“Simplicial Localisation” — Saul Glasman — 5/10/2011

Abstract

The message of simplicial localisation is that trying to invert morphisms in a category catapults the would-be-inverter into homotopy theory, whether they want to be there or not. This accounts, in part, for the ubiquity of homotopical concepts in modern mathematics. First I'll give a brief and bracing refresher on simplicial sets for those whose heads are not yet simplicial. I'll discuss two perspectives on simplicial localisation, first presenting the useful and picturesque hammock localisation and then teaching you how to take a free resolution of a category. I'll make a few remarks on how great this is and apply it to the theory of model categories. Most of the material on this talk is based on three seminal 1980 papers by Dwyer and Kan.

Group completion (do it levelwise?), classifying spaces, operads, etc. are all better in the category of simplicial sets. We have a Quillen adjunction:

$$|{-}| : \mathsf{sSet} \longleftrightarrow \mathsf{Top} : \mathrm{Sing}$$

These form a Quillen equivalence.

Example. The nerve of a category $\mathcal{C}$ is the simplicial set whose k-simplices are chains of k composable morphisms. Forming the realisation of the nerve of a group (as a category) you get the classifying space.

One drawback: Not everything is fibrant (unlike in Top). This is bad, for example if you'd like to calculate, say, self maps of S^1, if we view S^1 simply as the boundary of the 2-simplex. The fibrant objects in sSet are the Kan complexes, “characterised by the fact that they are enormous”.

Simplicial Localisation

Given a category $\mathcal{C}$ and a class $\mathscr{W}$ of “weak equivalences” in $\mathcal{C}$. $\mathscr{W}$ should have the 2 of 3 property, and maybe some other stuff. We want to localise $\mathcal{C} \longrightarrow \mathcal{C}[\mathscr{W}^{-1}]$ with the obvious universal property.

Let's write a candidate for $\mathcal{C}[\mathscr{W}^{-1}]$. The objects are the same, while the maps are equivalence classes of zigzags (under some relation where we at least allow ourselves to add in pairs of identities):

$$X \longrightarrow A_0 \xleftarrow{\sim} A_1 \longrightarrow A_2 \xleftarrow{\sim} \cdots \longrightarrow A_{2n} \xleftarrow{\sim} A_{2n+1} \longrightarrow Y$$

There are computational problems here. The reason for this difficulty is that we've skipped a step.

Definition. A simplicial category is one in which $\mathrm{Map}(X,Y)$ is a simplicial set, composition is a morphism of simplicial sets. Actually we could just say that a simplicial category is a functor $\Delta^{\mathrm{op}} \longrightarrow \mathsf{Set}$, such that for any $I, J \in \Delta$, $S(I)$ and $S(J)$ have the same objects, and $S(I) \longrightarrow S(J)$ is the identity on objects. The morphisms are free to do as they please.

There's a universal simplicial category $L^{\mathscr{W}}C$ in which $\mathscr{W}$ becomes invertible. I.e. if $\mathcal{D}$ is a simplicial category, and we have a functor sending $\mathscr{W}$ to isomorphisms, we get a unique (up to homotopy) factorisation

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{D} \\ L^{\mathscr{W}}\mathcal{C} \arrow[ur, dotted] & \end{tikzcd}$$

In particular, $\mathcal{D}$ could be $\mathcal{C}[\mathscr{W}^{-1}]$, so we have filled in our missing step.

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \mathcal{C}[\mathscr{W}^{-1}] \\
L^{\mathscr{W}}\mathcal{C} \arrow[ur, dotted] &
\end{tikzcd}$$

Moreover, the map on morphisms given by this factorisation is the projection to π_0 (after we realise the simplicial set of morphisms $L^{\mathscr{W}}\mathcal{C}(X,Y)$).

Constructing the simplicial localisation

Suppose we have $X, Y \in \operatorname{ob}\mathcal{C}$. The 0-simplices in $\operatorname{Mor}(X,Y)$ are just zig-zags (up to an equivalence relation in which we can add in identities):

$$X \longrightarrow A_0 \xleftarrow{\sim} A_1 \longrightarrow A_2 \xleftarrow{\sim} \cdots \longrightarrow A_{2n} \xleftarrow{\sim} A_{2n+1} \longrightarrow Y$$

1-simplices are going to be diagrams like:

$$\begin{tikzcd}
 & A_0 \arrow[d, "\sim"] & A_1 \arrow[l, "\sim"'] \arrow[r] \arrow[d, "\sim"] & A_2 \arrow[d, "\sim"] & \cdots \arrow[l, "\sim"'] \arrow[r] & A_{2n} \arrow[d, "\sim"] & A_{2n+1} \arrow[l, "\sim"'] \arrow[d, "\sim"] \arrow[dr] & \\
X \arrow[ur] \arrow[dr] & & & & & & & Y \\
 & B_0 & B_1 \arrow[l, "\sim"'] \arrow[r] & B_2 & \cdots \arrow[l, "\sim"'] \arrow[r] & B_{2n} & B_{2n+1} \arrow[l, "\sim"'] \arrow[ur] &
\end{tikzcd}$$

Higher simplices come from carrying on in this fashion — adding more and more rows to the diagram. A k-simplex will be a hammock of width $k+1$. Thus $\operatorname{Mor}(X,Y)$ is just the nerve of the category of zig-zags from X to Y and weak equivalences of zig-zags.

In a model category

I can get a factorisation

$$X \overset{\sim}{\twoheadleftarrow} X' \longrightarrow Y' \overset{\sim}{\hookleftarrow} Y$$

Where X' is cofibrant, and Y' is fibrant. So actually, we only need hammocks as in diag 3.

Another model for the simplicial localisation

There's a forgetful functor $\mathsf{Cat} \longrightarrow \mathsf{DirectedGraphs}$, which has a left adjoint, called 'Free'. The free category on a directed graph has morphisms given by composable words in the edges of the graph. Now suppose I want to localise $\mathcal{C} = \operatorname{Free}(V \cup W)$, where V and W are graphs with no shared edges (but probably with shared vertices). Then $\mathcal{C}[W^{-1}] = \operatorname{AlmostFree}(V \cup W^{\pm 1})$: the morphisms are reduced composable words in the arrows of V and of $W \cup W^{-1}$. That's easy, and if we could replace a category with something like this, we'd win.

Given $\mathcal{C}$, I can form $F(\mathcal{C})$, the free category on $\mathcal{C}$, by applying forget, then free. F is a monad, and I have natural transformations $FF \longrightarrow F$ and $1 \longrightarrow F$ satisfying the monoid axioms. So I'll construct a simplicial category M, whose category of k-simplices is $F^{k+1}(\mathcal{C})$, and whose faces and degeneracies are given by the monad maps.

Fact 1. M is equivalent to $\mathcal{C}$ as simplicial category.

But now to invert W is achievable levelwise. $M \longrightarrow M[W^{-1}]$ is the simplicial localisation.

A simplicial model category

should be a model category enriched in simplicial sets, which already has the 'right' space of morphisms, at least between its cofibrant and fibrant objects.

Definition. A simplicial model category is a model category, and a simplicial category, which:

1. is tensored and cotensored over sSet. [So that given $X \in \mathcal{C}$ and $K \in \mathsf{sSet}$, can take $X \otimes K$ and $\mathrm{Hom}(K, X)$, and there's an adjunction].
2. Given a cofibration $A \hookrightarrow B$ and a fibration $X \twoheadrightarrow Y$, there is a natural map:

$$c : \mathrm{Hom}(B, X) \longrightarrow \mathrm{Hom}(A, X) \times_{\mathrm{Hom}(A,Y)} \mathrm{Hom}(B, Y).$$

The codomain is the space of ways to complete the following into a commuting square:

$$\begin{tikzcd} A \arrow[d, hook] & X \arrow[d, two heads] \\ B & Y \end{tikzcd}$$

We demand that c is a fibration, and if either $X \twoheadrightarrow Y$ or $A \hookrightarrow B$ is acyclic, that c is an acyclic fibration.

Punchline. *If $\mathcal{C}$ is a simplicial model category, and $\mathcal{C}'$ is the underlying category (take* 0*-simplices as morphisms), then $\mathcal{C}$ and $L^{\mathscr{W}}(\mathcal{C}')$ are equivalent and simplicial categories.*[1]

Post talk ramblings

A weak equivalence of simplicial categories is a zig-zag of (simplicial) functors $F_i : \mathcal{C}_i \longrightarrow \mathcal{C}_{i+1}$ such that $\pi_0 F_i$ is a weak equivalence (of spaces), and for each $X, Y \in \mathrm{ob}\,(\mathcal{C}_i)$, $\mathrm{Mor}(X, Y) \longrightarrow \mathrm{Mor}(F_i X, F_i Y)$ is a weak equivalence in the model category of simplicial sets.

[1] $\mathscr{W}$ is the 0-simplices that are weak equivalences.

Abstract

The problem of localizing a model category M is that of enlarging the class of weak equivalences to include an additional set S of maps while still obtaining a model structure $S^{-1}M$, preferably one that can be related to the original model structure on M (meaning a Quillen adjunction is desirable). While such a localization needs not always exist, there are certain hairy technical conditions (which nonetheless hold in most model categories of interest) on M that always allow one to form $S^{-1}M$. This is the so-called left Bousfield localization, whose base category is M itself and whose cofibrations are precisely those of M, with an enlarged class of weak equivalences.

After discussing this process and some general ideas of how and why it works (this will include a rare sighting of the definition of the ubiquitous Reedy model structures of diagrams), we will illustrate it's importance with a wealth of examples: Bousfield's localization of spaces with respect to homology (the primal example), Dugger's hocolim localization of sM, showing a wide class of model categories can be made simplicial, Rezk's complete Segal spaces (a model for $(\infty, 1)$-categories) obtained from localizing simplicial spaces, and Dugger's theory of presentations of model categories, which are localizations of the so-called universal model categories.

Let M be a model category. This structure supports the inversion of the weak equivalences. Have $\mathrm{ho}M(X,Y) = M(X_C, Y_F)/\sim$. What if we want to further localise $\mathrm{ho}M$, by a bunch of maps S. Want:

$$\begin{tikzcd} M \arrow[r, dotted] \arrow[d] & N \arrow[d] \\ \mathrm{ho}M \arrow[r] & \mathrm{ho}M[S^{-1}] \end{tikzcd}$$

We'll get a Quillen pair, and there'll be a choice of which direction the adjunction goes:

$$L : M \longleftrightarrow N : R$$

We'll usually choose version that makes L the left adjoint!

Definition. A localisation of M on a class S of maps should be a Quillen pair $L : M \longrightarrow N$ such that L sends S to weak equivalences, and is initial with this property.

The first guess is to take $N = M$, and enlarge the weak equivalences by S as necessary, and, as we want the maps $M \longrightarrow M$ to be the identity, we try to use the same fibrations. (NB: Left Quillen functors preserve cofibrations, so they have to increase. It'd be nice if we could keep the class of cofibrations unchanged).

Proposition. *If such a model structure exists, it satisfies the universal property.*

Definition. These model structures are called left Bousfield localisations.

Definition. Let S be a set of maps. Then $X \in M$ is called S-local if X is fibrant and $\underline{M}(B,X) \longrightarrow \underline{M}(A,X)$ is a weak equivalence for all $A \longrightarrow B$ in S.

Definition. A map $\overline{A} \longrightarrow \overline{B}$ is called an S-local equivalence if $\underline{M}(\overline{B},X) \longrightarrow \underline{M}(\overline{A},X)$ is a weak equivalence for all $X \in M$ S-local.

Now we can redefine left Bousfield localisation, calling it $L_S M$:

- ⇝ The cofibrations are the same as those of M.
- ⇝ the weak equivalences are the S-local equivalences.
- ⇝ the fibrations are those with the right lifting property.
- ⇝ Note that the fibrant objects are now the S-local objects.

Homotopy function complexes (with a discussion of Reedy model categories)

$\underline{(X}, Y)$ is supposed to be such that $\pi_0 = \mathrm{ho}M(X,Y)$. One way to think about it is via Saul's talk. This is the simplicial mapping space. However, there are set-theoretical problems, for large categories, and model categories are always large — a small category doesn't have enough limits and colimits.

We thus need a new definition. Suppose that M is a simplicial model category. Then define $\underline{M}(X,Y)$ by $\underline{M}(X,Y)_m = M(X_C \otimes \Delta^m, Y_F)$.

But how can we define $\underline{M}(X,Y)$ for an arbitrary model category? We need analogues of the cosimplicial object X^*:

$$X_c \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; X_C \otimes \Delta^1 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; X_C \otimes \Delta^2$$

The first bit should be a cylinder object, giving $X \sqcup X \hookrightarrow X \otimes \Delta^1 \xrightarrow{\sim} X$. Ummm, in general, want $X \otimes \partial\Delta^m \hookrightarrow X \otimes \Delta^M \xrightarrow{\sim} X$ each time.

This is the same as saying that X^* is a cofibrant replacement of the constant diagram at X in the Reedy model structure on cM, the category of cosimplicial objects in M, i.e. functors $\Delta \longrightarrow M$.

The Reedy model structure

Consider Δ, and two subcategories: Δ^+ the injective maps, and Δ^- the injective maps. Maps in Δ factor uniquely by a map in Δ^1 then a map in Δ^+.

$$\begin{tikzcd}
\bullet \arrow[rr] \arrow[dr, "-"'] & & \bullet \\
& \bullet \arrow[ur, "+"'] &
\end{tikzcd}$$

Now we can define three model structure on cM:

1. $(cM)_{proj}$ in which the weak equivalences (fibrations) are the termwise weak equivalences (fibrations).

2. $(cM)_{inj}$ in which the weak equivalences (cofibrations) are the termwise weak equivalences (cofibrations).

3. $(cM)_{Reedy}$. Here the weak equivalences are the termwise weak equivalences. The cofibrations are the natural transformations whose restrictions to Δ^+ are cofibrations in the projective model structure on M^{Δ^+}. The fibrations are the natural transformations whose restrictions to Δ^- are fibrations in the injective model structure on M^{Δ^-}.

The upshot: we can define $\underline{M}(A,X) = M((A^*)_c, X_f)$.

Proposition. *For M a model category, a map $A \longrightarrow B$ is a weak equivalence* iff *the $\underline{M}(B,X) \longrightarrow \underline{M}(A,X)$ are weak equivalences for all X.*

Certainly, if $a \longrightarrow B$ is to be a weak equivalence in $L_S M$, then it must be true that $\underline{L_S M}(B,X) \longrightarrow \underline{L_S M}(A,X)$ is a weak equivalence. But: $\underline{L_S M}(B,X) = \underline{M}(B,X)$, provided that X is $L_S M$-fibrant.

Do the $L_S M$ exist?

Fortunately, the answer is often 'yes'.

Theorem. *$L_S M$ always exists provided (S is a set, and) M is*

1. *left proper combinatorial; or*
2. *left proper cellular.*

Definition. It's left proper if when you push out a weak equiv along a cofibration, you get a weak equiv.

Definition. It's combinatorial if it's cofibrantly generated and satisfies some set-theoretic conditions. Cellular is similar.

Definition. It's cofibrantly generated if there is a **set** I of cofibrations and a set J of trivial fibrations so that I determines the trivial fibrations by the right lifting property, and J does the same for the (cofibrations?). (the full classes always determine this stuff, but are proper classes (in general, at least))

Example. $\rightsquigarrow$ Top is cellular

- $\rightsquigarrow$ sSet is both cellular and combinatorial
- $\rightsquigarrow$ $(M^C)_{proj}$ exists and has whatever of these properties M has. (What's this?)
- $\rightsquigarrow$ $(M^R)_{reedy}$ exists and has whatever of these properties M has.

Lets have some examples

1. Let M be sSet or Top, and let S be the homology isomorphisms.

2. (m-types) Let $M = \mathsf{sSet}$, and let $S = \partial\Delta^{m+2} \longrightarrow \Delta^{m+2}$.

   What now are the fibrant objects? Should have $\mathrm{Map}(\Delta^{n+2}, X) \xrightarrow{\sim} \mathrm{Map}(S^{n+1}, X)$, which implies that $\pi_{n+1}(X) = 0$.

   If you smash $\partial\Delta^{m+2} \longrightarrow \Delta^{m+2}$ with $\partial\Delta^0 \longrightarrow \Delta^0$, you get that the higher inclusions $S^{m+2} \hookrightarrow \Delta^{m+3}$ are weak equivalences, and so on. So, $L_S M$ is the homotopy theory of m-types, and fibrant replacement consists of taking the m^{th} stage of the Postnikov tower.

3. (Dugger) He gets a Quillen equivalence $c_* : M \longleftrightarrow sM_{hc} : (—)_0$, where the right hand side is $((sM)_{reedy})_{hocolim\ equivalences}$.

4. (Rezk's complete Segal spaces) $M = sSet^{\Delta^{\mathrm{op}}}$ (the same as bisimplicial set, but we want to distinguish a coordinate). A Segal space is X_* such that there's a map $X_m \longrightarrow X_1 \times_{X_0} \cdots \times_{X_0} X_1$. Ummm X_0 is supposed to be the space of objects in an $(\infty, 1)$-cat. X_1 is supposed to be the space of arrows, X_2 is the space of composable pairs of arrows, etc. I.e. you take a simplex and you look at its spine. (?context?) These 'Segal spaces' are supposed to represent $(\infty, 1)$-categories.

   Ummmmmmmm localise with respect to $F_{\Delta^1} \sqcup_{F_{\Delta^0}} \cdots \sqcup_{F_{\Delta^0}} F_{\Delta^1} \longrightarrow F_{\Delta^n}$.

“(Simplicial) (model) categories” — Inna Zakharevich — 19/10/2011

Abstract

We’ve talked about model categories, we’ve talked about simplicial categories. Now we must tackle the two monsters... together. This talk will be about the interactions of the simplicial structure and the model structure on the category of simplicial objects in a model category: $SM7$, the Reedy structure, and (if we have time) the E^2 model structure.

We had the rather complicated hammock localisation:

$$\text{Model Cats} \xrightarrow{L_H} \text{Simplicial Cats}$$

If $\mathcal{C}$ is a model category with a simplicial enrichment, we want $\pi_0 \underline{\mathrm{Hom}}(A, B)$ to be the set of homotopy classes $\mathrm{Hom}_{\mathrm{ho}\mathcal{C}}(A, B)$ of maps A to B.

We’d rather like $L_H(A, B) \simeq \mathrm{Hom}_{\mathcal{C}}(A, B)$, maybe with some conditions of cofibrancy, etc.

In this talk, $\underline{\mathrm{Hom}}$ refers to a space of morphisms, but I didn’t notice this for a while, so some underlines might be missing.

Axiom SM7

$$\begin{tikzcd} A \arrow[r, dashed, "\alpha"] \arrow[d, hook, "i"'] & X \arrow[d, "P"] \\ B \arrow[r, dashed, "\beta"'] \arrow[ur, dotted] & Y \end{tikzcd}$$

Have, for i, P fixed, a function

$$\mathrm{Hom}(B, X) \xrightarrow{(i^*, P_*)} \mathrm{Hom}(B, Y) \times_{\mathrm{Hom}(A,Y)} \mathrm{Hom}(X, Y)$$

expressing the fact that “for every lift you get a diagram”. Note that we didn’t say that i or P had to by acyclic.

- **SM7:** (i^*, P_*) is always a fibration. If either i or P is acyclic, then so is (i^*, P_*).
  - $\rightsquigarrow$ We want the lift to be well defined, at least up to a contractible choice, when it is guaranteed to exist by the model category axioms.
  - $\rightsquigarrow$ You want there to be some continuity in the answer to the question “what lifts exist for (α, β)” to vary smoothly in (α, β), where $\alpha : A \longrightarrow X$ and $\beta : B \longrightarrow Y$ make the diagram commute. For example, if no lift exists, and you deform the diagram slightly, you don’t want to suddenly see a lift.

Definition. A *simplicial model category* is a model category enriched over sSet satisfying SM7.

There are the best things that are both simplicial and model.

How to simplicially enrich sSet

Well, we start be setting

$$\underline{\mathrm{Hom}}(A, B)_0 := \mathrm{Hom}(A, B).$$

We really want the 1-simplices to be homotopies, so define:

$$\underline{\mathrm{Hom}}(A,B)_1 := \mathrm{Hom}(A\times\Delta^1, B).$$

We carry on this way, using Δ^k instead of $(\Delta^1)^k$, the former being more simplicial:

$$\underline{\mathrm{Hom}}(A,B)_k := \mathrm{Hom}(A\times\Delta^k, B).$$

Theorem (Quillen?)**.** *This works, making* sSet *a simplicial model category.*

Proof. See Goerss and Jardine, chapter II. ☐

We can generalise this. Suppose that $\mathcal{C}$ has all coproducts (for example, if $\mathcal{C}$ is a model category). We construct a functor $\mathcal{C}\times\mathsf{sSet}\xrightarrow{\otimes} s\mathcal{C}$, where $s\mathcal{C} = [\Delta^{\mathrm{op}},\mathcal{C}]$, by setting[2] (any further identification in this colimit?)

$$(X\otimes K)_n = \coprod_{K_n} X_n.$$

Now in general, given a functor $\mathcal{C}\times\mathsf{sSet}\xrightarrow{\otimes}\mathcal{C}$, (satisfying some conditions), we can give a simplicial enrichment on $\mathcal{C}$, via the recipe:

$$\underline{\mathrm{Hom}}(A,B)_k := \mathrm{Hom}(A\otimes\Delta^k, B).$$

All the simplicial structure here comes from the Δ^k coordinate.

$\mathcal{C}$ is a model category

Consider $s\mathcal{C}$. We have a simplicial structure on $s\mathcal{C}$ (from previous). We also have the projective and injective model structures

Why do we care about model structures on diagram categories?

Suppose we have a diagram, like D: $\bullet\longrightarrow\bullet\longleftarrow\bullet$. We have the functor

$$\text{pullback} : \mathsf{Top}^D \longrightarrow \mathsf{Top}\,.$$

Both of these are model categories, but this isn't a Quillen functor [i.e. doesn't preserve weak equivs]. Pullback is not compatible with homotopy.[3] *However*, if the two maps in are fibrations, even if you change either the objects or the maps up to homotopy, you get a weak equiv on pullbacks.[4]

There is a model structure on Top^D, in particular, it's the injective structure you need for pullbacks. (I don't get how some of the comments hereabouts are relating to each other.)

The injective and projective structures aren't right (for what?). To demonstrate this, I'll try to characterise the (acyclic?) cofibrations on the model category $s\mathcal{C}$ given the projective model structure, between objects A_* and B_*.

[2] A common alternative way to think of this: it's the diagonal in the bisimplicial set $(X\otimes K)_{ik} = \coprod_{K_i} X_j$.

[3] For example, the pullback of $*\longrightarrow S^1\longleftarrow *$ is either a point or the empty set.

[4] In spaces, you get lucky and only need one to be a fibration.

To get a lift, we at least need one on the 0-level, so we'll need $A_0 \longrightarrow B_0$ to be an acyclic cofibration in $\mathcal{C}$:

$$\begin{tikzcd}
A_0 \arrow[r] \arrow[d, hook, "\simeq"'] & X_0 \arrow[d] \\
B_0 \arrow[r] \arrow[ur, dotted] & Y_0
\end{tikzcd}$$

Let's try to go a little higher. The lift on the next level is a rather more complicated piece of cheese:

$$\begin{tikzcd}
A_0 \arrow[rrr] \arrow[drr, hook] \arrow[ddd] & & & X_0 \arrow[ddd] \\
& & B_0 \arrow[ur, dotted] \arrow[dl] \arrow[ddd] & \\
& \text{Push} \arrow[ddr, dashed] & & \\
A_1 \arrow[ur] \arrow[rrr] \arrow[drr] & & & X_1 \\
& & B_1 \arrow[ur, dotted] &
\end{tikzcd}$$

Somehow, things work when the dashed map on the left face from the pushout to B_1 is a weak equivalence. (I haven't figured this out yet.) Anyway, this is all getting complicated rather quickly, so we should move on instead to:

The Reedy model structure

Theorem. *We have a model structure on $s\mathcal{C}$, where the weak equivalences are levelwise, and*

$$\text{proj} \xrightarrow{\sim} \text{Reedy} \xrightarrow{\sim} \text{inj}$$

are left Quillen equivalences.[5]

The downside is that the Reedy structure is not simplicial.

Theorem. *If $\mathcal{C} \otimes \mathsf{sSet} \longrightarrow \mathcal{C}$ ($\mathcal{C}$ a model category) satisfies, for all $A \hookrightarrow B \in \mathcal{C}$:*

1. *$(A \otimes \Delta^n) \cup_{A \otimes \partial\Delta^n} (B \otimes \partial\Delta^n) \longrightarrow B \otimes \Delta^n$, the "box product"*[6]*, is a cofibration, which is acyclic whenever i is acyclic; and*
2. *The same box product for $\Delta^0 \overset{\sim}{\hookrightarrow} \Delta^1$ is always an acyclic cofibration;*

then $\mathcal{C}$ is a simplicial model category.

In the Reedy structure, it's the easier condition, 2, that fails.

There's another structure, the E^2 model structure, for which this works, in a paper by Dwyer, Kan and Stover. They say $X \xrightarrow{\sim} Y$ is a weak equivalence *iff* $\pi_i(\pi_j X) \xrightarrow{\cong} \pi_i(\pi_j Y)$ is iso for all i, j (the homotopy of a simplicial group). (That is, we get an E^2-page iso (??)).

[5] Reedy's paper on this is hard to find, Goerss and Jardine's account is hard to read.

[6] Suppose that $\mathcal{C} = \mathsf{Top}$ and $\mathsf{Top} \otimes \mathsf{sSet} \longrightarrow \mathsf{Top}$ is $A \otimes K \longmapsto A \times |K|$. Suppose given a cofibration $A \hookrightarrow B$. The "box product" maps out of $A \times D^n \cup B \times S^{n-1}$, a cylinder $B \times D^n$ with $B \setminus A \times (D^n)^\circ$ removed. It is then the inclusion of this into the full cylinder $B \times D^n$. It seems likely that this functor satisfies the criteria of the theorem.

The Reedy structure

Definition (Formal definition)**.** Let Δ_n be the full subcat of Δ of objects $\leq n$. Have inclusion $\Delta_n \longrightarrow \Delta$, giving

$$\mathcal{C}^{\Delta^{op}} \longrightarrow \mathcal{C}^{\Delta_n^{op}}$$

This functor has both a left and right adjoint:

- ⇝ The left adjoint is the n^{th} skeleton sk_n. This is the object in which you throw out all the nondegenerate simplices whose dimension exceeds n. Note that the skeleton is the minimal object with the correct cells at dimensions n and below, which suits the fact that it is a left adjoint — it should at least like supporting maps.
- ⇝ The right adjoint is the n^{th} coskeleton cosk_n. This is the object in which you add in all possible $(n+1)$-simplices, then all possible $(n+2)$-simplices, and so on forever. Note that the coskeleton is the maximal object with the correct cells at dimensions n and below, which suits the fact that it is a right adjoint — it should at least like receiving maps.

Now let $L_nX = (\mathrm{sk}_{n-1}X)_n \in \mathcal{C}$, and $M_nX = (\mathrm{cosk}_{n-1}X)_n \in \mathcal{C}$ "latching and matching". Have $L_nX \longrightarrow X_n$ and $X_n \longrightarrow M_nX$. The cofibrations are the maps $A \longrightarrow B$ where $A_n \sqcup_{L_nA} L_nB \longrightarrow B_n$ is a cofibration for all n. The fibrations are the things $A_n \longrightarrow M_nA \times_{M_nB} B_n$ are fibrations for all n.

Corollary. *These maps are acyclic for all n* iff *$A \longrightarrow B$ is.*

The pushout is the map from the first latching object. This model structure is self-dual, halving the number of proofs!

Q: Why is $s\mathcal{C}$ interesting? A: If you want to define Andre-Quillen cohomology, you really need simplicial objects, which are relevant to resolutions of things in some settings.

“Stable model categories are spectral” — Michael Donovan — 26/10/2011

Abstract

A stable model category is a pointed model category for which the suspension map is an auto-equivalence of the homotopy category. Given a model category enriched over pointed simplicial sets, we’ll construct a stabilisation, which is Quillen equivalent when the original category is stable. We’ll also have it that the stabilisation is enriched over the category of symmetric spectra, so that every stable model category is spectral. Given time, we’ll also discuss a characterisation given by Schwede and Shipley of stable model categories with a set of compact generators.

Overview

- ⇝ Most of the material for this talk comes from Schwede and Shipley, “Stable model categories are categories of modules”.
- ⇝ Define a stable model category.
  - ⇝ A stable model category is a pointed model category $\mathcal{C}$ in which the evident suspension functor (homotopy push out $* \leftarrow X \rightarrow *$) is an auto-equivalence of ho$\mathcal{C}$.
- ⇝ Schwede and Shipley give lots of examples:
  - ⇝ Any category of spectra (Bousfield Friedlander, symmetric, orthogonal, co-ord free,...).
  - ⇝ Equivariant stable homotopy theory — G-eq coordinate-free spectra and G-eq orthogonal spectra, for G compact Lie.
  - ⇝ Motivic stable homotopy: “$\mathbb{A}^1$-local model category structure for schemes over a base”.
  - ⇝ Unbounded chain complexes of left R-modules, R a ring.
  - ⇝ More generally, the homotopy category of any abelian category $\mathcal{A}$. There’s a model structure on the unbounded shain complexes in $\mathcal{A}$ whose weak equivalences are chain homotopy equivalences.
- ⇝ All model categories will be **nice**, where by nice we mean cofibrantly generated, proper and simplicial[7].
- ⇝ Our purpose is to see that spectral model categories are the same as stable model categories, by constructing a stabilisation of any nice model category, Quillen equivalent when the category is stable.

$\mathcal{D}$-model categories

- ⇝ Suppose $\mathcal{D}$ is a **closed symmetric monoidal category**, i.e.
  - ⇝ $\mathcal{D}$ has a bifunctor $\otimes : \mathcal{D} \times \mathcal{D} \longrightarrow \mathcal{D}$ with unit, associativity commutativity and unitality isomorphisms.
  - ⇝ $\mathcal{D}$ has a self-enrichment: $\underline{\mathrm{Hom}}_{\mathcal{D}} : \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \longrightarrow \mathcal{D}$ giving right adjoints to $-\otimes d$.
- ⇝ If $\mathcal{D}$ is also a model category, call it a **symmetric monoidal model category** if:

[7] Every cofibrantly generated, proper (*weak equivalences are preserved under pulling back (pushing out) along (co)fibrations*), stable model category is in fact Quillen equivalent to a simplicial model category.

$\rightsquigarrow$ An SM7 analogue holds. For fixed cofibration i and fibration p:

$$\begin{tikzcd} A \arrow[r, dashed, "\alpha"] \arrow[d, hook, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, dashed, "\beta"'] \arrow[ur, dotted] & Y \end{tikzcd}$$

$$\underline{\mathrm{Hom}}_{\mathcal{D}}(B,X) \xrightarrow{(i^*,p_*)} \underline{\mathrm{Hom}}_{\mathcal{D}}(A,X) \times_{\underline{\mathrm{Hom}}_{\mathcal{D}}(A,Y)} \underline{\mathrm{Hom}}_{\mathcal{D}}(B,Y)$$

is always a fibration in $\mathcal{D}$, acyclic if either i or P is acyclic.

$\rightsquigarrow$ For $X \in \mathcal{D}$ cofibrant, $-\otimes X$ and $X \otimes -$ both send the cofibrant replacement of $\mathbf{1}_{\mathcal{D}}$ to a weak equivalence.
[If $Q\mathbf{1}_{\mathcal{D}} \xrightarrow{q} \mathbf{1}_{\mathcal{D}}$ is the cofibrant replacement for the unit, then for all cofibrant X, the natural maps $Q\mathbf{1}_{\mathcal{D}} \otimes X \xrightarrow{q\otimes 1} \mathbf{1}_{\mathcal{D}} \otimes X$ and $X \otimes Q\mathbf{1}_{\mathcal{D}} \xrightarrow{1\otimes q} X \otimes \mathbf{1}_{\mathcal{D}}$ are weak equivalences.]

$\rightsquigarrow$ A category $\mathcal{M}$ is a **closed $\mathcal{D}$-module category** if it has natural constructions:

$\rightsquigarrow$ $\mathcal{M} \times \mathcal{D} \longrightarrow \mathcal{M}$ written $(X,d) \longmapsto X \otimes d$. (tensored)

$\rightsquigarrow$ $\mathcal{D}^{\mathrm{op}} \times \mathcal{M} \longrightarrow \mathcal{M}$ written $(d,Z) \longmapsto Z^d$. (cotensored)

$\rightsquigarrow$ $\mathcal{M}^{\mathrm{op}} \times \mathcal{M} \longrightarrow \mathcal{D}$ written $(X,Z) \longmapsto \underline{\mathrm{Hom}}_{\mathcal{D}}(X,Z)$. ($\mathcal{D}$-enriched)

satisfying

$$\mathcal{M}(X \otimes d, Z) \cong \mathcal{M}(X, Z^d) \cong \mathcal{D}(d, \underline{\mathrm{Hom}}_D(X,Z)),$$

so that in particular,

$$\mathcal{M}(X,Z) \cong \mathcal{D}(\mathbf{1}_{\mathcal{D}}, \underline{\mathrm{Hom}}_{\mathcal{D}}(X,Z)).$$

We get induced[8] composition morphisms

$$\underline{\mathrm{Hom}}_{\mathcal{D}}(Y,Z) \otimes \underline{\mathrm{Hom}}_{\mathcal{D}}(X,Y) \longrightarrow \underline{\mathrm{Hom}}_{\mathcal{D}}(X,Z)$$

and the induced map (as follows) is exactly composition in $\mathcal{M}$:

$$\begin{tikzcd} \mathcal{D}(\mathbf{1}_{\mathcal{D}}, \underline{\mathrm{Hom}}_{\mathcal{D}}(Y,Z)) \times \mathcal{D}(\mathbf{1}_{\mathcal{D}}, \underline{\mathrm{Hom}}_{\mathcal{D}}(X,Y)) \arrow[r] \arrow[dr] & \mathcal{D}(\mathbf{1}_{\mathcal{D}}, \underline{\mathrm{Hom}}_{\mathcal{D}}(Y,Z) \otimes \underline{\mathrm{Hom}}_{\mathcal{D}}(X,Y)) \arrow[d] \\ & \mathcal{D}(\mathbf{1}_{\mathcal{D}}, \underline{\mathrm{Hom}}_{\mathcal{D}}(X,Z)) \end{tikzcd}$$

$\rightsquigarrow$ $\mathcal{M}$ is a **$\mathcal{D}$-model category** if:

$\rightsquigarrow$ it's a model category and a closed $\mathcal{D}$-module category

$\rightsquigarrow$ $\otimes : \mathcal{M} \times \mathcal{D} \longrightarrow \mathcal{M}$ is a Quillen bifunctor[9]

$\rightsquigarrow$ If $Q\mathbf{1}_{\mathcal{D}} \xrightarrow{q} \mathbf{1}_{\mathcal{D}}$ is the cofibrant replacement for the unit, then for all cofibrant $X \in \mathcal{M}$, the natural maps $Q\mathbf{1}_{\mathcal{D}} \otimes X \xrightarrow{q\otimes 1} \mathbf{1}_{\mathcal{D}} \otimes X$ and $X \otimes Q\mathbf{1}_{\mathcal{D}} \xrightarrow{1\otimes q} X \otimes \mathbf{1}_{\mathcal{D}}$ are weak equivalences.

$\rightsquigarrow$ $\mathcal{M}$ is a **spectral model category** if it's a Sp^Σ-model category, where Sp^Σ is the category of symmetric spectra.

[8] Take the adjoint of $X \otimes \underline{\mathrm{Hom}}_D(X,Y) \otimes \underline{\mathrm{Hom}}_D(Y,Z) \longrightarrow Y \otimes \underline{\mathrm{Hom}}_D(Y,Z) \longrightarrow Z$. The map $X \otimes \underline{\mathrm{Hom}}_{\mathcal{D}}(X,Y) \longrightarrow Y$ is adjoint to the identity of $\underline{\mathrm{Hom}}_{\mathcal{D}}(X,Y)$.

[9] Should preserve fibrations and trivial fibrations.

Spectral model categories are stable

- Under the Quillen adjunction $\mathsf{sSet}_* \leftrightarrow \mathrm{Sp}^\Sigma$ (more to follow), have simplicial structure on a spectral model category.
- To form Σ (for X cofibrant), normally take a replacement for $X \longrightarrow *$, and push out.
  - Have a cofibration $\partial\Delta[1] \longrightarrow \Delta[1]$ in sSet_*, and have pushout diagram at left:

$$\begin{tikzcd} \partial\Delta[1] \arrow[r] \arrow[d] & \Delta[1] \arrow[d] \\ \Delta[1] \arrow[r] & S^1 \end{tikzcd} \qquad \begin{tikzcd} \partial\Delta[1]\otimes X \arrow[r] \arrow[d] & \Delta[1]\otimes X \arrow[d] \\ \Delta[1]\otimes X \arrow[r] & S^1\otimes X \end{tikzcd}$$

  - The diagram on the right is still a pushout, since $- \otimes X$ is a left adjoint.
  - The two maps $\partial\Delta[1]\otimes X \longrightarrow \Delta[1]\otimes X$ are still cofibrations, by SM7.
  - $\partial\Delta^1 = S^0$ is the unit for the SMP in sSet_*, so have the diagram for calculating ΣX.
  - Thus $\Sigma X = X \otimes S^1$.
- S^1 is invertible up to stable equivalence in Sp^Σ, so smashing with its inverse gives the inverse automorphism.

Symmetric spectra on a simplicial model category

- Let $\mathcal{C}$ be cocomplete, tensored and cotensored over sSet_*.
- Define $S^1 = \Delta[1]/\partial\Delta[1]$, a simplicial circle, so that $S^n = (S^1)^{\wedge n}$ has Σ_n-action.
- A **symmetric sequence** over $\mathcal{C}$ is $X = \{X_n\}$ with a left action of Σ_n on X_n. Write $\mathcal{C}^\Sigma$ for the category of symmetric sequences with equivariant maps.
- A **symmetric spectrum** over $\mathcal{C}$ is such with coherently associative $\Sigma_p \times \Sigma_q$-equivariant morphisms $S^p \otimes X_q \longrightarrow X_{p+q}$. A morphism of such is just a compatible collection of Σ_n-equivariant maps. Call this category $\mathrm{Sp}^\Sigma(\mathcal{C})$.
- The category $\mathcal{C}^\Sigma$ is monoidal, and tensored, cotensored and enriched over sSet_*^Σ. In fact, we define, by analogy with taking tensor products of chain complexes:

$$(K\otimes X)_n = \bigvee_{p+q=n} \Sigma_n^+ \otimes_{\Sigma_p\times\Sigma_q} (K_p \otimes X_q).$$

  Now $S = (S^0, S^1, \ldots)$ is a commutative monoid in the symmetric monoidal category $(\mathsf{sSet}_*^\Sigma, \otimes)$.

  - The unit is $u = (S^0, *, \ldots)$, and the unit map $\eta : u \longrightarrow S$ is the obvious.
  - We define $m : S\otimes S \longrightarrow S$ using the $\Sigma_p\times\Sigma_q$-equivariant map $S^p \wedge S^q \longrightarrow S^n$:

$$(S\otimes S)_n = \bigvee_{p+q=n} \Sigma_n^+ \wedge_{\Sigma_p\times\Sigma_q} (S^p\wedge S^q) \longrightarrow S^n.$$

    As we've thrown in the Σ_n^+, this multiplication is *strictly commutative*.
- In fact, a symmetric spectrum is just a left S-module — it's the same to give the pairing,

$$\text{[which amounts to giving:}\quad \Sigma_{p+q}^+ \otimes_{\Sigma_p\times\Sigma_q} S^p\otimes X_q \underset{\Sigma_{p+q}}{\longrightarrow} X_{p+q}, \quad \text{i.e.}\quad S^p\otimes X_q \underset{\Sigma_p\times\Sigma_q}{\longrightarrow} X_{p+q}],$$

  as to give a symmetric spectrum with these terms.

- Define the smash product $K \wedge X \in \mathrm{Sp}(\mathcal{C})$ of $K \in \mathrm{Sp}(\mathsf{sSet}_*)$ and $X \in \mathrm{Sp}(\mathcal{C})$ as the coequaliser of the following maps in $\mathcal{C}^\Sigma$ induced by the actions of S on K and X:

$$\begin{tikzcd} K \otimes S \otimes X \arrow[r, shift left, "m\otimes 1"] \arrow[r, shift right, "1\otimes m"'] & K \otimes X \end{tikzcd}$$

  Define the smash product $X \wedge Y$ of $X, Y \in \mathrm{Sp}(\mathcal{C})$ in the same way, when $\mathcal{C}$ is symmetric monoidal!
- For $X, Y \in \mathrm{Sp}(\mathcal{C})$, define $\underline{\mathrm{Hom}}_{\mathrm{Sp}(\mathcal{C})}(X, Y) \in \mathrm{Sp}^\Sigma$ as follows.
  - Define $\mathrm{sh}_n : \mathrm{Sp}(\mathcal{C}) \longrightarrow \mathrm{Sp}(\mathcal{C})$ by $(\mathrm{sh}_n(X))_m = X_{n+m}$. This has a Σ_n action left over.
  - Define $\underline{\mathrm{Hom}}_\Sigma(X, Y) \in \mathsf{sSet}_*^\Sigma$ by $\underline{\mathrm{Hom}}_\Sigma(X, Y)_n := \underline{\mathrm{Hom}}_{\mathsf{sSet}_*}(X, \mathrm{sh}_n Y)$, (has Σ_n action).
  - Define $\underline{\mathrm{Hom}}_{\mathrm{Sp}^\Sigma}(X, Y) \in \mathrm{Sp}^\Sigma$ as the equaliser[10] of the two maps:

$$\begin{tikzcd} \underline{\mathrm{Hom}}_\Sigma(X, Y) \arrow[r, shift left, "(m_X)^*"] \arrow[r, shift right, "(m_Y)_*"'] & \underline{\mathrm{Hom}}_\Sigma(S \otimes X, Y) \end{tikzcd}$$

- We define the cotensor structure $(—)^K$ as the right adjoint of $K \otimes —$.
- There's an adjunction:
$$F_n : \mathcal{C} \longleftrightarrow \mathrm{Sp}(\mathcal{C}) : \mathrm{Ev}_n\,.$$
  Set $\mathrm{Ev}_n(X) = X_n$, and $(F_n X)_m = \Sigma_m^+ \otimes_{\Sigma_{m-n}} S^{m-n} \otimes X$.

The level model structure

- We should suppose that $\mathcal{C}$ is simplicial model.
- A map $f : X \longrightarrow Y$ is a **level equivalence** (**fibration**) if each f_n is a weak equivalence (fibration) in $\mathcal{C}$ (ignoring the Σ_n-action). Level cofibrations are those with the LLP wrt acyclic fibrations.
- *If $\mathcal{C}$ is a simplicial, cofibrantly generated model category, this gives a cofibrantly generated model category. This spectrum is a* $(\mathrm{Sp}^\Sigma)^{\mathrm{lv}}$*-model category (and so has a* $(\mathrm{SP}^{\mathrm{lv}})$ *in the level model structure on symmetric spectra).*
  - The first sentence must first be applied to the case $\mathcal{C} = \mathsf{sSet}_*$, to give a level model structure on symmetric spectra Sp^Σ.
  - Rereading shows that $(\mathrm{Sp}^\Sigma)^{\mathrm{lv}}$ is self-enriched with an $\mathrm{SP}^{\mathrm{lv}}$.
  - This also makes Sp^Σ a simplicial model category, under the adjunction $\mathsf{sSet}_* \leftrightarrow \mathrm{Sp}^\Sigma$.

The stable model structure

- Let $\lambda : F_1 S^1 \longrightarrow F_0 S^0 \cong S$ be the map of symmetric spectra adjoint to the identity map on the first level.
- $Z \in \mathrm{Sp}(\mathcal{C})$ is an **Ω-spectrum** if it is level-wise fibrant, and $Z \cong Z^{F_0 S^0} \longrightarrow Z^{F_1 S^1}$ is a level equivalence (i.e. $Z_n \longrightarrow \Omega Z_{n+1}$ is a weak equivalence in $\mathcal{C}$ for all n).

[10] Think: f is linear *iff* the following commutes:
$$\begin{tikzcd} S \otimes X \arrow[r, "m_X"] \arrow[d, "1\otimes f"] & X \arrow[d, "f"] \\ S \otimes Y \arrow[r, "m_Y"] & N \end{tikzcd}$$

$\rightsquigarrow$ $g : A \longrightarrow B$ is a **stable equivalence** if

$$\mathrm{Hom}_{\mathrm{Sp}(C)}(A^c, Z) \longrightarrow \mathrm{Hom}_{\mathrm{Sp}(C)}(B^c, Z)$$

is a level equivalence of symmetric spectra for all Ω-spectra $Z \in \mathrm{Sp}(\mathcal{C})$, where $(—)^c$ is cofibrant replacement.

$\rightsquigarrow$ A map is a **stable cofibration** if it is a cofibration in the level model structure. The stable fibrations follow.

$\rightsquigarrow$ *Let $\mathcal{C}$ be a simplicial, cofibrantly generated, proper model category. Then these specifications define a spectral model category structure on* $\mathrm{Sp}(\mathcal{C})$, *and the functors* $\Sigma^\infty = F_0$ *and* $\Omega^\infty = \mathrm{Ev}_0$ *give a Quillen adjunction (so that Σ^∞ is a stabilisation):*

$$\Sigma^\infty : \mathcal{C} \longleftrightarrow \mathrm{Sp}(\mathcal{C}) : \Omega^\infty.$$

In addition, if $\mathcal{C}$ is already stable, this is a Quillen equivalence.

$\rightsquigarrow$ **Disclaimer:** Want to define stable equivalences f to be those maps for whom Uf is a stable equivalence of spectra, where $U : \mathrm{Sp}^\Sigma \longrightarrow \mathrm{Sp}^\mathbb{N}$ is the forgetful functor. I.e. if Uf induces an isomorphism of stable homotopy groups. *This fails!* Instead, f is a stable equivalence *iff* E^*f is an isomorphism for every generalised cohomology theory E. *This gets repaired in the category of orthogonal spectra.*

Proof. It's enough to show:

1. That if $f : X \longrightarrow Y$ is a map of fibrant objects (Ω-spectra) and $\mathrm{Ev}_0 f$ is a weak equivalence, then so is f.

   $\rightsquigarrow$ Suppose that f_0 is a weak equivalence, for X, Y Ω-spectra. Then have

$$\begin{tikzcd} X_0 \arrow[r, "\sim"] \arrow[d, "\sim"'] & Y_0 \arrow[d, "\sim"] \\ \Omega^n X_n \arrow[r] & \Omega^n Y_n \end{tikzcd}$$

   so that the bottom map $\Omega^n f_n$ is a weak equivalence. As Ω is part of a Quillen equivalence (**stability**), this shows f_n is a weak equivalence. Thus f is even a level equivalence!

2. If $A \in \mathcal{C}$ is fibrant, $A \longrightarrow \mathrm{Ev}_0((\Sigma^\infty A)^f)$ is a weak equivalence (fibrantly replace in $\mathrm{Sp}^\Sigma(\mathcal{C})$).

   $\rightsquigarrow$ For 2, as $\mathcal{C}$ is stable, $\Sigma^\infty_f A$ (take suspension spectrum and level-wise fibrantly replace) is in fact an Ω-spectrum, thus stably fibrant. Then need to look at $A \longrightarrow \mathrm{Ev}_0 \Sigma^\infty_f A = A^f$, which is just fibrant replacement of A, a weak equivalence. $\square$

"An Overview Of $(\infty, 1)$-Categories" — Saul Glasman — 2/11/2011

Abstract

The many-faceted formalism of $(\infty, 1)$-categories is, much like the theory of Quillen adjunctions between model categories, a way of discussing relations between homotopy theories. First I'll discuss why we might want to think of homotopy theories as higher categories, and I'll mention some examples we've already seen. We'll see how this perspective can clarify the concept of homotopy limits. Our camera does not have a big enough lens to take a comprehensive picture of the theory of $(\infty, 1)$-categories, but we'll survey some common models for this theory and sketch the constructions of some functors between them. If time permits, we will mention stable $(\infty, 1)$-categories.

What's a homotopy theory?

Options

- $\rightsquigarrow$ A model category?
- $\rightsquigarrow$ A simplicial model category?
- $\rightsquigarrow$ A simplicial category? (It often feels that this is good enough)
- $\rightsquigarrow$ A category enriched in topological spaces? (as this should be the same)

Common features

- $\rightsquigarrow$ Mapping spaces
- $\rightsquigarrow$ Homotopy colimits and limits
- $\rightsquigarrow$ being tensored and cotensored over spaces

The idea of inf1cats is to tie this all together somehow.

To motivate further, we have the notion that spaces shold be the same as ∞-groupoids — infinity categories where all of the morphisms of a certain level are invertible.

By an inftycat I mean any notion where you have n-morphisms for each n.

Given a space, have its fundamental ∞-groupoid, $\underline{\pi}_1 X$, whose objects are points, 1-morphisms are paths, 2-morphisms are homotopies, etc. There should be some realisation functor the other way.

Maybe a category enriched over spaces should be an infty1cat:

Definition. An $(\infty, 1)$-category is an ∞-category with all n-morphisms for $n \geq 2$ invertible. [Figuring out what invertible means is a feature of the model.]

Simplicial categories fit the bill, taking fibrant replacements, but taking homotopy limits is hard. Morally, the problem with simplicial categories is that there is a composition function $\underline{\mathrm{Map}}(X, Y) \times \underline{\mathrm{Map}}(Y, Z) \longrightarrow \underline{\mathrm{Map}}(X, Z)$, and this is actually defined, not defined up to homotopy. This is too much rigidity, and we must pay the price.

Kan complexes

Kan complexes are the fibrant simplicial sets in the usual model structure, characterised by the following. Define $\Lambda^n_k \subset \Delta^n$ to be Δ^n less the k^{th} codimension one face and everything that attaches to it (i.e. the interior). These inclusions form a generating set of acyclic cofibrations.

Nerves of categories

There's a similar characterisation of nerves of categories. A simplicial set X is $N\mathcal{C}$ for some category $\mathcal{C}$ *iff* we have a unique lifting property for inner horns $(0 < k < n)$:

$$\begin{tikzcd}
c1 & \Lambda^n_k \arrow[r] \arrow[d] & X & c4 & c5 & c6 \\
d1 & \Delta^n \arrow[ur, "!"] & d3 & d4 & d5 & d6
\end{tikzcd}$$

Given a simplicial set like this, I can get a category back by saying that the objects are the 0-simplices, the morphisms are the 1-simplices, and then evrything else is determined.

Definition. A quasicategory (popularised by Joyal, Lurie) is a simplicial set with the inner horn lifting property. We cal the 0-simplices 'objects', and the 1-simplices 'maps'.

[We forgot uniqueness, since we don't *want* uniqueness.] Every Kan complex is one of these, in fact an ∞-groupoid.

We have mapping spaces, which are $\underline{Map}(X,Y)_n$ the $n+1$-simplices with vertices $x, y, y, y, y,$, and the face opposite x totally degenerate — you get it from a 0-simplex... ?? Think later

Theorem (joyal)**.** *X is a quasicategory* iff $\underline{Map}(\Delta^2, X) \overset{\sim}{\twoheadrightarrow} \underline{Map}(\Lambda^2_1, X)$ *is an acyclic fibration of simplicial sets.*

In particular it has a section, which corresponds to a composition law. There are choices of composition, there's no canonical choice — just a contractible choice.

Homotopy limits and colimits

Imagine we've succeeded in turning some bunch of interesting objects (e.g. spaces) into a quasicategory. We'll start with the easiest limit, the ho-terminal object.

Suppose that $a \in \operatorname{ob} X$, define X/a (the Qcat of objects over a) so that the n-simplices are $n+1$-simplices of X with $n+1$th vertex a. There's a map back from here to X given by taking the first n vertices. This is still a qcat.

Theorem (ThmDef)**.** *Suppose X a qcat and $a \in X$. TFAE:*

1. *for any $x \in \operatorname{ob} X$, $\underline{Map}(x, a)$ is contractible.*
2. *Say $f : \partial\Delta^n \longrightarrow X$ has $f(a) = a$. Then f extends to Δ^n.*
3. *The natural $X/a \longrightarrow X$ is a weak equivalence of qcats.*

In this setting, we call a a terminal object. The dual is true for initial objects.

This is not unique, but the mapping spaces are contractible, and all the maps therein are 'invertible'. [Once you take the homotopy category, all the sections give the same composition law, and a map is an iso if its image in $\underline{\pi}_0$ is an isomorphism).

Now, defining homotopy limits is not too much more work. Let $D \longrightarrow X$ be a diagram in X (here, D is the nerve of a category, the one I'd normally think of). We can form the qcat of cones over D, in the same spirit as that of the objects over a. [Here, a 0-simplex is a cone over D (a map from the projective cone $1 * D$ on D (join D with a point as a simp set, which is basically D with an initial object adjoined).] This is still a qcat and we define

Definition. *holimD* is defined to be the terminal object of X/D, which is defined up to weak equivalence.

The model structure on qcats

This is not a model structure on the cat of qcats, but one on the cat of ssets.

Theorem (Thmdef)**.** *There is a 'Joyal' model structure on* sSet *where the cofibrations are the inclusions (as usual), the fibrant objects are the qcats, and the weak equivalences are 'the weak equivs according to the quasi-categories'. That is, maps* $Y \longrightarrow Z$ *such that* $\underline{Map}(Z, X) \xrightarrow{\sim} \underline{Map}(Y, X)$ *(a w.e. in usual sSets) for all QCs* X.

If you're a weak equiv for QCs you're a weak equiv for kan complexes, so that the condition of being a weak equivalence is stronger than the classical condition. This is a Bousfield delocalisation of the classical model structure. That is, if we B localise w.r.t. the classical weak equivalences, we get the standard model structure. Thus there must be more fibrations in the Joyal structure, so that at least the

The weak equivalence with simplicial categories

We have a model category of simplicial categories, with the Bergner model structure. This is Quillen equivalent to the Joyal model structure on simplicial sets.

The problem with simp cats is that they were strict, so we must destrictify. We'll bring in the 'coherent nerve'. The n-simplices of the coherent nerve $N_h(\mathcal{C})$ are "paths up to homotopy" in $\mathcal{C}$. Define $P^h[n]$ to be the category whose objects are $\{0, \ldots, n\}$, and $\mathrm{Map}(i, j) = N(\mathcal{P}(i{+}1, \ldots, j{-}1))$. A k-simplex says 'I went from i to j and stopped k times'. I can take the nerve as I'm thinking of this as a poset, and thus a category.

The homotopy coherent nerve $N^h(\mathcal{C})$ is the simplicial set with k-simplices functors $P^h[n] \longrightarrow \mathcal{C}$.

Theorem. *If* $\mathcal{C}$ *is fibrant, then* $N^h(\mathcal{C})$ *is a QC, and the coherent nerve is the right half of a Quillen equivalence*

$$QC \longrightarrow sC.$$

“Weak Associativity and Complete Segal Spaces” — Geoffroy Horel — 9/11/2011

Abstract

In this talk I will try to show that the category Δ arises very naturally when we want describe homotopy coherent associativity in simplicial categories. After quickly reviewing the equivalence between Segal Δ-spaces and A_∞-spaces, I will describe a model category discovered by Rezk on simplicial spaces that is Quillen equivalent to the model category of quasi-categories. The fibrant objects are called complete Segal spaces and provide another model for $(\infty, 1)$-categories. If there is enough time I will also talk about the higher dimensional version that gives a model for (∞, n)-categories.

Definition. A Segal space is a functor $\Delta^{\mathrm{op}} \longrightarrow \mathscr{S} = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{Set})$ which is Reedy fibrant and satisfies the Segal condition.

Example. Suppose that $\mathcal{C}$ is a simplicial category. One can take its nerve, a simplicial space:

$$N(\mathcal{C})_n = \coprod_{x_1,\ldots,x_n} \mathrm{map}_{\mathcal{C}}(x_0, x_1) \times \cdots \times \mathrm{map}_{\mathcal{C}}(x_{n-1}, x_n)$$

Lemma. *The category Δ^{op} is isomorphic to the category I whose objects are totally ordered finite sets with at least two elements, and whose morphisms are order preserving maps which preserve the smallest and largest elements.*

Proof. A fun exercise. □

The nerve has two properties:

1. $(N\mathcal{C})_0$ is discrete.
2. $\varphi_n : (N\mathcal{C})_n \longrightarrow (N\mathcal{C})_1 \times_{(N\mathcal{C}_0)} \cdots \times_{(N\mathcal{C}_0)} (N\mathcal{C})_1$ is an isomorphism.

To construct φ_n, we give n maps $\alpha_i : [n] \longrightarrow [1]$. We should think of $[n]$ as $\{0, \ldots, n+1\}$ (!). Then α_i sends $\{0, \ldots, i\}$ to 0, and the rest to 1. We can check that $d_0\alpha_i = d_1\alpha_{i-1}$, and so for any object $X_* \in s\mathscr{S}$, we obtain such a φ_n.

Theorem (Segal, May, Thomason)**.** $\mathrm{ho}(A_\infty{-}\mathsf{spaces})$ *is the same as the homotopy category of Reedy fibrant simplicial spaces satisfying the Segal condition with $X_0 = *$.*

Definition. A Segal space is a Reedy fibrant object of $s\mathscr{S}$ with $\varphi_n : M_n \xrightarrow{\sim} W_1 \times_{W_0} \cdots \times_{W_0} W_1$ a weak equivalence.

Definition. Let W be a Segal space. Define $\mathrm{Ob}(W) := (W_0)_0$. For two objects x, y of W, as (fact) $(d_0, d_1) : W_1 \longrightarrow W_0 \times W_0$ is a fibration, we define $\mathrm{Map}_W(x, y)$ to be the fibre over (x, y) of (d_0, d_1). More generally, given $x_0, \ldots, x_n \in \mathrm{Ob}(W)$, we can form $(\delta_0, \ldots, \delta_n) : W_1 \longrightarrow W_0^{n+1}$, a fibration (in the ordinary model structure on simplicial sets), and define

$$\mathrm{Map}_W(x_0, \ldots, x_n) = \text{the fiber above } (x_0, \ldots, x_n) \text{ of this map.}$$

Now

$$\begin{tikzcd}
W_1 & & \\
\mathrm{Map}_W(x_0,x_n) \arrow[u, hook] & \mathrm{Map}_W(x_0,\dots,x_n) \arrow[l] \arrow[r, hook] \arrow[d] & W_n \arrow[ull] \arrow[d, "\sim, fibrat"', "\varphi_n"] \\
& \mathrm{Map}(x_0,x_1)\times\cdots\times\mathrm{Map}(x_{n-1},x_n) \arrow[r, hook] & W_1\times_{W_0}\cdots\times_{W_0} W_1
\end{tikzcd}$$

The space of sections of φ_n is contractible, so that there is a well defined composition up to homotopy.

The homotopy category of a Segal space

ho$W$ has objects $(W_0)_0$, and $\mathrm{Hom}_{\mathrm{ho}W} = \pi_0 \,\mathrm{Map}_W(x,y)$.

Completeness

We need a functor $\mathsf{Cat} \longrightarrow (\infty,1)-\mathsf{Cat}$. If $\mathcal{C}$ is an ordinary category, then $\mathrm{ob}\,\mathcal{C}$ is not an invariant under equivalence. Weak equivalence between Segal spaces then *cannot* be levelwise.

Taking the connected components of $\mathcal{C}$ *is* an invariant. More generally, suppose that $(\mathcal{C}, W)$ is a pair with $\mathcal{C}$ a category, and W is a subcategory which contains all identities (like a set of weak equivalences). I'd like a 'nerve' functor, taking as input such data, and returning a Segal space $N(\mathcal{C}, W)$. Let's try to write the 0 space. It should be like the space of objects of $\mathcal{C}$ up to weak equiv.

$$N(\mathcal{C}, W)_0 := N(W).$$

$$N(\mathcal{C}, W)_1 = N\mathcal{D}$$

Where the objects of $\mathcal{D}$ are arrows of $\mathcal{C}$ (i.e. objects of $\mathcal{C}^{[1]}$), and whose morphisms are commuting squares with verticals lying in W. I.e. arrows up to weak equivalences.

More generally, $N(\mathcal{C}, W)_{m,n}$ is the set of $n \times n$ grids whose points represent objects of $\mathcal{C}$, with vertical arrows in W, and horizontal arrows in $\mathcal{C}$. This is a bisimplicial set, by taking composition in each direction. This gives a construction for the nerve we wanted. One can check that $N(\mathcal{C}, W)$ is a Segal space, and it is also **complete**.

If $\mathcal{C}$ is a category, can form $N(\mathcal{C}, \text{identities})$, and $N^h(\mathcal{C}, \text{isomorphism})$, which is the equivalence invariant notion.

Completeness, ctd.

One has a map $W_1 \longrightarrow (\mathrm{ho}W)^{[1]}$.

Definition. A point in W_1 is a homotopy equivalence if it is sent to an isomorphism in the homotopy category (ho$W$). $(W_1)_{hoequiv} \subset W_1$ is the union of all components of W_1 containing a homotopy equivalence.

We have $W_0 \xrightarrow{s_0} W_1$, and there's a lift:

$$\begin{tikzcd} W_0 \arrow[r, dashed] \arrow[dr] & (W_1)_{hoequiv} \arrow[d] \\ & W_1 \end{tikzcd}$$

Definition. A Segal space is complete if this lift is a weak equivalence.

Theorem (Rezk)**.** *There is a model category structure on $s\mathscr{S}$ whose cofibrations are monomorphism, whose fibrant objects are complete Segal spaces, and whose weak equivalences are the maps $f : X \longrightarrow Y$ such that for all complete Segal spaces W, $\mathrm{Map}(Y, W) \xrightarrow{\sim} \mathrm{Map}(X, W)$ is a weak equivalence.*

If X, Y are complete Segal spaces, then $X \longrightarrow Y$ is a weak equivalence iff *it's a levelwise equivalence. If X, Y are Segal spaces, we can characterise weak equivalences $f : X \longrightarrow Y$ as the maps such that for all $x, y \in \mathrm{ob}\,(X)$, $\mathrm{Map}_X(x, y) \xrightarrow{\sim} \mathrm{Map}_Y(fx, fy)$ and the induced map* $\mathrm{ho}X \longrightarrow \mathrm{ho}Y$ *is an equivalence of categories.*

Theorem (Joyal, Tierney)**.** $p_1 : \Delta \times \Delta \longrightarrow \Delta$ *has a right adjoint i_1, where $i_1[m] = ([m][0])$, taking opposites, one obtains an adjunction*

$$s\mathscr{S} \underset{i_1^*}{\overset{p_1^*}{\rightleftarrows}} \mathscr{S}$$

This is a Quillen equivalence between the model category of CSS and quasi-categories.

[All you need to know is the bottom row of your bisimplicial set, and you can reconstruct the rest.

$$\begin{tikzcd} \mathsf{Top} \arrow[r] \arrow[dr] & s\mathsf{Set} \\ & s\mathsf{Top} \arrow[u] \end{tikzcd}$$

]

Try Julie Bergner, 'a survey of $(\infty, 1)$-categories'.

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & B \arrow[d] \\ & C \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$