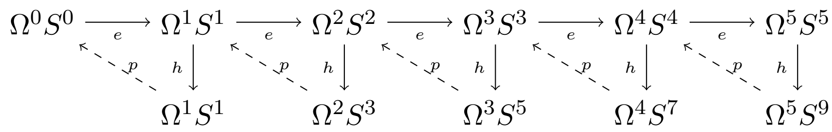
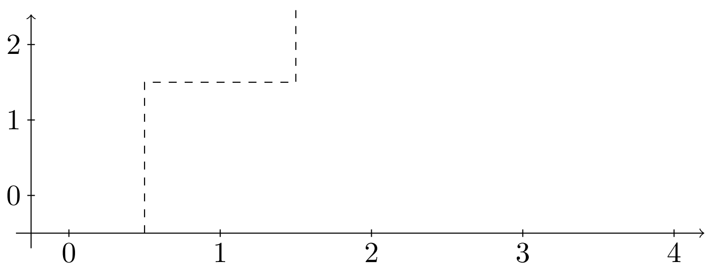
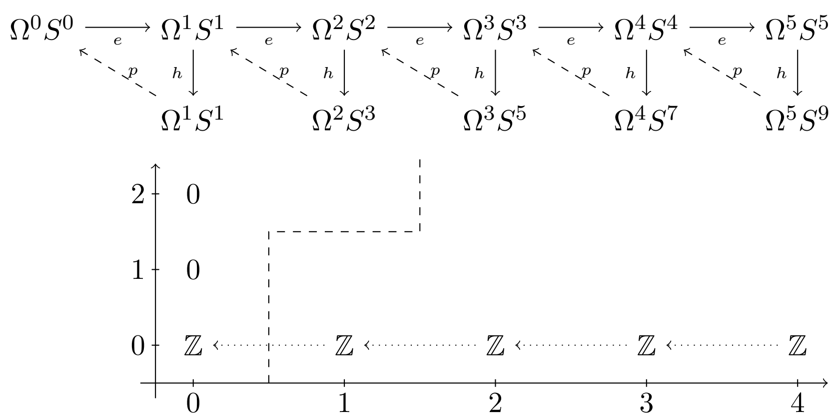


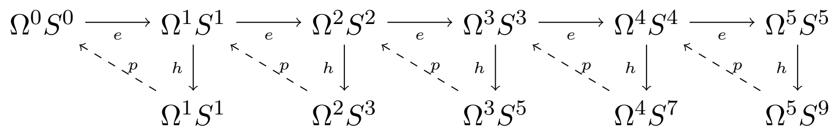
$\Omega^0 S^0$
$\Omega^1 S^1$
$\Omega^2 S^2$
$\Omega^3 S^3$
$\Omega^4 S^4$
$\Omega^5 S^5$
e
p
h
$\Omega^1 S^1$
$\Omega^2 S^3$
$\Omega^3 S^5$
$\Omega^4 S^7$
$\Omega^5 S^9$

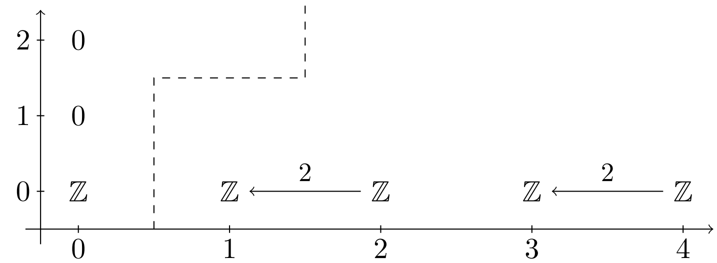


2
1
0
0
1
2
3
4



We know that the 0-stem is $\mathbb{Z}$ on any sphere S^n with $n > 1$, by the Hurewicz theorem. We know the homotopy groups of S^1, by covering space theory.



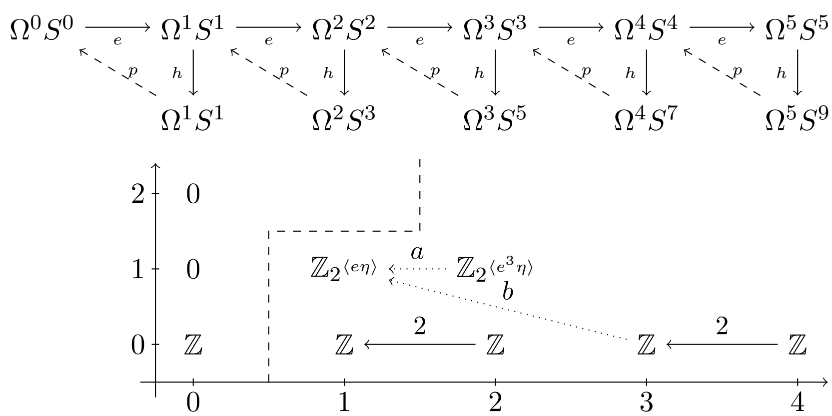


1-stem: $0 \longrightarrow \mathbb{Z}\langle\eta\rangle \twoheadrightarrow \mathbb{Z}_2\langle e\eta\rangle$

To calculate the differential $E^1_{p,0} \longrightarrow E^1_{p-1,0}$, we take the element $\iota \in \pi_p(\Omega^{p+1}S^{2p+1})$, apply p, to obtain $p(\iota) = w_p \in \pi_{p-1}(\Omega^p S^p)$, and then apply h, to obtain $h(w_p) \in \pi_{p-1}(\Omega^p S^{2p-1})$. The Hopf invariant of the whitehead square w_p is two when p is even and zero when p is odd.

Now we are able to calculate the 1-stem. Truncating to one column calculates the 1-stem on S^1, and we see (again) that this is zero, as there is nothing in the $p+q=1$ diagonal. Truncating to two columns, we obtain a copy of $\mathbb{Z}$. Truncating at three columns, we allow an additional differential, which kills $2\mathbb{Z} \subset \mathbb{Z}$. After this, no more differentials will ever be involved, and so the 1-stem is stable.

Note also that the generator of $\pi_1(\Omega^2 S^2)$ maps to $\iota \in \pi_1(\Omega^2 S^3)$ under h. Thus it is a Hopf invariant one element on S^2, commonly denoted η. Moreover, $\pi_1(\Omega^3 S^3)$ is generated by $e\eta$.



1-stem: $0 \longrightarrow \mathbb{Z}\langle\eta\rangle \twoheadrightarrow \mathbb{Z}_2\langle e\eta\rangle$

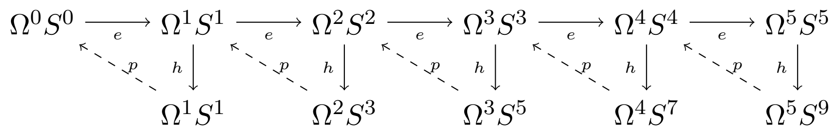
As we have calculated the 1-stem on S^3 and all higher spheres, we can fill in $E^1_{1,1}$ and $E^1_{2,1}$. Then two more differentials are possible between groups that we have written down.

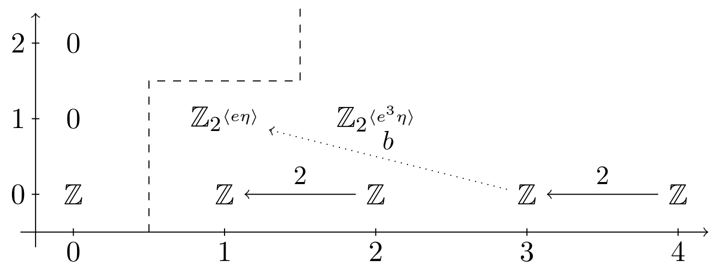
Firstly, a is zero. To see this, we note that $p(e^3\eta) = w_2 \circ (e\eta)$, where $w_2 \in \pi_3(S^2)$ and $e\eta \in \pi_4(S^3)$. Now we have calculated the 1-stem on S^2 to be $\mathbb{Z}$, generated by the Hopf invariant one element, η. As w_2 has Hopf invariant two, $w_2 = 2\eta$. Thus:

$$p(e^3\eta) = (2\eta) \circ (e\eta) := \eta \circ (2\iota_3) \circ (e\eta) \overset{*}{=} \eta \circ (e\eta) \circ (2\iota_4) =: 2(\eta \circ (e\eta)).$$

So $d_1(e^3\eta) := h(p(e^3\eta)) = 2\,h(\eta \circ (e\eta))$, and as the target is $\mathbb{Z}_2$, this is zero.

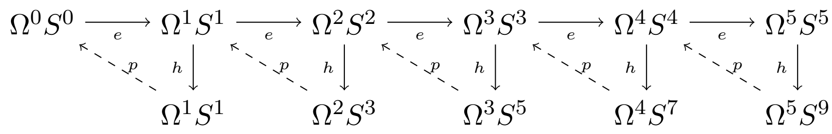
Note that there is a subtlety in the above, at the symbol $\overset{*}{=}$. We can only make this manipulation since $e\eta$ is a suspension. I'll write this up later.

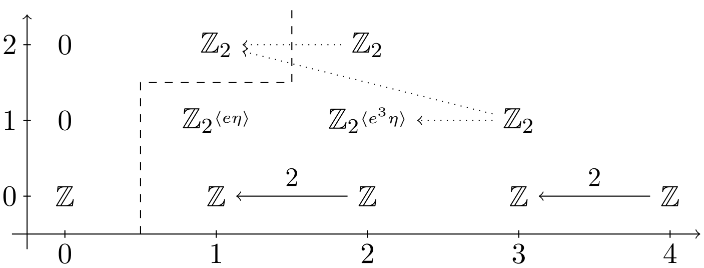




1-stem: $\quad 0 \longrightarrow \mathbb{Z}\langle\eta\rangle \longrightarrow\!\!\!\!\rightarrow \mathbb{Z}_2\langle e\eta\rangle$

To see that the differential b vanishes, recall that there is an element ν of Hopf invariant one in $\pi_3(\Omega^4S^4)$, so that $h(\nu) = \iota \in \pi_3(\omega_4S^7)$. In particular, $p : \pi_3(\omega_4S^7) \longrightarrow \pi_2(\omega_3S^3)$ is zero, so that no nonzero differentials ever leave $E^1_{3,0}$.

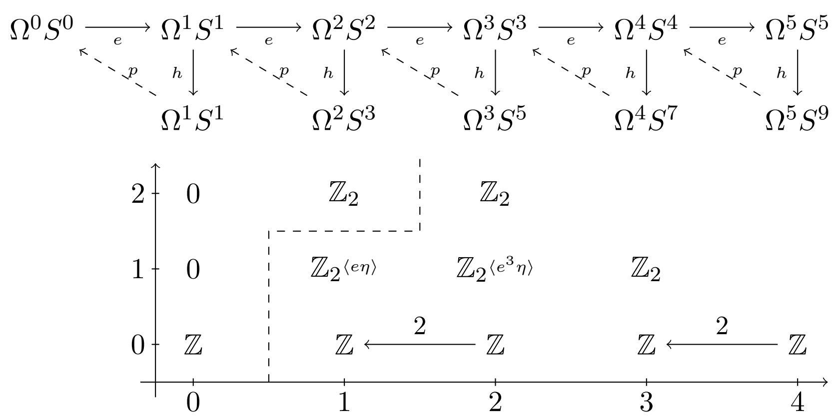




1-stem: $0 \longrightarrow \mathbb{Z}\langle\eta\rangle \twoheadrightarrow \mathbb{Z}_2\langle e\eta\rangle$

2-stem: $0 \longrightarrow \mathbb{Z}_2 \xrightarrow{\simeq} \mathbb{Z}_2 \xrightarrow{\simeq} \mathbb{Z}_2$

With all of the differentials involving the $p+q=2$ diagonal calculated, we can read off the 2-stem. Knowledge of the 2-stem allows us to fill in the row $E^1_{*,2}$, and we write in two groups soon to be of interest. We also write in another group of interest in the 1-stem, and add in all the possible nonzero differentials between groups we can see.



$$\text{1-stem:} \qquad 0 \longrightarrow \mathbb{Z}_{\langle \eta \rangle} \longrightarrow\!\!\!\!\!\rightarrow \mathbb{Z}_{2\langle e\eta \rangle}$$

$$\text{2-stem:} \qquad 0 \longrightarrow \mathbb{Z}_2 \xrightarrow{\simeq} \mathbb{Z}_2 \xrightarrow{\simeq} \mathbb{Z}_2$$

$$\text{3-stem:} \qquad 0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_4 \oplus \mathbb{Z} \longrightarrow \mathbb{Z}_8$$

In order to see that these three differentials all vanish, we cheat a little, assuming that we know that the stable 3-stem is $\mathbb{Z}_8$. Given that information, none of these three differentials could be nonzero, as if they were, the cardinality of the stable 3-stem would have to be strictly less than 8.

The extension problems can all be solved easily given that we are aiming at $\mathbb{Z}_8$, so we can calculate the whole 3-stem.