrum is such in which the spaces are CW-complexes and the maps are CW-inclusions.

Let $E^*$ be a generalised cohomology theory. That is, to each CW-pair and integer $n$ an abelian group, contravariant in the pair, with excision $E^n(X, A) = E^n(X - C, A - C)$, connecting maps $E^n(A) \to E^{n+1}(X, A)$ giving a long exact sequence, and a wedge axiom. We’ll define $\widetilde{E}^n(X) = E^n(X, *)$.

Let $E_n$ be the space representing $\widetilde{E}^n(\cdot)$. Then we have

$$E^n(X, *) \cong E^{n+1}(CX, X) \cong E^{n+1}(SX, CX) \cong E^{n+1}(SX, *)$$

Thus we have a natural isomorphism $[X, E_n] \to [SX, E_{n+1}] \cong [X, \Omega E_{n+1}]$, and by Yoneda’s lemma, we have maps $E_n \to \Omega E_{n+1}$, which are homotopy equivalences. This gives an $\Omega$-spectrum, which can be made a CW-spectrum (i.e. with CW-inclusion structure maps) using a telescoping argument.

On the other hand, suppose we have a spectrum $E$, we get a cohomology theory by defining (for finite CW-complexes $X$ only):

$$E^n(X) = \lim\to([X, E_n] \to [SX, E_{n+1}] \to \cdots)$$

We’ll also define

$$\pi_r(E) = \lim\to(\pi_{n+r}(E_n)).$$

Define a function $E \to F$ to be the strictest possible notion of map — levelwise commuting pointed maps. Call $E' \subset E$ dense (or cofinal) if every cell of $E$ is eventually in $E'$. A map is an equivalence class of functions defined on cofinal subspectra.

Define $\text{Cyl}(E) = E \wedge I_+$ (you can always smash with a space on the right, as it doesn’t interfere with the suspensions). A map $\text{Cyl}(E) \to F$ gives a homotopy of maps of spectra. A morphism is a homotopy class of maps. A morphism of degree $n$ is a homotopy equivalence class of maps which lower degree by $n$.

We now define

$$E^n(X) = [X, E]_{-n} \quad \text{and} \quad E_n(X) = [S, E \wedge X]_n.$$ 

Note that homology has a wedge axiom because $S$ is compact! [By the way, this generalises my normal concept of singular homology — try $E$ an Eilenberg-MacLane spectrum and $X$ a sphere to check this. In particular, smashing with an Eilenberg-MacLane spectrum turns homology to stable homotopy.]

I’d rather like a commutative, associative monoidal product, so that given $E \wedge F \to G$, we obtain a natural pairing $E^s(X) \otimes F^s(X) \to G^{n+s}(X)$ for any space $X$. In particular, given $E \wedge E \to E$ we get a product on cohomology. For example, we can use the maps $MO(n) \wedge MO(m) \to MO(n + m)$ to get a map $MO \wedge MO \to MO$.

Universal coefficients theorem

Adams shows that cofibre sequences and fibre sequences coincide, and that smash products preserve cofibre sequences. Given a free resolution $0 \to R \to F \to G \to 0$ of a group $G$, form the cofibre sequence $\bigvee S \to \bigvee S \to MG$, to get the Moore spectrum. Define $EG := E \wedge MG$. 

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Theorem (Universal coefficients). There is a short exact sequence:

\[ 0 \to E_n(F) \otimes G \to (EG)_n(F) \to \Tor^Z(E_{n-1}(F), G) \to 0 \]

Proof. We get a long exact sequence:

\[
\begin{array}{c}
[S, \bigvee E \wedge F]_n \to [S, \bigvee B E \wedge F]_n \to [S, E \wedge F]_n \to [S, \bigvee E \wedge F]_{n-1} \\
\bigvee A \otimes [S, E \wedge F] \to \bigvee B \otimes [S, E \wedge F]
\end{array}
\]

Splitting this into short exact sequences gives the UCT.

Consider now the case \( G = \mathbb{Q} \) and \( F = S \). Then there’s no Tor term, and we get

\[ \pi_*(E) \otimes \mathbb{Q} \xrightarrow{\sim} \pi_n(E \mathbb{Q}). \]

Consider the map \( S \to H \) representing a generator of \( \pi_0(H) = \mathbb{Z} \). This gives a commutative diagram

\[
\begin{array}{c}
\pi_*(S) \otimes \mathbb{Q} \to \pi_n(S \mathbb{Q}) \\
\pi_*(H) \otimes \mathbb{Q} \to \pi_n(H \mathbb{Q})
\end{array}
\]

But \( \pi_n(S) \otimes \mathbb{Q} = 0 \) for \( n \neq 0 \), so the left vertical is an iso. The horizontals are isomorphisms by the UCT, thus the right vertical is an iso.

[Haynes: can always write \( E = \colim \Sigma^{-n}E_n \).

There are maps of spectra which induce zero on every homology theory. Call these ‘phantoms’. This doesn’t happen for cohomology theories.]
We showed that for a compact Lie group $G$, $K(G) = \widehat{R(G)}$, an object which is purely algebraic.

**Question.** Can we describe $H^*(BG, \mathbb{Z}/p\mathbb{Z})$ algebraically?

We’ll assume these coefficients from now on.

**Conjecture** (Atiyah-Swan). The Krull dimension of $H^*(BG)_{red}$ is equal to the maximal rank of an elementary abelian $p$-subgroup of $G$.

Recall that an elementary abelian $p$-subgroup of rank $r$ is a subgroup of the form $(\mathbb{Z}/p\mathbb{Z})^r$.

By analogy with the approach of Atiyah and Segal, instead of studying $H^*(BG)$, we should look at $H^*_G(X) := H^*(EG \times_G X)$ for a $G$-space $X$.

[This is the default definition of an equivariant cohomology thoery given a normal cohomology theory. For $K$-theory, you can do better, and you get a completion on the left...] Then we’re interested in $H^*_G(\ast) = H^*(BG)$, which we abbreviate as $H^*_G$ following Quillen.

Some elementary remarks on equivariant cohomology

Suppose that $u : G \rightarrow G'$ is a homomorphism of compact Lie groups and $f : X \rightarrow X'$ is $u$-equivariant, so that $f(gx) = u(g)f(x)$. Now we can draw a diagram:

\[
\begin{array}{ccc}
(EG \times EG') \times_G X & \xrightarrow{p_2} & EG' \times_G X \\
p_1 & \downarrow{H^*\text{-iso}} & \downarrow{f} \\
EG \times_G X & & EG' \times_{G'} X'
\end{array}
\]

$p_1$ has contractible fibre $EG'$, and so there is a natural map $H^*_G(X') \rightarrow H^*_G(X)$. Suppose moreover that $v : EG \rightarrow EG'$ is $u$-equivariant, we have a map $v \times f$ as follows:

\[
\begin{array}{ccc}
(EG \times EG') \times_G X & \xrightarrow{p_2} & EG' \times_G X \\
p_1 & \downarrow{\text{use } v'} & \downarrow{f} \\
EG \times_G X & - & EG' \times_{G'} X'
\end{array}
\]

In this diagram, we use $v$ to obtain a section of $p_1$, which must also be a cohomology isomorphism. Then the two paths from $EG \times_G X$ to $EG' \times_{G'} X'$ are equal, so that $v \times f^* : H^*_G(X') \rightarrow H^*_G(X)$ is the same map as we described earlier.

**Conjugation invariance**

Fix $g_0 \in G$, and suppose that $u : G \rightarrow G$ is given by $g \mapsto g_0 g g_0^{-1}$. For any $G$-space $X$ we have a $u$-equivariant map $f : X \rightarrow X$ given by $x \mapsto g_0 x$. We can contemplate the induced map $H^*_G(X) \rightarrow H^*_G(X)$. We claim that it is the identity.

To see this, note that we have $v : EG \rightarrow EG$ given by $p \mapsto g_0 p$. Now the map $v \times f : (p, x) \mapsto (g_0 p, g_0 x)$ has $v \times f^*$ equal to the identity, so that so that conjugation has no effect on equivariant cohomology.
The Leray Spectral Sequence

We have a fibre bundle

\[ X \longrightarrow EG \times_G X \]
\[ p \downarrow \]
\[ BG \]

Unfortunately, the Serre spectral sequence isn’t all that useful in general. Instead, we’ll use the map:

\[ EG \times_G X \stackrel{p_2}{\longrightarrow} X/G \]

The fibre at \( y \in X/G \) is \( EG \times_G O \), where \( O \) is the orbit of \( y \), which looks radically different for varying \( y \), so we need something more robust! We’ll use the Leray spectral sequence for sheaf cohomology.

\[ E_2^{st} = H^s(X/G, \mathcal{H}_G^t) \Rightarrow H^{s+t}(X). \]

The sheaf \( \mathcal{H}_G^t \) is the sheafification of

\[ U \mapsto \leftarrow H^t(\pi^{-1}(U)) \]

The stalk at \( y \in X/G \) is just \( H^t_G(O) \). [Which is also \( H^t(BG_x) \), with \( G_x \) the stabiliser of \( x \), a preimage of \( y \).]

Example. If \( G \) acts freely on \( X \), then \( H^*_G(O) = H^*_G(G) = H^i(EG) = 0 \) for \( i \neq 0 \), so that the stalks are zero, and the sheaves are zero. Thus \( H^*(X/G) = H^*_G(X) \).

Nilpotence of elements of positive filtration

We would like a setting in which we can say something interesting about:

\[ E_2^{st} = H^*(X/G, \mathcal{H}_G^t) \Rightarrow H^{*+t}(X). \]

Suppose that \( X \) is a \( G \)-CW-complex with bounded dimension. [Actually, \( X \) anything of bounded topological dimension.] Anything with positive filtration, cup-product it with itself enough and it’ll die. So everything with positive filtration will die!

Now if \( s \in E_2^{0s} \), then \( s^p \in E_2^{0s+1} \), as automatically, \( d_r(s^p) = ps^{p-1}d_r(s) = 0 \). This means that for any \( s \in E_2^{0s} \), \( s^{p^d} \in E_2^{0s} \). This is good, as we’re only interested in the Krull dimension, so we should view any element whose power lives in the image as good enough.

Thus, the edge homomorphism \( \varphi : H^*_G(X) \longrightarrow H^0(X/G, \mathcal{H}_G^t) \) is an \( F \)-isomorphism. That is, \( \ker \varphi \) consists only of nilpotent elements, and all \( s \in H^0(X/G, \mathcal{H}_G^t) \) have some power \( s^{p^d} \in \text{im} \varphi \).

This is good, as \( F \)-isomorphism preserves the set of prime ideals.

Onwards!

Quillen’s result here is that the following composite is an \( F \)-isomorphism:

\[ H^*_G(X) \overset{Fiso}{\longrightarrow} H^0(X/G, \mathcal{H}_G^t) \longrightarrow A^*_G(X) \]
where $\mathcal{A}_G^*(X)$ is the ring of ‘compatible families’ of locally constant functions $f_A : X^A \to H_A^*$ where $A$ runs over all elementary abelian $p$-subgroups of $G$.

The compatibility conditions are as follows. Suppose that $A, A'$ are elementary $p$-subgroups and $g \in G$ are such that $g^{-1}Ag \subset A'$. Then $s_A(ga') = \theta^*s_A(a')$ where $\theta$ sends $k \in A$ to $g^{-1}kg \in A'$, $(\theta^* : H_{A'}^i \to H_A^i)$. These are the conditions on a compatible family.

What is the map $H^0(X/G, \mathcal{H}_G^1) \to \mathcal{A}_G^*(X)$? Choose a section $s \in H^0(X/G, \mathcal{H}_G^1)$. We give $s_A : X^A \to H_A^i$ by mapping $x$ (whose image in $X/G$ is $\pi$) to the image of the value $s(\pi)$ of $s$ in the stalk at $\pi$ under the map $H_G^i(Gx) \to H_A^i$ induced by $A \subset G$ and $\{x\} \hookrightarrow Gx$.

**An example**

$H^*(BO(n), \mathbb{Z}/p\mathbb{Z})$ has Krull dimension $n$. We should compare this to that of $\mathcal{A}_{O(n)}^*(pt)$, which is the set of maps $pt \to H_A^*$. There’s a natural elementary abelian 2-subgroup $S$.

**Claim.** *Every* $A \subset O(n)$ *is subconjugate to* $S$.

**Proof.** Any $a \in A$ has order 2, so that $a^T = a^{-1} = a$, so that $a$ is diagonalisable in $O(n)$. As this is abelian, all the elements of $A$ are simultaneously diagonalisable.

Thus $\mathcal{A}_{O(n)}^* \simeq (H_{(\mathbb{Z}/p\mathbb{Z})^n}^*)_{\Sigma_n}$, which has the correct Krull dimension.

**The proof**

**Proposition.** If $X$ is ‘locally nice’, $H^0(X/G, \mathcal{H}_G^1) \to \mathcal{A}_G^*(X)$ (a map of anti-commutative $\mathbb{Z}/p\mathbb{Z}$-algebras) is a genuine isomorphism if every isotropy group of $X$ is an elementary abelian $p$-subgroup.

**Proof.** Given $S_A : X^A \to H_A^i$, we can take $\pi \in X/G$, and lift it to $x \in X$, and apply $s_{Ga_x}$ to obtain an element of $H_G^i$, which is the stalk of $\mathcal{H}_G^1$. Quillen checks that these glue to give a genuine section, giving an inverse.

We now have a sequence

$$H_G^i(X) \xrightarrow{Fiso} H^0(X/G, \mathcal{H}_G^1) \to \mathcal{A}_G^*(X)$$

and we wish to show that the composite is an $F$-iso.

Now embed $G$ in $U(n)$, and let $F$ denote $U(n)/S$ where $S$ is the group generated by all elements of order dividing $p$. Now the idea is

$$H_G^i(X) \to H_G^i(X \times F) \to H_G^i(X \times F \times F)$$

Now isotropy subgroups of $X \times F$ are automatically elementary $p$-subgroups. So we get $F$-isos in the right two columns (being the composite of an $F$-iso and a genuine composite). Moreover, Quillen checks that the rows are exact, so that the first row is an $F$-iso, and we’re done.