Hartshorne I.1 — Affine Varieties

⇝ A nonempty open subset of an irreducible space is irreducible and dense.
⇝ The closure of an irreducible subspace is irreducible.
⇝ For $a \subset A$, $I(Z(a)) = \sqrt{a}$, by the Nullstellensatz. $Z(I(Y)) = \overline{Y}$.
⇝ Closed sets correspond to radical ideals. A closed set is irreducible iff it corresponds to a prime ideal. (A topological space is irreducible if it is not the union of two proper closed subsets).
⇝ Affine varieties correspond to finitely generated $k$-algebras which are domains.
⇝ In a noetherian topological space, closed sets are uniquely finite unions of irreducible ones.
⇝ The dimension of a space is one less than the length of the longest chain of distinct irreducible closed subsets. This coincides with the dimension of the coordinate ring.
⇝ Theorem 1.8A: For $k$ a field, $B$ a f.g. $k$-algebra which is a domain:
   • $\dim(B)$ is the transcendence degree of $K(B)/k$.
   • $\text{height } p + \dim B/p = \dim B$.
⇝ Theorem 1.11A: (Hauptidealsatz) Let $A$ be noetherian, and $f \in A$ be neither zero divisor nor unit. Then if $\mathcal{I}$ is the set of prime ideals containing $f$, any minimal ideal in $\mathcal{I}$ in fact has height one.
   Interpretation if $A$ is a UFD — $Z(f)$ can be written uniquely as a union of hypersurfaces $Z(f_i)$. Any maximal closed irreducible subset of $Z(f)$ has codimension one in Spec $A$ — the only larger irreducible is all of $A$.
⇝ Proposition 1.12A: A noetherian domain $A$ is a UFD iff every prime ideal of height one is principal.
   Interpretation — a noetherian domain $A$ is a UFD iff every maximal proper closed irreducible subset of Spec $A$ is a hypersurface, the zero set of one element.
⇝ A variety $Y \subset A^n$ has dimension $n - 1$ iff it is a hypersurface: $Z(f)$ for some nonconstant irreducible $f$. The same holds for a projective variety in $\mathbb{P}^n$ of dimension $n - 1$.

Hartshorne I.2 — Projective Varieties

⇝ To talk about $\mathbb{P}^n$, introduce graded ring $S = k[x_0, \ldots, x_n]$, zero sets of homogeneous polynomials, and of homogeneous ideals.
⇝ Constructed homeomorphism $U_i \rightarrow \mathbb{A}^n$. Note that if $Y$ is a (quasi)-projective variety, it is covered by the (quasi)-affine varieties $Y \cap U_i$.
⇝ Closed subsets of $\mathbb{P}^n$ correspond to homogeneous radical ideals of $S$ not equal to $S_+$, the irrelevant maximal ideal.
⇝ Given a variety $Y \subset A^n$, the closure $\overline{Y}$ of $Y$ in $\mathbb{P}^n$ is called the projective closure. Its ideal is generated by $\beta(I(Y))$, where $\beta : A \rightarrow S$ maps $x_i$ to $x_i/x_0$.
⇝ Have $d$-uple embedding $\mathbb{P}^n \rightarrow \mathbb{P}^N$ sending $[x_0 : \cdots : x_n]$ to all the degree $d$ monomials. The image of the 2-uple embedding $\mathbb{P}^2 \rightarrow \mathbb{P}^5$ is called the Veronese surface. The image of the 3-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$ is the twisted cubic curve.
⇝ Have the Segre embedding $\mathbb{P}^r \times \mathbb{P}^s \rightarrow \mathbb{P}^{rs+r+s}$. The quadric surface in $\mathbb{P}^3$, defined by $xy = zw$, is the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3$. 

1
Hartshorne I.3 — Morphisms

⇝ A function \( f : Y \rightarrow k \) is regular if it is locally a quotient of (homogeneous in projective case) polynomials of the same degree with nonvanishing denominator. A regular function is continuous.

⇝ A morphism of varieties is a continuous map \( f : X \rightarrow Y \) such that if \( \varphi : V \rightarrow k \) is a regular function for open \( V \subset Y \), then \( \varphi \circ f \) is regular of \( f^{-1}(V) \). That is, \( f \) must map the sheaf \( \mathcal{O}_Y \) of regular functions on \( Y \) into \( \mathcal{O}_X \) (as subsheaves of the sheaves of discontinuous functions).

⇝ For \( p \in Y \), have the local ring \( \mathcal{O}_{p,Y} \), the stalk of \( \mathcal{O}_Y \) at \( p \). The function field is the local ring at the generic point — here we should define it to be the direct limit of the \( \mathcal{O}_Y(U) \) over all nonempty open subsets \( U \) of \( Y \). Elements of the function field are called rational functions. Note that by restriction, we have injections \( \mathcal{O}_Y(Y) \rightarrow \mathcal{O}_{p,Y} \rightarrow K(Y) \).

⇝ Note that the function field of a product is the quotient field of the tensor product of the function fields.

⇝ **Theorem 3.2:** For \( Y \) affine, \( \mathcal{O}_Y(Y) = A(Y) \). \( \mathcal{O}_{p,Y} \cong A(Y)_{m_p} \), and \( \dim \mathcal{O}_{p,Y} = \dim Y \). \( p \mapsto m_p \) gives a bijection between points of \( Y \) and maximal ideals of \( A(Y) \). Finally, \( K(Y) \cong A(Y)_{(0)} \), so that \( K(Y) \) is a finitely generated extension of \( k \), of transcendence degree \( \dim Y \).

⇝ **Theorem 3.4:** For \( Y \) projective, \( \mathcal{O}_Y(Y) = k \), \( \mathcal{O}_{p,Y} = S(Y)_{(m_p)} \), and \( K(Y) = S(Y)_{(0)} \). Here we are using degree zero localizations.

⇝ **Proposition 3.5:** There is a natural isomorphism, for varieties \( X \) and affine varieties \( Y \), \( \text{Hom}(X,Y) \cong \text{Hom}(A(Y),\mathcal{O}(X)) \). In fact, if \( V \) is the category of varieties over \( k \), and \( D \) is the category of finitely generated integral domains over \( k \), there is a contravariant adjunction \( \mathcal{O} : V \leftrightarrow D^{\text{op}} : \text{Spec} \), which induces an isomorphism of categories between \( D^{\text{op}} \) and the full subcategory of \( V \) consisting of affine varieties.

Hartshorne I.4 — Rational Maps

⇝ Morphisms which agree on a nonempty open subset of a variety are equal.

⇝ A rational map \( X \rightarrow Y \) is an equivalence class of morphisms defined on a nonempty open subset of \( X \). A rational map is dominant if its image is dense (i.e. for some (equivalently any) choice of open subset). A birational map is a rational map with an inverse as rational maps. A variety is simply rational if it is birational to a projective space.

⇝ The complement of a hypersurface \( Z(f) \subset \mathbb{A}^n \) is a hypersurface \( Z(x_{n+1}f) \subset \mathbb{A}^{n+1} \) with coordinate ring \( k[x_1, \ldots, x_n]_f \).

⇝ A variety has a base for its topology consisting of open affine subsets.

⇝ **Theorem 4.4:** A dominant rational map induces a map of function fields, and the resulting correspondence is bijective, giving an contravariant equivalence of categories from “varieties with DRMs” to “finitely generated field extensions of \( k \)”.

⇝ **Corollary 4.5:** Varieties are birational iff they have isomorphic open subsets iff they have isomorphic function fields.

⇝ The blowup is defined — to blow up \( 0 \in Y \subset A^n \), use composite \( \varphi : X \rightarrow \mathbb{A}^n \times \mathbb{P}^n \rightarrow \mathbb{A}^n \), where \( X \) is defined by equations \( x_iy_j = x_jy_i \). Take \( Y = \varphi^{-1}(Y - 0) \).

⇝ Rational functions and more generally rational maps have a maximum domain of definition.

⇝ If \( p \in X \) and \( q \in Y \) are such that \( \mathcal{O}_{p,X} \) and \( \mathcal{O}_{q,Y} \) are isomorphic \( k \)-algebras, then there is a
Hartshorne I.5 — Nonsingular Varieties

⇝ If \( Y \in \mathbb{A}^n \) is an affine variety with ideal generated by \( f_1, \ldots, f_t \), then \( Y \) is nonsingular at \( p \in Y \) if the rank of the Jacobian \( \| (\partial f_i/\partial x_j)(p) \| \) is \( n - \dim Y \). (It may be lower).
⇝ If \( A \) is a noetherian local ring, we say \( A \) is a regular local ring if \( \dim_{A/m} m/m^2 = \dim A \). Note that \( \geq \) holds always.
⇝ Theorem 5.1: \( Y \in \mathbb{A}^n \) is nonsingular at \( p \) iff \( \mathcal{O}_{p,Y} \) is a regular local ring. Thus the notion of nonsingularity is intrinsic and extends to all varieties.
⇝ Theorem 5.3: For any variety \( Y \), the set \( \text{Sing} \ Y \) of singular points is a proper closed subset.
⇝ To study very local behaviour, we take completions. The completion of a local ring \( (A, \mathfrak{m}) \) is \( \hat{A} := \varprojlim A/m^n = \varprojlim \{ \cdots \to A/m^3 \to A/m^2 \to A/m \} \).
⇝ Theorem 5.4A: Suppose \( A \) is noetherian. \( \hat{A} \) is local with maximal ideal \( \mathfrak{m} \hat{A} \), and \( A \to \hat{A} \) is injective. For finitely generated \( A \)-modules \( M \), \( M \otimes_A \hat{A} \) is the \( \mathfrak{m} \)-adic completion of \( M \).
⇝ Theorem 5.5A: (Cohen Structure Theorem) The only complete regular local ring of dimension \( n \) containing a field \( k \) is the power series ring in \( n \) variables over \( k \). In particular, any two nonsingular points of the same dimension are analytically isomorphic.
⇝ Theorem 5.7A: (elimination theory) Given homog. polynomials \( f_1, \ldots, f_r \in k[x_0, \ldots, x_n] \) with indeterminate coefficients \( a_{ij} \), we want to know when they have a common root other than \( (0,0,\ldots,0) \). There a polynomials \( g_1, \ldots, g_t \in \mathbb{Z}[a_{ij}] \), homogeneous in the coefficients of each \( f_i \) separately, such that there is a common nonzero root for the \( f_i \) iff there is a common root for the \( g_i \).
⇝ Exercise 5.4: We define the intersection multiplicity at \( P \in \mathbb{A}^2 \) of the curves \( C_1 \) and \( C_2 \) defined by \( f,g \in k[x,y] \) to be the length of the \( \mathcal{O}_P \)-module \( \mathcal{O}_P/(f,g) \). This makes sense, for we are pulling back:

\[
\begin{array}{ccc}
C_1 \cap C_2 & \longrightarrow & C_2 \\
\downarrow & & \downarrow \\
C_1 & \longrightarrow & \mathbb{A}^2
\end{array}
\]

We have a closed immersion \( C_1 \cap C_2 \hookrightarrow \mathbb{A}^2 \), and the length of the \( \mathcal{O}_P \)-module \( \mathcal{O}_{C_1 \cap C_2,P} \) measures the multiplicity at \( P \). However, \( \mathcal{O}_{C_1 \cap C_2,P} = \mathcal{O}_{C_1,P} \otimes_{\mathcal{O}_P} \mathcal{O}_{C_2,P} \), and \( \mathcal{O}_{C_1,P} = \mathcal{O}_P/f \), so that \( \mathcal{O}_{C_1 \cap C_2,P} = \mathcal{O}_P/(f,g) \) as desired.

In higher dimensions, this is not the correct formula for the intersection multiplicity — there are some derived functors which get involved. Maybe the intersection itself should really get ‘derived’.
Hartshorne I.6 — Nonsingular Curves

There is a unique nonsingular projective curve in each birational equivalence class of curves.

For each finitely generated field extension $K/k$ of transcendence degree one, there is then a unique nonsingular projective curve $C_K$ with function field $K$.

Moreover, and homomorphism $K_2 \rightarrow K_1$ over $k$ is represented by a morphism $C_{K_1} \rightarrow C_{K_2}$.

Let $K$ be a field and $G$ be a totally ordered abelian group. A valuation on $K$ with values in $G$ is homomorphism $v : K^\times \rightarrow G$ such that for all nonzero $x, y$: $v(x + y) \geq \min\{v(x), v(y)\}$.

The valuation ring of $v$ is then $\{0\} \cup v^{-1}(\geq 0)$, a local ring whose units are $\ker v$. The valuation ring always has quotient field $K$. Call $v$ a valuation of $K/k$ if $v(k^\times) = \{0\}$.

If $A, B$ are local rings in a field $K$, then $B$ dominates $A$ if $A \subseteq B$ and $m_B \cap A = m_A$.

Theorem 6.1A: A local ring $R$ contained in a field $K$ is a valuation ring of $K$ iff it is a maximal local ring w.r.t. domination. Every local ring inside $K$ is dominated by some valuation ring of $K$.

Theorem 6.2A: Let $A$ be a noetherian local domain of dimension one. TFAE:

1. $A$ is a discrete valuation ring (i.e. comes from a valuation with values in $\mathbb{Z}$);
2. $A$ is integrally closed (i.e. normal);
3. $A$ is a regular local ring (implies 2, even in higher dimensions);
4. The maximal ideal of $A$ is principal.
5. $A$ is a UFD (see next point).

Note that is always the case that a regular local ring is normal. In fact:

$$\text{Regular local ring} \implies \text{UFD} \implies \text{normal/integrally closed}$$

Note that being a UFD is a natural demand for smoothness — an obvious example of local failure to be factorial arises at the origin of the cone $xy = z^2$. Note that this example is normal!

A Dedekind domain is an integrally closed noetherian domain of dimension one. As being integrally closed is a local property, every localization of a Dedekind domain is a DVR.

For $K/k$ a finitely generated extension of transcendence degree one, let $C_K$ be the set of all discrete valuation rings of $K/k$.

Given a point $y$ on a nonsingular curve $Y$ with function field $K$, there’s an injective function $Y \rightarrow C_K$ given by $p \mapsto \mathcal{O}_p \in C_K$.

Theorem 6.5: For any $x \in K$, all but finitely map $R \in C_K$ contain $x$.

Interpretation — $x \notin R$ means that $x$ will have a pole at $R$ (as an element of the ‘curve’ $C_K$). As $K$ is one dimensional, each $x$ has only finitely many poles.

Corollary 6.6: Any DVR of $K/k$ is the local ring of a point on a nonsingular affine curve.

Make $C_K$ a topological space using the cofinite topology. If $U \subseteq C_K$ is open, define $\mathcal{O}(U) := \cap_{P \subseteq U} R_P$. An element $f \in \mathcal{O}(U)$ defines a function $U \rightarrow k$ by taking $P$ to the image of $f$ in the residue field of $\mathcal{O}_P$, which must be $k$ by 6.6.

$f \in \mathcal{O}(U)$ can be recovered from its function $U \rightarrow k$, as if $0 \neq f \mapsto 0$, $f \in \mathfrak{m}_p$ for all $p$, so $f^{-1} \notin R_p$ for all $p$, contradicting 6.5, as there must be infinitely many $R_p$ by 6.6. By 6.5, any $f \in K$ is a regular function on some open $U$. Thus the quotient field of $C_K$ is $K$.

An abstract nonsingular curve is an open subset of $C_K$ with the induced topology and sheaf of regular functions. A morphism of such is a continuous map such that regular functions
Proposition 6.7: Every nonsingular quasi-projective curve is isomorphic to an abstract nonsingular curve.

Theorem 6.9: \(C_K\) is isomorphic to a nonsingular projective curve.

Corollary 6.10: Every abstract nonsingular curve is isomorphic to a quasi-projective curve. Every nonsingular quasi-projective curve is isomorphic to an open subset of a nonsingular projective curve.

Corollary 6.11: Every curve is birationally equivalent to a nonsingular projective curve.

Corollary 6.12: The following categories are equivalent:

(i) Nonsingular projective curves and dominant morphisms;
(ii) Quasi-projective curves and dominant rational maps;
(iii) Function fields of dimension 1 over \(k\) and \(k\)-homomorphisms.

Proposition 6.8: Let \(X\) be a nonsingular curve, \(p \in X\), let \(Y\) be a projective variety, and let \(\varphi: X \setminus \{p\} \to Y\) be a morphism. Then there is a unique extension of \(\varphi\) to a morphism \(X \to Y\). (This fails if \(Y\) is not projective, or if \(\dim X > 1\).)

Using proposition 6.8, one can show that every automorphism of \(\mathbb{P}^1\) is in \(\text{PGL}(1)\).

Hartshorne I.7 — Intersections in Projective Space

Hartshorne II.1 — Sheaves

A presheaf on \(X\) is a contravariant functor from the category of open subsets of \(X\) to abelian groups. There is some contention (Bjorn) as to whether to require \(\emptyset \mapsto 0\). It’s a sheaf if:

3. If \(U\) is open with open cover \(\{V_i\}\), and \(s \in \mathcal{F}(U)\) has \(s|_{V_i} = 0\) for all \(i\), then \(s = 0\).
4. If \(s_i \in \mathcal{F}(V_i)\) agree on overlaps, then they glue (uniquely).

Any subpresheaf of a sheaf has (3). Any presheaf maps to the sheaf of its discontinuous sections, and the kernel of this map consists of sections which violate (3).

A morphism of sheaves is an isomorphism iff it is an isomorphism on the stalks.

The presheaf kernel, cokernel and image are exactly as one would hope.

Given a presheaf \(\mathcal{F}\) there is a universal morphism \(\mathcal{F} \to \mathcal{F}^+\) into a sheaf \(\mathcal{F}^+\) called the sheafification. \(\mathcal{F}\) and \(\mathcal{F}^+\) have the same stalks. The sections of \(\mathcal{F}^+(U)\) can be viewed as sections of \(\bigcup_{p \in U} \mathcal{F}_p \downarrow U\) which are locally induced by sections of \(\mathcal{F}\). If \(\mathcal{F}\) is a sub-presheaf of a sheaf \(\mathcal{G}\), then its sheafification is the intersection of all subsheaves of \(\mathcal{G}\) containing \(\mathcal{F}\).

Given a morphism of sheaves \(\mathcal{F} \to \mathcal{I}\), the kernel is a sheaf. The image need not be a sheaf, so we define the sheaf image to be the smallest subsheaf of \(\mathcal{I}\) containing the presheaf image of \(\mathcal{F}\). We are thus already equipped to discuss exact sequences of sheaves, by demanding that kernel equals sheaf image.

If \(\mathcal{F}' \subset \mathcal{F}\) is an inclusion of sheaves, the quotient sheaf \(\mathcal{F}/\mathcal{F}'\) is defined to be the sheafification of the quotient presheaf. The quotient presheaf has (3) but not (4), as if \([f_i] \in \mathcal{F}(V_i)/\mathcal{F}'(V_i)\) agree on \(V_i \cap V_j\), this only means that they \(f_i - f_j \in \mathcal{F}'(V_i \cap V_j)\), and this element may not extend to \(\mathcal{F}'(V_i \cup V_j)\).

Sections of \((\mathcal{F}/\mathcal{F}')(U)\) are equivalence classes of collections of sections of the presheaf on coverings of \(U\) which agree on overlaps.
A sequence of sheaves is exact iff it is exact on the stalks.

Constructing new sheaves from old, given \( f : X \to Y \):

- **direct image**: \( f_* : \text{Shf}_X \to \text{Shf}_Y \), \( (f_*F)(V) := F(f^{-1}(V)) \).
- **inverse image**: \( f^* : \text{Shf}_Y \to \text{Shf}_X \), \( (f^*G)(U) := \lim_{V \supseteq f(U)} G(V) \).

If \( f \) is the inclusion of a subspace, we call \( f^{-1}G \) the restriction of \( G \) to \( X \), denoted \( G|_X \).

Note that there are maps \( f^{-1}f_*F \to F \) and \( G \to f_*f^{-1}G \) which are easy to write down, and are the counit and unit of an adjunction \( f^{-1} : \text{Shf}_Y \to \text{Shf}_X : f_* \).

Given an inverse system of sheaves on \( X \), the inverse limit presheaf is in fact a sheaf, and is the inverse limit in the category of sheaves on \( X \).

**extension by zero**: Let \( i : Z \to X \) be the inclusion of a closed subset, and \( j : U \to X \) be the inclusion of its complement.

- **from sheaf on closed subset**: Suppose \( F \) is a sheaf on \( Z \). Then its extension by zero is simply \( i_*F \).
- **from sheaf on open subset**: Suppose \( F \) is a sheaf on \( U \). Then its extension by zero is the sheafification \( j_*F \) of \( j^{-1}F \in \text{PreShf}_X \), where \( j^{-1}F(V) := \begin{cases} \{ F(V) \}, & V \subseteq U \\ 0, & \text{ otherwise}. \end{cases} \)

The stalks on \( U \) are unchanged, and are zero on \( Z \). \( j^{-1}F \) satisfies (3) and so embeds in its discontinuous sections, which are identified with the discontinuous sections of \( F \). The sheafification consists of sections of \( F \) whose support (as a subset of \( U \)) has closure in \( X \) not intersecting \( Z \). In this sense, there are fewer sections of \( j_*F \) than of \( F \).

**sheaf Hom**: Given \( F, G \in \text{Shf}_X \), define \( \text{Hom}(F, G) \in \text{Shf}_X \) by \( U \mapsto \text{Hom}(F|_U, G|_U) \).

The functor \( \Gamma(U, -) \) is left exact from \( \text{Shf}_X \to \text{AbGp} \). It is exact on short exact sequences where the kernel is flasque.

The support of a section of a sheaf is the set of points at which it is nonzero in the stalk. This is a closed set!

**Hartshorne II.2 — Schemes**

Define \( \text{Spec} \, A \) to be the set of prime ideals of \( A \). For \( \mathfrak{a} \subseteq A \) any ideal, let \( V(\mathfrak{a}) \) be the set of prime ideals which contain \( \mathfrak{a} \). These sets are the closed subsets of \( \text{Spec} \, A \). Note that \( V(\mathfrak{a}) \subseteq V(\mathfrak{b}) \) iff \( \sqrt{\mathfrak{a}} \subseteq \sqrt{\mathfrak{b}} \). Note that \( \mathfrak{p} \in V(f) \) iff \( f \) vanishes at \( \mathfrak{p} \), that is, \( f \) maps to zero in the residue field \( A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \cong (A/\mathfrak{p})_{(0)} \) at \( \mathfrak{p} \).

Define a sheaf \( \mathcal{O} \) on \( \text{Spec} \, A \) by assigning \( \mathfrak{p} \) the stalk \( A_\mathfrak{p} \), and requiring that a section be locally a quotient of elements of \( A \). Together, this data is the spectrum of \( A \).

**Proposition 2.2**: Let \( D(f) \) be the complement of \( V(f) \). Then the ring \( \mathcal{O}(D(f)) \) is isomorphic to \( A_f \). In particular, \( \Gamma(\mathcal{O}) \cong A \).

A ringed space is a pair \((X, \mathcal{O}_X)\) of a space and a sheaf of rings. A morphism of ringed spaces is a pair \((f, f^\#)\) of a map \( f : X \to Y \) and a map \( f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X \). A locally ringed space is a ringed space wherein the stalks are all local rings. A morphism of locally ringed spaces is a morphism of ringed spaces where the maps on stalks are local (i.e. the preimage of the maximal ideal is the maximal ideal).

**Proposition 2.3**: Each spectrum is a locally ringed space, and there is a natural bijective
correspondence between homomorphisms $A \rightarrow B$ and morphisms $\text{Spec} \, B \rightarrow \text{Spec} \, A$ of locally ringed spaces.

$\Rightarrow$ An affine scheme is any ringed space isomorphic to the spectrum of a ring. A scheme is a locally ringed space with an open cover of affine schemes.

$\Rightarrow$ Suppose $S$ is a graded ring. Let $\text{Proj} \, S$ be the set of all homogeneous prime ideals which do not contain all of $S_+$. If $a$ is a homogeneous ideal of $S$, let $V(a)$ be the set of elements of $\text{Proj} \, S$ which contain $a$. The $V(a)$ are the closed sets.

Define a sheaf $O$ on $\text{Proj} \, S$ by assigning $p$ the stalk $S_{(p)}$, the degree zero localization, and requiring that a section be locally a quotient of homogeneous elements of $S$ of the same degree.

$\Rightarrow$ Proposition 2.5: Let $D_+(f)$ be the complement of $V(f)$ for any homogeneous $f$. Then the ringed space $(D_+(f), O(D_+(f)))$ is isomorphic to $\text{Spec} \, A(f)$, so that $\text{Proj} \, S$ is a scheme.

$\Rightarrow$ Proposition 2.6: Let $k$ be an algebraically closed field. There is a natural fully faithful functor $\text{Var} \, (k) \rightarrow \text{Sch} \, (k)$. For any variety $V$, its topological space is homeomorphic to the subspace of closed points of $t(V)$, and its sheaf of regular functions is obtained by restricting the structure sheaf of $t(V)$.

Proposition 4.10: The image of $t$ is exactly the set of quasi-projective integral schemes over $k$. The image of the set of projective varieties is the set of integral projective schemes. If $V$ is a projective variety with homogeneous coordinate ring $S$, then $t(V) \cong \text{Proj} \, S$.

$\Rightarrow$ A scheme is reduced if $O_X(U)$ has no nilpotent elements for every open subset $U$, or equivalently if $O_{X,P}$ has no nilpotent elements for every $P \in X$.

Interpretation of being reduced — we need only interpret this on an affine scheme. Suppose that $n \in R$ is a nilpotent. Then as $n \in \sqrt{0} = \bigcap p$, the global section $n$ of $O_{\text{Spec} \, R}$ maps to zero in the residue field at every point $p$. Thus the nilpotents are exactly the global sections which are zero when viewed as a function from $\text{Spec} \, R$ to its residue fields.

Given a scheme $X$, there is a universal reduced scheme $X_{\text{red}}$ mapping to $X$. The map is a homeomorphism on spaces.

$\Rightarrow$ The functor $\text{Spec}$ is right adjoint to global sections, $\Gamma : \text{Sch} \leftrightarrow \text{Rings}^{\text{op}} : \text{Spec}$, i.e.:

$$
\text{Hom}_{\text{Sch}}(X, \text{Spec} \, A) \cong \text{Hom}_{\text{Rings}}(A, \Gamma(O_X)).
$$

In particular $\text{Spec} \, Z$ is terminal in $\text{Sch}$.

$\Rightarrow$ To give a map $\text{Spec} \, K \rightarrow X$, it is equivalent to choose a point of $x$ and specify an inclusion $k(x) \rightarrow K$, the residue field $O_x/m_x$ at $x$.

$\Rightarrow$ For $x \in X$, define the Zariski tangent space at $x$ to be the $k(x)$-vector space $T_x$ dual to $m_x/m_x^2$. Let $D = k[e]/e^2$ be the ring of dual numbers over $k$. If $X$ is a scheme over $k$, then to give a $k$-morphism $\text{Spec} \, D \rightarrow X$ is to give a point $x \in X$ rational over $k$ (i.e. $k(x) = k$), and an element of $T_x$. In particular, rational points map to rational points under $k$-maps.

$\Rightarrow$ Every non-empty irreducible closed subset $Z$ of $X$ has a unique generic point, a point $\zeta$ such that $\{\zeta\} = Z$.

$\Rightarrow$ A scheme is called quasi-compact if its underlying topological space is so. Any affine scheme is quasi-compact.

$\Rightarrow$ A space is noetherian iff all of its closed subsets are quasi-compact. The spectrum of a noetherian ring is a noetherian topological space.

$\Rightarrow$ $\text{Proj} \, S$ is empty iff $\text{Rad}(S) \supseteq S_+$. Given a map $\varphi : S \rightarrow T$ of graded rings, we only get a map $f : U \rightarrow \text{Proj} \, S$, where $U$ is the set of homogeneous prime ideals in $\text{Proj} \, T$ whose
preimage is not all of $S_+$. $f$ only depends on the $\varphi_d$ for $d \gg 0$, so can be an isomorphism even if $\varphi$ is not.

$\rightsquigarrow$ Suppose $\varphi : A \to B$ corresponds to $f : Y = \text{Spec} B \to \text{Spec} A = X$. Then:

- $\varphi$ is injective iff $f^* : \mathcal{O}_X \to f_* \mathcal{O}_Y$ is injective, in which case $f$ is dominant.
- $\varphi$ is surjective iff $f$ is a homeomorphism onto a closed subset and $f^*$ is surjective.

### Hartshorne II.3 — First Properties of Schemes

$\rightsquigarrow$ A scheme $X$ is:

- **connected:** if its topological space is connected.
- **reduced:** if none of the rings $\mathcal{O}_X(U)$ (equiv. $\mathcal{O}_{X,p}$) have nilpotent elements.
- **integral:** if $\mathcal{O}_X(U)$ is a domain for all open $U \subseteq X$.
  - iff it is reduced and irreducible.

- **locally noeth.:** if it can be covered by open affine $\text{Spec} A_i$ with $A_i$ noetherian rings.
  - (a local property) iff whenever $\text{Spec} A$ is an open subscheme of $X$, $A$ is noetherian (3.2)
- **noetherian:** if it is locally noetherian and quasi-compact.

$\rightsquigarrow$ A morphism $X \to Y$ of schemes is:

- **locally of finite type:** if $Y$ can be covered by open affine $V_i = \text{Spec} B_i$ such that each $f^{-1}(V_i)$ can be covered by $\text{Spec} A_{ij}$ making $A_{ij}$ a finitely generated $B_i$-algebra.
- **of finite type:** if only finitely many $A_{ij}$ are required for each $i$ in the previous.
  - $\iff$ locally of finite type and quasi-compact.
- **quasi-compact:** if $Y$ can be covered with open affines each with compact preimage.
- **finite:** if $Y$ can be covered by open affine $V_i = \text{Spec} B_i$ such that each $f^{-1}(V_i)$ is affine equal to $\text{Spec} A_i$, making $A_i$ a finitely generated $B_i$-module.
  - $\implies$ proper; closed, and quasi-compact, separated (and affine).
- **affine:** if $Y$ can be covered by open affine $V_i = \text{Spec} B_i$ with $f^{-1}(V_i)$ affine.
  - $\implies$ quasi-compact and separated.
- **separated:** if $\Delta : X \to X \times_Y X$ is a closed immersion.
- **univ. closed:** if any base extension $f' : X' \to Y'$ is closed.
- **proper:** if it is separated, of finite type and universally closed.

All of these properties are local on the base — the requirements end up satisfied for any open cover $V_i$.

$\rightsquigarrow$ An open immersion is simply the inclusion of an open subscheme.

$\rightsquigarrow$ A closed immersion is a morphism $f : Y \to X$ such that $f$ is a homeomorphism onto a closed subset of $X$, and $f^* : \mathcal{O}_X \to f_* \mathcal{O}_Y$ is surjective. Note that the surjectivity condition simply says that any function on $Y$ should lift to a function on $X$ locally.

A closed subscheme of a scheme $X$ is an equivalence class of closed immersions, where closed immersions $f_i : Y_i \to X$ are equivalent if there is an isomorphism $Y_1 \to Y_2$ making the triangle commute.

For $X$ affine, the closed subsets are the $V(\mathfrak{a})$. There is a different closed subscheme structure on $V(\mathfrak{a})$ for each $\mathfrak{a}'$ with $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{a}'}$. In fact, every closed subscheme arises in this way.

$\rightsquigarrow$ Given a scheme $X$ and closed subset $Y$, there is a ‘smallest’ closed subscheme structure for $Y$, the reduced induced closed subscheme structure. We construct this when $X$ is affine and glue. If $X = \text{Spec} A$ and $Y = V(\mathfrak{a})$, take $\text{Spec}(X/\sqrt{\mathfrak{a}}) \to \text{Spec} X$. (Note that $\sqrt{\mathfrak{a}}$ is the largest
ideal available, so there are no redundant functions in the structure. Allowing a function in $\sqrt{a} \setminus a$ allows a nilpotent section.)

$\rightsquigarrow$ There is a fibre product $X \times_{S} Y$ of schemes. If $X = \text{Spec } A$, $Y = \text{Spec } B$, and $S = \text{Spec } R$, then $X \times_{S} Y$ is $\text{Spec } (A \otimes_{R} B)$.

$\rightsquigarrow$ The fibre of $f : X \to Y$ over a point $y \in Y$ is $X_{y} := X \times_{Y} \text{Spec } k(y)$. Moreover, we have base extension $\text{Sch}_{S} \to \text{Sch}_{S'}$ whenever we have a morphism $S' \to S$. The following types of morphisms are stable under base extension (not exhaustive):

- those of finite type
- closed and open immersions
- separated and proper

$\rightsquigarrow$ Integral schemes are very unstable under base extension — it is easy to give examples where irreducibility ($xy = t$) and reducedness ($ty = x^2$) are not preserved.

$\rightsquigarrow$ Given a morphism $f : Z \to X$, there is a universal closed subscheme $Y$ of $X$ through which $f$ factors, called the scheme theoretic image — if $f$ also factors through $Y'$, then $Y \to X$ factors as $Y \to Y' \to X$.

$\rightsquigarrow$ Properties of morphisms of finite type:

1. if $f : X \to Y$ has finite type, $\text{Spec } B \subset Y$ and $\text{Spec } A \subset f^{-1}(\text{Spec } B)$ are open subsets, then $A$ is always a finitely generated $B$-algebra.
2. A closed immersion is of finite type.
3. A quasi-compact open immersion is of finite type.
4. A composition of two morphisms of finite type is of finite type.
5. If $X$ and $Y$ have finite type over $S$, so does $X \times_{S} Y$.
6. If $X \to Y \to Z$ is of finite type, then $X \to Y$ is locally of finite type. (Use 1.)
7. If $X \to Y$ has finite type, and $Y$ is noetherian, then $X$ is noetherian.

$\rightsquigarrow$ Noetherian induction: if a property $\mathcal{P}$ of closed subsets of a noetherian space $X$ holds for $Z$ whenever it holds for all proper closed subsets of $Z$, then it holds for all closed subsets of $X$.

**Hartshorne II.4 — Separated and Proper Morphisms**

$\rightsquigarrow$ A morphism $X \to Y$ is separated if $\Delta : X \to X \times_{Y} X$ is a closed immersion.

$\rightsquigarrow$ Any morphism of affine schemes is separated, as $A \otimes_{B} A \to A$ is surjective.

$\rightsquigarrow$ It’s enough that $\text{im}(\Delta(X))$ is a closed subset of $X \times_{Y} X$.

(It’s all local on $Y$. Surjectivity on sheaves is local on $X$, so we can pretend $X$ is affine to get a closed immersion. As $X \to X \times_{Y} X \to X$ is the identity, $\Delta$ is always a homeomorphism onto its image.)

$\rightsquigarrow$ **Theorem 4.3:** (valuative criterion of separatedness) Let $f : X \to Y$ be any morphism where $X$ is noetherian. Then $f$ is separated iff the following condition holds. Suppose $R \to K$ is the inclusion of a valuation ring into its field of fractions. Then there is at most
one lifting (dotted) in any commuting diagram of the following form:

\[
\begin{array}{ccc}
(0) & \in & \text{Spec } K \\
\downarrow & & \downarrow \\
\xi & \in & \text{Spec } R
\end{array}
\xrightarrow{f}
\begin{array}{ccc}
& X \\
\downarrow & & \downarrow \\
& f
\end{array}
\xrightarrow{\downarrow}
\begin{array}{ccc}
& Y
\end{array}
\]

**Interpretation** — If a map to \( Y \) can be lifted to \( X \) at the generic point \( \xi \), as everything in \( \text{Spec } R \) is so close to \( \xi \), any extension of this lift to all of \( \text{Spec } R \) must be unique.

\[\sim\textbf{ Theorem 4.7:} \quad \text{(valuative criterion of properness)} \]
Let \( f : X \rightarrow Y \) be a morphism of finite type, where \( X \) is noetherian. Then \( f \) is proper iff there is exactly one lifting (dotted) in:

\[
\begin{array}{ccc}
(0) & \in & \text{Spec } K \\
\downarrow & & \downarrow \\
\xi & \in & \text{Spec } R
\end{array}
\xrightarrow{f}
\begin{array}{ccc}
& X \\
\downarrow & & \downarrow \\
& f
\end{array}
\xrightarrow{\downarrow}
\begin{array}{ccc}
& Y
\end{array}
\]

**Interpretation** — As before, except that as \( X \rightarrow Y \) is proper, a lift should exist.

**Example** (Exercise 4.1). A finite morphism is proper.\[^1\]

**Proof.** As properness is local on the base, we can check this for a map of affine schemes induced by a finite ring homomorphism \( B \rightarrow A \). Now there’s a correspondence between dashed arrow in the diagrams:

\[
\begin{array}{ccc}
\text{Spec } K & \longrightarrow & \text{Spec } A \\
\downarrow & & \downarrow \\
\text{Spec } R & \longrightarrow & \text{Spec } B
\end{array}
\xrightarrow{K \xleftarrow{a}}
\begin{array}{ccc}
A & \longrightarrow & \text{Spec } A \\
\downarrow & & \downarrow \\
B & \longrightarrow & \text{Spec } B
\end{array}
\]

As the left vertical is injective, we can assume that the right vertical in injective. Now \( R \alpha (A) \) is finite over \( R \), in particular it’s integral over \( R \), so that \( R \alpha (A) = R \), (as a valuation ring is integrally closed). This gives the dashed arrow.\[\Box\]

**Example** (Exercise 4.6). A proper morphism of affine varieties over a field is finite.

**Proof.** Suppose that the morphism is induced by a ring map \( B \rightarrow A \). Denote the epi-mono factorisation as in the diagram. Whenever \( A' \subset R \subset A_{(0)} \) for a valuation ring \( R \), we have the dashed arrows:

\[
\begin{array}{ccc}
A_{(0)} & \xleftarrow{\alpha} & A \\
\downarrow & & \downarrow \\
A' & \xleftarrow{\beta} & B
\end{array}
\]

By properness we obtain the dotted arrow, and thus \( A \subset R \). In particular, \( A \) is a subring of the integral closure \( \overline{A'} \) of \( A' \) in \( A_{(0)} \), and by finiteness of integral closure, \( \overline{A'} \) is finite over \( A' \). But \( A' \) is Noetherian, so that \( A \) is finite over \( A' \).\[\Box\]

\[^1\]We should just note that finite morphisms are closed under base extension, closed, separated and of finite type.
Theorem I.3.9A: (finiteness of integral closure) Suppose that $A$ is an integral domain which is a finitely generated $k$-algebra. Let $L$ be a finite algebraic extension of the quotient field $A(0)$. Then the integral closure of $A$ in $L$ is a finite $A$-module and finitely generated $A$-algebra:

\[
\begin{array}{c}
A \quad \longrightarrow \quad A_L \quad \hookrightarrow \quad L \\
\text{finite type} \quad \text{finite} \quad \text{finite algebraic}
\end{array}
\]

Corollaries 4.6 & 4.8: When everything is noetherian:

- Open and closed immersions are separated.
- Closed immersions are proper.
- A composition of (separated/proper) morphisms is (separated/proper)
- (Separated/proper) morphisms are stable under base extension.
- Products of (separated/proper) morphisms are (separated/proper):
  - If $X \rightarrow Y \rightarrow Z$ is separated, so is $X \rightarrow Y$.
  - If $X \rightarrow Y \rightarrow Z$ is proper, and $Y \rightarrow Z$ is separated, then $X \rightarrow Y$ is proper.
- Being (separated/proper) is local on the base. All of these statements remain true if you replace ‘proper’ with ‘projective’.

Define $P^n_A$ to be $\text{Proj} \ A[x_0, \ldots, x_n]$ for any ring $A$. Given a map $A \rightarrow B$ of rings, using the corresponding map $\text{Spec} B \rightarrow \text{Spec} A$ we have $P^n_B \cong P^n_A \times_{\text{Spec} A} \text{Spec} B$. Thus we could define $P^n_A := P^n_Z \times \text{Spec} B$.

For any scheme $Y$, we define the projective $n$-space $P^n_Y$ over $Y$, to be $P^n_Z \times Y$.

A map $X \rightarrow Y$ is called projective if it is essentially the restriction to a closed subscheme of $P^n_Y$ of the map $P^n_Y \rightarrow Y$. (i.e. it factors as a closed immersion $X \rightarrow P^n_Y$ and then the projection.)

A map $X \rightarrow Y$ is called quasi-projective if it is essentially the restriction of $P^n_Y \rightarrow Y$ to an open subscheme of a closed subscheme of $P^n_Y$. (i.e. it factors as an open immersion into $X'$ followed by a projective map $X' \rightarrow Y$.)

Theorem 4.9: Projective morphisms of noetherian schemes are proper. Quasi-projective morphisms of noetherian schemes are of finite type and separated.

A variety is an integral separated scheme of finite type over an algebraically closed field $k$. If it is proper over $k$, we call it complete.

Let $X$ be reduced and let $Y$ be separated (over some base $S$). If $f, g : X \rightarrow Y$ are $S$-morphisms which agree on an open dense subset of $X$, then $f = g$.

In a separated scheme over an affine scheme, the intersection of affine open subsets is affine. (To see this, use the following pullback to see that $U \cap V \rightarrow U \times V$ is a closed immersion, and note that $U \times V$ is affine when $U, V$ and the base are affine.)

\[
\begin{array}{ccc}
U \cap V & \longrightarrow & X \\
\downarrow & & \downarrow \\
U \times V & \longrightarrow & X \times X
\end{array}
\]

A proper morphism of affine varieties over $k$ is a finite morphism.
Hartshorne II.5 — Sheaves of Modules

A sheaf of \( \mathcal{O}_X \)-modules on a ringed space \( X \) is a sheaf \( \mathcal{F} \) on \( X \) such that \( \mathcal{F}(U) \) is an \( \mathcal{O}_X(U) \)-module for each open \( U \), compatibly with restrictions.

The category of \( \mathcal{O}_X \)-modules is closed under taking kernels, cokernels, images, quotients, direct limits, inverse limits, direct sums and products. There is a sheaf Hom for \( \mathcal{O}_X \)-modules, requiring no sheafification: \( \mathcal{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) := \mathcal{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}|_U, \mathcal{G}|_U) \).

The tensor product of \( \mathcal{O}_X \)-modules is the sheafification of \( U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U) \).

(I should think about whether the presheaf has (3).)

An \( \mathcal{O}_X \)-module \( \mathcal{F} \) is free if it is a direct sum of copies of \( \mathcal{O}_X \). It is locally free if \( \mathcal{F}|_U \) is a free \( \mathcal{O}_X|_U \)-module for some open \( U \) around any point of \( X \). The rank is well defined on each connected component. If it is locally free of rank one, it is called invertible.

A sheaf of ideals on \( X \) is a sheaf of modules \( \mathcal{I} \) which is a sub-\( \mathcal{O}_X \)-module of \( \mathcal{O}_X \). That is, \( \mathcal{I}(U) \) is an ideal of \( \mathcal{O}_X(U) \) for all open \( U \).

Constructing new sheaves of modules from old, given \((f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)\):

- **direct image:** \( f_* : \mathcal{O}_X\text{-Mod} \to \mathcal{O}_Y\text{-Mod} \), using the following module structure:
  \[
  \mathcal{O}_Y \times f_* \mathcal{F} \to f_* \mathcal{O}_X \times f_* \mathcal{F} \to f_* \mathcal{F}.
  \]

- **inverse image:** \( f^* : \mathcal{O}_Y\text{-Mod} \to \mathcal{O}_X\text{-Mod} \):
  \[
  \mathcal{O}_X \in (f^{-1}\mathcal{O}_Y)\text{-Mod} \text{ (via } (f^{-1}\mathcal{O}_Y) \to \mathcal{O}_X), \text{ and } f^{-1}\mathcal{G} \in (f^{-1}\mathcal{O}_Y)\text{-Mod}. \text{ Define:}
  \]
  \[
  f^* \mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.
  \]

There is an adjunction \( f^* : \mathcal{O}_Y\text{-Mod} \leftrightarrow \mathcal{O}_X\text{-Mod} : f_* \).

Given an \( M \in A\text{-Mod} \), we obtain \( \tilde{M} \in \mathcal{O}_X\text{-Mod} \), where \( X = \text{Spec } A \). Assign \( p \) the stalk \( M_p \), and require that a section be locally a quotient \( m/a \), for \( m \in M \) and \( a \in A \) non-vanishing.

**Proposition 5.1:** For any \( f \in A \), \( \tilde{M}(D(f)) \cong M_f \), and thus \( \Gamma(X, \tilde{M}) = M \).

**Proposition 5.2:** Let \( X = \text{Spec } A \). Then \( \tilde{M} \) gives an exact, fully faithful functor \( A\text{-Mod} \to \mathcal{O}_X\text{-Mod} \), preserving tensor products and direct sums. Moreover, for \( f : \text{Spec } B \to \text{Spec } A \), \( f_* \) and \( f^* \) correspond to the usual adjunction \( f_* : A\text{-Mod} \leftrightarrow B\text{-Mod} : f^* \).

A sheaf \( \mathcal{F} \) of \( \mathcal{O}_X \)-modules is quasi-coherent if \( X \) can be covered by open affine subsets \( \text{Spec } A_i \) such that \( \mathcal{F}|_{\text{Spec } A_i} \) is isomorphic to \( \tilde{M}_i \) for some \( M_i \in A_i\text{-Mod} \). It is coherent if the \( M_i \) can be taken to be finitely generated \( A_i \)-modules.

**Proposition 5.4:** An \( \mathcal{O}_X \)-module \( \mathcal{F} \) is quasi-coherent iff for every open affine \( U = \text{Spec } A \subset X \), \( \mathcal{F}|_U \cong \tilde{M} \) for some \( M \in A\text{-Mod} \). The corresponding test for coherence can be applied when \( X \) is noetherian.

**Proposition 5.5:** If \( X = \text{Spec } A \), \( M \mapsto \tilde{M} \) is an equivalence \( A\text{-Mod} \to \text{QCoh}_X \). As long as \( A \) is noetherian, it is also an equivalence \( A\text{-Mod}^{f.g.} \leftrightarrow \text{Coh}_X \).
Proposition 5.6: If $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$ is an exact sequence of $O_X$-modules, where $X$ is affine, then $0 \rightarrow \Gamma(F') \rightarrow \Gamma(F) \rightarrow \Gamma(F'') \rightarrow 0$ is exact whenever $F$ is quasicoherent. (Note that in fact, $H^1(X, F') = 0$, when $X$ is affine and $F \in \text{QCoh}_X$).

Proposition 5.7: $\text{QCoh}_X$ is closed under taking kernels, cokernels, images and extensions. So if $\text{Coh}_X$ when $X$ is noetherian.

Proposition 5.8: Let $f : X \rightarrow Y$ be a morphism of schemes.

(a) If $F \in \text{QCoh}_Y$, then $f^*F \in \text{QCoh}_X$.
(b) If $F \in \text{Coh}_Y$, then $f^*F \in \text{Coh}_X$, assuming $X$ and $Y$ are both noetherian.
(c) Assume that either $X$ is noetherian, or $f$ is quasi-compact and separated. If $F \in \text{QCoh}_X$, then $f_*F \in \text{QCoh}_Y$. (Here, $f$ could be proper, or finite.)

Exercise 5.5: If $f : X \rightarrow Y$ is a finite morphism of noetherian schemes, and $F \in \text{Coh}_X$, then $f_*F \in \text{Coh}_Y$. (Easy.)

If $i : Y \rightarrow X$ is a closed subscheme, the kernel of the map $O_X \rightarrow i_*O_Y$ is called the ideal sheaf $\mathcal{I}_Y$ of $Y$. That is, $\mathcal{I}_Y$ is the ideal of functions on $X$ which vanish on $Y$.

By 5.7 and 5.8, $\mathcal{I}_Y$ is always quasi-coherent, and coherent when $X$ is noetherian.

Proposition 5.9: Any quasi-coherent sheaf of ideals $F$ on $X$ is the ideal sheaf of a uniquely determined closed subscheme of $X$. (In particular, closed subschemes of $\text{Spec} \ A$ correspond to ideals of $A$.)

Proof. Let $Y = \text{supp}(O_X/F)$, giving a ringed space $(Y, O_X/F)$. To check that this is a closed subscheme, we may assume $X = \text{Spec} \ A$. Then as $F$ is quasi-coherent, $F = \widetilde{\mathcal{A}}$ for some ideal $\mathcal{A} \subseteq \mathcal{A}$. Then $O_X/F = \mathcal{A}/\mathcal{A} = (\mathcal{A}/\mathcal{A})$, which is simply $O_{\text{Spec} \ A/\mathcal{A}}$.

Suppose that $S$ is a graded ring and $M$ is a graded $S$-module. Then we can form a quasi-coherent sheaf $\widetilde{M}$ of $O_{\text{Proj} \ S}$-modules by the standard construction, defining the stalk at $p$ to be $M(p)$, the degree zero localisation of $M$, and requiring sections to be fractions locally.

Proposition 5.9: For homogeneous $f \in S_+$, we have $\widetilde{M}|_{D_+(f)} = \widetilde{M(f)}$, where we note that $D_+(f) \cong \text{Spec} \ S(f)$, and $M(f)$ is an $S(f)$-module. Of course $\widetilde{M}$ is always quasi-coherent, and is coherent if $S$ is noetherian and $M$ is finitely generated.

Given a graded $S$-module $M$, we define the twisted module $M(n)$ by $M(n)_d := M_{n+d}$. If $X = \text{Proj} \ S$, we define $O_X(n) := \widetilde{S(n)}$. Serre’s twisting sheaf is $O_X(1)$. We twist other sheaves by tensoring them with this one, defining $F(n) := F \otimes O(n)$.

Proposition 5.12: Suppose that $S$ and $T$ are graded rings, such that $S(T)$ is generated by $S_1(T_1)$ as an $S_0$-algebra ($T_0$-algebra). Let $X = \text{Proj} \ S$ and $Y = \text{Proj} \ T$. Then:

- The sheaf $O_X(n)$ is invertible for all $n$.
- For any graded $S$-module $M$, $\widetilde{M(n)} \cong \widetilde{M(n)}$. Thus $O_X(n) \otimes O_X(m) = O_X(n + m)$.
- Suppose $S \xrightarrow{\varphi} T$ is a graded ring homomorphism, defining a map $X \xleftarrow{\psi} U \subseteq Y$. Then $(f, O_U(n)) \xleftarrow{\varphi} O_Y(U)(n)$ and $O_X(n) \xrightarrow{\psi} O_Y(n)|_U$.
- Suppose $F$ is a sheaf of $O_X$-modules, where $X = \text{Proj} \ S$. The graded module associated to $F$ is $\bigoplus \Gamma(X, F(n))$. This becomes a graded $S$-module, as any $s \in S_d$ can be viewed as an element of $\Gamma(X, O_X(d))$, and there is a map:

\[ \Gamma(X, O_X(d)) \otimes \Gamma(X, F(n)) \rightarrow \Gamma(X, O_X(d) \otimes O_F(n)) = \Gamma(X, F(n + d)). \]
This is great, since proposition 5.15 states that if \( S \) is finitely generated by elements of \( S_1 \) as an \( S_0 \)-algebra, and \( \mathcal{F} \) is quasi-coherent, then \( \Gamma_*(\mathcal{F}) \cong \mathcal{F} \).

**Proposition 5.13:** If \( S = A[x_0, \ldots, x_n] \), so that \( X = \text{Proj} \, S = \mathbb{P}^n_A \), then \( \Gamma_*(\mathcal{O}_X) \cong S \).

**Corollary 5.16:** Let \( A \) be a ring.

- Every closed subscheme \( Y \) of \( \mathbb{P}^n_A \) arises from a homogeneous ideal of \( S = A[x_0, \ldots, x_n] \).
  (We noted above that closed subschemes correspond precisely to saturated ideals: \( \Gamma_*(\mathcal{I}_Y) \) is the saturated ideal.)

- A scheme over \( \text{Spec} \, A \) is projective iff it is \( \text{Proj} \, S \) for some graded ring \( S \) with \( S_0 = A \) and \( S \) finitely generated by \( S_1 \) as an \( A \)-algebra. (Almost by definition.)

To produce the ideal corresponding to \( Y \), note that \( \mathcal{I}_Y \subset \mathcal{O}_{\mathbb{P}^n_A} \), twisting is exact, and global sections is left exact, so that \( \Gamma_* (\mathcal{I}_Y) \subset \Gamma_* (\mathcal{O}_{\mathbb{P}^n_A}) = S \).

**Corollary 5.18:** If \( Y = \text{Spec} \, A \), then \( \mathcal{I}_Y \subset \mathcal{O}_{\mathbb{P}^n_A} \), twisting is exact, and global sections is left exact, so that \( \Gamma_* (\mathcal{I}_Y) \subset \Gamma_* (\mathcal{O}_{\mathbb{P}^n_A}) = S \).

**Theorem 5.17:** Let \( X = \text{Proj} \, S \) be a projective scheme over \( \text{Spec} \, A \), and let \( \mathcal{F} \) be a coherent \( \mathcal{O}_X \)-module. Then \( \mathcal{F}(n) \) is generated by global sections for \( n \geq 0 \). This yields the following:

**Corollary 5.18:** \( \mathcal{F} \) can be written as a quotient of a finite direct sum \( \bigoplus \mathcal{O}(n_i) \).

**Theorem 5.19:** Let \( k \) be a field, \( A \) a finitely generated \( k \)-algebra, \( X \) a projective scheme over \( A \), and \( \mathcal{F} \) a coherent \( \mathcal{O}_X \)-module. Then \( \Gamma(X, \mathcal{F}) \) is a finitely generated \( A \)-module.

In particular, if \( X \) is projective over \( k \), then \( \Gamma(X, \mathcal{F}) \) is finite dimensional for all \( \mathcal{F} \in \text{Coh}_X \). **Corollary 5.20:** Let \( f : X \to Y \) be a projective morphism, where \( X \) and \( Y \) are both of finite type over a field \( k \). Then \( f_* \mathcal{F} \in \text{Coh}_Y \) for any \( \mathcal{F} \in \text{Coh}_X \).

**Proof.** We may assume \( Y = \text{Spec} \, A \). We already saw that \( f_* \mathcal{F} \) is quasi-coherent, so we only need to check that \( \Gamma(Y, \mathcal{F}) \) is finitely generated \( A \)-module.

Suppose that \( \mathcal{E} \) is a locally free \( \mathcal{O}_X \)-module of finite rank. We define the dual \( \mathcal{E}^* \) of \( \mathcal{E} \) to be \( \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X) \). Then as usual, (for any \( \mathcal{F}, \mathcal{G} \in \mathcal{O}_X \text{-Mod} \):

\[
\mathcal{E}^* \cong \mathcal{E}, \quad \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \cong \mathcal{E}^* \otimes \mathcal{F}, \quad \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E} \otimes \mathcal{F}, \mathcal{G}) \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{G})).
\]
We have the following projection formula, given \( f : X \rightarrow Y, \mathcal{F} \in \mathcal{O}_X\text{-Mod} \) and \( \mathcal{E} \in \mathcal{O}_Y\text{-Mod} \), with \( \mathcal{E} \) locally free:

\[
fs(\mathcal{F} \otimes \mathcal{O}_X f^*(\mathcal{E})) \cong fs(\mathcal{F}) \otimes \mathcal{O}_Y \mathcal{E}.
\]

(That is, given \( f : X \rightarrow Y \), a sheaf \( \mathcal{F} \) on the X and a locally free sheaf \( \mathcal{E} \) on Y, we can form two tensor products. We can pull \( \mathcal{E} \) back to X and tensor with \( \mathcal{F} \), or push \( \mathcal{F} \) forward to Y and tensor with \( \mathcal{E} \). The two outcomes are the same when they are compared on Y.)

Let \( X \) be a noetherian scheme, and let \( \mathcal{F} \) be a coherent sheaf. Then \( \mathcal{F} \) is locally free iff the stalks \( \mathcal{F}_x \) are free \( \mathcal{O}_{X,x} \)-modules for all \( x \in X \). Moreover, \( \mathcal{F} \) is invertible (locally free of rank one) iff there exists \( \mathcal{G} \in \text{Coh}_X \) such that \( \mathcal{F} \otimes \mathcal{G} \cong \mathcal{O}_X \) (justifying the name ‘invertible’). Of course, \( \mathcal{G} = \mathcal{F} \).

**Proof.** Suppose that \( \mathcal{F}, \mathcal{G} \) are coherent (over a noetherian scheme \( X \)) such that \( \mathcal{F} \otimes \mathcal{G} \cong \mathcal{O}_X \).

It is enough to prove that \( \mathcal{F}_x \) is a free \( \mathcal{O}_{X,x} \)-module of rank one for all \( x \in X \), as this shows that \( \mathcal{F}_x \) is locally free of rank one. Now write \( M = \mathcal{F}_x, \ N = \mathcal{G}_x \) and \( L = \mathcal{O}_{X,x} \). We know that \( L \) is local, and \( M \otimes_L N = L \). All we are to do is show that in this setting, \( M \) is a free module of rank 1. But we have \((M \otimes_L N)/m(M \otimes_L N) = M/mM \otimes_L N/mN \), and this tensor product has dimension one! So \( M/mM \) has dimension one over the residue field, and Nakayama’s lemma shows that \( M \) is generated by one element. Of course, this element must have annihilator zero, so that \( M \) is free of rank one.

**Exercise 5.8:** Let \( X \) be noetherian, and \( \mathcal{F} \in \text{Coh}_X \). Consider \( \phi(x) := \dim_{k(x)} \mathcal{F}_x \otimes \mathcal{O}_x k(x), \) where \( k(x) = \mathcal{O}_x/m_x \) is the residue field at \( x \).

(a) \( \phi \) is upper semicontinuous (i.e. it exceeds any given value on a closed set).

(b) If \( \mathcal{F} \) is locally free, then \( \phi \) is locally constant.

(c) Conversely, if \( X \) is reduced, and \( \phi \) is locally constant, then \( \mathcal{F} \) is locally free.

**Interpretation of \( \phi(x) \) —** suppose that \( \mathcal{F} \) is the sheaf of sections of an \( n \)-bundle (which is locally free). Then \( \mathcal{F}_x \) is the \( \mathcal{O}_x \)-module of germs of sections at \( x \), where \( \mathcal{O}_x \) is the local ring of germs of functions at \( x \). Tensoring with \( k(x) \) removes the word ‘germs’: we have \( \mathcal{F}_x \otimes k(x) = \mathcal{F}_x/m_x \mathcal{F}_x \), which is the \( k(x) \)-vector space of values of the sections can take over \( x \). Thus, we obtain the rank of \( \mathcal{F} \) at \( x \). Had we not assumed that \( \mathcal{F} \) was locally free, then we would still have obtained a nice number measuring the size of the stalk.

We’ll prove this result using **Nakayama’s lemma:** suppose that \((A, m)\) is a local ring and \( M \) is a finitely generated \( A \)-module. Then any basis for the \( A/m \)-vector space \( M/mM \) lifts to a (minimal) generating set for \( M \) as an \( A \)-module.

In the context of the exercise, we use the local ring \((\mathcal{O}_x, m_x)\), and the finitely generated module \( \mathcal{F}_x \). Thus we may reinterpret \( \phi(x) \) as the minimal number of generators required for the module \( \mathcal{F}_x \).

**Proof.** To prove (a), we’ll observe that generators of \( \mathcal{F}_x \) extend to sections \( s_i \in \Gamma(U, \mathcal{F}) \), where \( U \) is an open neighbourhood of \( x \), and that the image of the \( s_i \) generates the stalks \( \mathcal{F}_y \) for \( y \) in some smaller neighbourhood of \( x \). For this, we may assume that in fact \( U = \text{Spec} \, R \) for \( R \) noetherian in which case \( \mathcal{F}|_U = M \) for some \( M \in R\text{-Mod}^{f.g.} \), generated by \( g_1, \ldots, g_n \). Then the \( s_i \) are in fact elements of \( M \), which generate \( M_x \) as an \( R_x \)-module. That is to say,
there are elements $a_{ij} \in R$ and $t \in R \setminus x$ such that:

$$m_i = \sum_j \frac{a_{ij}}{t} \cdot s_j, \text{ for each } j.$$  

Of course, this expression makes sense wherever $t \in R$ remains invertible — on $D(t) \subset R$.

(b) follows from basic considerations of rank. For (c), we reduce to the case where $X = \text{Spec } A$, $\mathcal{F} = M$, and we have $m_1, \ldots, m_r \in M$ which minimally generate $M_p$ in each localisation. Then we have a map $\gamma : A^r \rightarrow M$, given by $(x_1, \ldots, x_r) \mapsto \sum x_im_i$, such that $(A_p/pA_p)^r \rightarrow M_p/pM_p$ is an isomorphism for all $p$. Of course, if $(x_1, \ldots, x_r) \in \ker \gamma$, then $x_i \in p$ for all $p$, so that $x_i = 0$ (as $R$ is reduced). So $\gamma$ is an isomorphism.  

A note on vector bundles: a finitely generated module over a noetherian ring is locally free if and only if it is projective, so over an affine noetherian scheme $\text{Spec } A$, the vector bundles and the finitely generated projective $A$-modules coincide.

Hartshorne II.6 — Divisors

A scheme is said to be regular in codimension one if every local ring of dimension one is regular. (This includes nonsingular varieties, and noetherian normal schemes, as a noetherian local domain of dimension one is integrally closed iff it is regular.)

In our discussion of Weil divisors, we consider schemes satisfying the following conditions:

(*) $X$ is a noetherian integral separated scheme, regular in codimension one.

Let $X$ be a scheme satisfying (*).

- A prime divisor is a closed integral subscheme of codimension one. (A closed integral subscheme is determined by its set-theoretic image.)
- A Weil divisor is an element of the free abelian group $\text{Div}$ on the prime divisors.
- A Weil divisor is effective if it is a nonnegative linear combination of prime divisors.

Suppose that $X$ satisfies (*), and that $f \in K^*$, where $K$ is the function field. Each prime divisor $Y$ with generic point $\eta$ gives a DVR $\mathcal{O}_{X,\eta}$ with quotient field $K$. We denote by $v_Y$ the valuation on $K$ corresponding to this DVR. (When $v_Y(f)$ is positive, it measures the highest power of the maximal ideal $f$ lies in.) Depending on whether $v_Y(f)$ is positive or negative, we say that $f$ has a zero or a pole on $Y$.

We define a divisor $(f) = \sum v_Y(f) \cdot Y$. This is called a principal divisor, (lemma 6.1 states that it is actually a finite sum as required) and the map $K^* \rightarrow \text{Div}(X)$ is a homomorphism.

The divisor class group $\text{Cl}(X)$ is the cokernel of this homomorphism. That is, $\text{Cl}(X)$ is the group $\text{Div}(X)$ modulo linear equivalence, where two divisors are linearly equivalent if they differ by a principal divisor.

Proposition 6.2: Let $A$ be a noetherian domain. Then $A$ is a UFD iff $X = \text{Spec } A$ is normal and $\text{Cl}(X) = 0$. (The proof uses proposition 6.3A, which essentially states that if $A$ is a noetherian normal domain, $a \in A_{(0)}$ defines a function on all of $\text{Spec } A$ iff it has no poles on any prime divisor.)

\[\text{Everything always needs to be separated and noetherian. } X \text{ needs to be integral so that there is a function field. } X \text{ needs to be regular in codimension one so that we can define the degree of vanishing of an element of the function field using the DVR structure. }\]
This shows that $\text{Cl}(\mathbb{A}^2_k) = 0$.

When $A$ is a Dedekind domain, $\text{Cl}(	ext{Spec } A)$ is the ideal class group of $A$.

Proposition 6.4: Let $X = \mathbb{P}^n_k$ ($k$ a field). The homomorphism $\text{deg} : \text{Div}(X) \rightarrow \mathbb{Z}$ which sends a prime divisor to its degree (the degree of a polynomial used to generate its homogeneous ideal) has kernel the subgroup of principal divisors. Thus $\text{Cl}(X) = \mathbb{Z}$, generated by the class of a hyperplane.

Proof. Note that if $D$ is a divisor of degree $d$, then $D \sim dH$, where $H$ is the hyperplane $x_0 = 0$. Moreover, it is clear (as $K = k[x_0, \ldots, x_n]_{(0)}$), that any $f \in K^*$ has $\text{deg}(f) = 0$. $\square$

Proposition 6.5: Suppose that $X$ satisfies (*), and that $Z$ is a proper closed subset. Let $U = X - Z$. Then the map $\text{Div}(X) \rightarrow \text{Div}(U)$ sending a prime divisor $Y$ to $Y \cap U$ (or zero if this is empty) descends to a surjective homomorphism $\text{Cl}(X) \rightarrow \text{Cl}(U)$.

- If $Z$ has codimension more that one, this is an isomorphism.
- If $Z$ is irreducible of codimension one, the kernel is generated by $Z$. That is, there is an exact sequence $\mathbb{Z} \rightarrow \text{Cl}(X) \rightarrow \text{Cl}(U) \rightarrow 0$, where $1 \mapsto 1 \cdot Z \in \text{Cl}(X)$.

If $Y$ is an irreducible curve of degree $d$ in $\mathbb{P}^2$, then $\text{Cl}(\mathbb{P}^2 - Y) = \mathbb{Z}_d$.

Example 6.5.2: the quadric cone, $xy = z^2$. Let $A = k[x, y, z]/xy - z^2$, and let $X = \text{Spec } A$, the quadric cone. Let $Y$ be the divisor specified by $y = z = 0$.

- We determine that $X - Y$ is Spec $A_y$, as $y = 0$ determines that $z = 0$. Now $A_y = k[y^\pm 1, z]$ is a UFD so that $\text{Cl}(X - Y) = 0$. Thus the above exact sequence tells us that $\text{Cl}(X)$ is generated by the image of the prime divisor $Y$.
- We calculate the principal divisor $(y)$. To find $v_Y(y)$, we note that the generic point of $Y$ is the ideal $\eta = (y, z) \subset A$. In the localisation $A_\eta$, this becomes the maximal ideal, any polynomial which is nonzero upon substituting $y = z = 0$ becomes invertible. That includes $x$, so we can write $y = x^{-1}z^2$. In particular, $z$ generates the maximal ideal, and $y$ has valuation two. The function $y$ has no poles, and has no other zeros, so that $(y) = 2Y$.
- For a noetherian domain, being normal is a local property ($A$ is normal iff the $A_p$ are normal iff the $A_m$ are normal), and $A$ is integrally closed (exercise 6.4 — essentially as $xy - z^2$ is squarefree). Thus it is enough to show that $A$ is not a UFD, in order to see that $Y$ is not principal, and that $\text{Cl}(X) = \mathbb{Z}_2(Y)$. It was obvious from the beginning that $A$ was not a UFD: $xy = z^2$!

Proposition 6.6: If $X$ satisfies (*), then so does $X \times \mathbb{A}^1$, and $\text{Cl}(X \times \mathbb{A}^1) = \text{Cl}(X)$.

Proof. The key point is to observe that we can reduce to divisors which are the product of a codimension one point of $X$ with $\mathbb{A}^1$. To see this, let $K$ be the function field of $X$. Then we have pullbacks:

$$
\begin{array}{c}
\text{Spec } K[t] \longrightarrow X \times \mathbb{A}^1 \\
\downarrow \\
\text{Spec } K \xrightarrow{\eta_X} X \\
\downarrow \\
\text{Spec } \mathbb{A}^1_k \longrightarrow \text{Spec } Z
\end{array}
$$

Now given a prime divisor $Z \subset X \times \mathbb{A}^1$, we intersect with $\text{Spec } K[t]$, the fibre of $\eta_X$, to obtain a divisor $Z'$ on $\text{Spec } K[t] = \mathbb{A}^1_k$. As $K[t]$ is a UFD, we have that $\text{Cl}(\mathbb{A}^1_k) = 0$, so that $Z'$ is principal, equal to $(f)$ for some $f \in K(t)$. In particular, back on $X \times \mathbb{A}^1$, $Z - (f)$ has zero
intersection with the fibre of \( \eta_X \), so that its components are supported \( C \times \mathbb{A}^1 \), for \( C \subset X \) a proper closed subset. But then, for dimension reasons, we must have \( Z - (f) \) equal to a sum of divisors of the form \( y \times \mathbb{A}^1 \), where \( y \in X \) has codimension one.

Now we have a map \( \pi_1^* : \operatorname{Cl}(X) \to \operatorname{Cl}(X \times \mathbb{A}^1) \) given by taking the preimage under the first projection. This map is surjective, by the previous argument. To see that it is injective, suppose that \( \sum n_y(y \times \mathbb{A}^1) \) is zero in \( \operatorname{Cl}(X \times \mathbb{A}^1) \). Then \( \sum n_y(y \times \mathbb{A}^1) = (f) \) for \( f \) a nonzero element of the function field of \( X \times \mathbb{A}^1 \). This function field is in fact the quotient field of \( K[t] \), so that \( f = g/h \) for \( g, h \in K[t] \setminus \{0\} \). We can assume that \( (g, h) = 1 \), so that there is no cancellation when taking the difference \( (g) - (h) \). We want to see that \( f, g \) are in fact constant polynomials, so that the sum \( \sum n_y(y) \) was already zero in \( \operatorname{Cl}(X) \).

As \( (g, h) = 1 \), we have that \( (g) \) contains only prime divisors of the form \( y \times \mathbb{A}^1 \). But this should imply that \( g \) is a constant polynomial. If \( g \) is not constant, let \( \operatorname{Spec} R \in X \) be any open affine subset. Then form the intersection:

\[
\begin{array}{c}
\text{Spec } K[t]/g \\
\downarrow \\
\text{Spec } R[t]/g \\
\downarrow \\
\text{Spec } K \\
\end{array}
\]

That this intersection (of \( (g) \) with \( \eta_X \times \mathbb{A}^1 \)) is nonempty gives a contradiction. \( \square \)

\( \sim \) **Exercise 6.1:** Let \( X \) satisfy (\( * \)). Then \( X \times \mathbb{P}^n \) satisfies (\( * \)), and \( \operatorname{Cl}(X \times \mathbb{P}^n) = \operatorname{Cl}(X) \times \mathbb{Z} \).

*Proof.* For proposition 6.5, we have an exact sequence, and it splits:

\[
\begin{array}{c}
\mathbb{Z} \\
\downarrow \\
\operatorname{Cl}(X \times \mathbb{P}^n) \\
\downarrow \\
\operatorname{Cl}(X \times \mathbb{A}^n) \\
\end{array} \to 0
\]

All that remains is to check that the image of \( 1 \in \mathbb{Z} \), being the class of \( X \times H \) (\( H \) a hyperplane), has infinite order. But this will follow by an argument as above, noting that the class of a hyperplane in \( \operatorname{Cl}(\mathbb{P}^n) \) has infinite order, by proposition 6.4. \( \square \)

\( \sim \) To save a bit of time, I’m skipping the next few examples. Note, however, that the nonsingular quadric surface \( xy = zw \) in \( \mathbb{P}^3 \), which is just the Segre embedding of \( \mathbb{P}^1 \times \mathbb{P}^1 \), has class group \( \mathbb{Z} \times \mathbb{Z} \), with the obvious generators.

\( \sim \) Let \( k \) be algebraically closed. A curve over \( k \) is an integral separated scheme \( X \) of finite type over \( k \), of dimension one. If \( X \) is proper over \( k \), we say it is complete.

\( \sim \) **Proposition 6.7:** Let \( X \) be a nonsingular curve over \( k \) with function field \( K \). Then \( X \) is projective iff \( X \) is complete iff \( X = t(C_K) \).

\( \sim \) **Proposition 6.8:** Let \( X \) be a complete nonsingular curve over \( k \), \( Y \) be any curve over \( K \), and let \( f : X \to Y \) be a nonconstant morphism. Then \( f(X) = Y \), \( K(X) \) is a finite extension of \( K(Y) \), \( f \) is a finite morphism, and \( Y \) is complete.

\( \sim \) To study divisors on a nonsingular curve \( X \), we define a map \( \deg : \operatorname{Div}(X) \to \mathbb{Z} \) which sends each prime divisor (i.e. each closed point) to 1.

\( \sim \) If \( f : X \to Y \) is a finite morphism of curves, we define the degree of \( f \) to be the degree of the extension \( [K(X) : K(Y)] \).
Given a finite map \( f : X \to Y \), we obtain a \( f^* : \text{Div}(Y) \to \text{Div}(X) \) by the following method. To define \( f^*(Q) \), let \( t \in \mathcal{O}_Q \) be a local parameter at \( Q \); an element of \( K(Y) \) with \( v_Q(t) = 1 \). Calculate \( v_P(f^*t) \) for each of the (finitely many) points \( P \in f^{-1}(Q) \), and use these as the coefficients of said \( P \) in \( f^*(Q) \). This induces a homomorphism \( f^* : \text{Cl}(Y) \to \text{Cl}(X) \). Moreover, Proposition 6.9 states that \( \deg(f^*(Q)) = \deg(f) \) for all \( Q \in Y \), so that:

\(~ \) Corollary 6.10: A principal divisor on a complete nonsingular curve \( X \) has degree zero, so that the degree function gives an epimorphism \( \text{Cl}(X) \to \mathbb{Z} \).

\(~ \) Example 6.10.1: By proposition 6.8 and 6.9: a complete nonsingular curve is rational (birational to \( \mathbb{P}^1 \)) if and only if there exist two distinct linearly equivalent points.

\[ \text{Proof.} \] If \( X \) is rational, it’s already projective, by 6.7, so we’re done. Suppose instead that \( P \sim Q \). Then for some \( f \in K(X) \), viewed as a morphism \( X \to \mathbb{P}^1 \), we have \( f^{-1}(0) = P \), \( f^{-1}(\infty) = Q \). By proposition 6.9, \( \deg(f) = 1 \), so that \( [K(X) : K(\mathbb{P}^1)] \) has degree one, and \( X \) is rational. \( \square \)

\(~ \) Example 6.10.2: Let \( X \) be the nonsingular cubic \( y^2z = x^3 - xz^2 \) in \( \mathbb{P}^2 \) (char \( k \neq 2 \)). \( X \) is not rational. Let \( \text{Cl}^0 \) \( X \) be the kernel of \( \deg : \text{Cl}(X) \to \mathbb{Z} \), a nonzero group. We’ll see that the group \( \text{Cl}^0(X) \) can be put in bijection with the points of \( X \), making \( X \) an abelian group variety.

Whenever \( P, Q \) and \( R \) are three points of \( X \), we will write \( \overrightarrow{PQR} \) as shorthand for \( "\ P, Q \ \text{and} \ \ R \ \text{are collinear}"^4. The observation which generates all others is that if \( \overrightarrow{PQR} \) and \( \overrightarrow{ABC} \), then \( P + Q + R \), and \( A + B + C \) are linearly equivalent divisors.

Now let \( P_0 \) be any fixed point of \( X \)^5. Given any two points \( R \) and \( S \), find \( I \) such that \( \overrightarrow{RP_0I} \) and then \( D \) such that \( \overrightarrow{ISD} \). Then \( R - S = D - P_0 \). Given any two points \( P \) and \( Q \), find \( I \) such that \( \overrightarrow{PQI} \) and then \( S \) such that \( \overrightarrow{IS} \). Then \( P + Q = S + P_0 \). We can rewrite these equations as:

\[ (R - P_0) - (S - P_0) = (D - P_0); \quad (P - P_0) + (Q - P_0) = (S - P_0). \]

In particular, every element of the group \( \text{Cl}^0(X) \) can be written as \( (P - P_0) \) for some \( P \in X \). This element is unique, as otherwise, \( P \sim Q \) for some \( P \neq Q \), which (by 6.10.1) would contradict the fact that \( X \) is not rational. Thus \( X \) is in bijection with \( \text{Cl}^0(X) \) via \( P \leftrightarrow (P - P_0) \), and the two constructions of \( S \) and \( D \) used above describe the sum and difference in the induced group structure on \( X \).

If \( X \) is a complete nonsingular curve, then \( \text{Cl}^0(X) \) is isomorphic to the set of closed points of the Jacobian variety \( J(X) \) of \( X \). The dimension of \( J(X) \) equals the genus of \( X \)! The whole of \( \text{Cl}(X) \) is an extension of \( \mathbb{Z} \) by the group of closed points of \( J(X) \).

**Cartier Divisors**

First, we will need an analogue for general schemes \( X \) of the function field of an integral scheme. This will be the sheaf \( \mathcal{K} \) of total quotient rings. It has an associated sheaf of

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^4Note that if \( f \) is unramified at \( P \), the coefficient \( v_P(f^*(t)) \) of \( P \) in \( f^*(Q) \) is one.

^5i.e. \( \overrightarrow{PQR} \) iff \( P, Q \) and \( R \) are the three points of intersection of a line in \( \mathbb{P}^2 \) with \( X \), counted with multiplicity.

^5One might choose \( P_0 = [0 : 1 : 0] \), the point of inflection, so that \( P_0P_0P_0 \).
(multiplicative) abelian groups, $\mathcal{K}^*$, the invertible elements in each ring, and there is an evident short exact sequence, $1 \rightarrow \mathcal{O}^* \rightarrow \mathcal{K}^* \rightarrow \mathcal{K}^*/\mathcal{O}^* \rightarrow 1$, yielding:

\[ \Gamma(X, \mathcal{K}^*) \xrightarrow{\pi^*} \Gamma(X, \mathcal{K}^*/\mathcal{O}^*) \rightarrow H^1(X, \mathcal{O}^*) \rightarrow H^1(X, \mathcal{K}^*) \]

A Cartier divisor is an element of $\Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$, and the principal Cartier divisors are those in the image of $\pi_*$. We define the Cartier class group $\text{CaCl}(X)$ to be the cokernel of $\pi_*$. As $H^1(X, \mathcal{O}^*)$ is the Picard group $\text{Pic}(X)$ of invertible sheaves on $X$, we obtain a natural injective map $\mathcal{L} : \text{CaCl}(X) \rightarrow \text{Pic}(X)$.

\[ \Rightarrow \text{There are two ways to think about the sheaf } \mathcal{H} \text{ of total quotient rings.} \]

- For each open affine $U = \text{Spec } A$, let $K(U)$ be the total quotient ring of $A$: the localisation of $A$ by the multiplicative system of non-zero-divisors in $A$. These rings form a presheaf on the basis of open affine subsets, and the sheafification is the sheaf of total quotient rings $\mathcal{H}$.

- For each open subset $U$, let $S(U)$ be the set of elements of $\Gamma(U, \mathcal{O})$ which are map to non-zero-divisors in each local ring $\mathcal{O}_x$ for $x \in U$. The rings $\Gamma(U, \mathcal{O})[S(U)^{-1}]$ form a presheaf, whose associated sheaf of rings is also $\mathcal{H}$.

\[ \Rightarrow \text{A Cartier divisor can be specified by giving an open cover } \{U_i\} \text{ of } X \text{ and elements } f_i \in \Gamma(U_i, \mathcal{K}^*) \text{ such that each } f_i/f_j \text{ is an element of } \Gamma(U_i \cap U_j, \mathcal{O}^*). \]

\[ \Rightarrow \text{Proposition 6.11: Let } X \text{ be an integral, separated noetherian scheme, all of whose local rings are UFDs, thus } X \text{ satisfies } (*). \text{ Then } \text{Div}(X) \cong \text{CaDiv}(X), \text{ and principal Weil divisors correspond to principal Cartier divisors, so that } \text{Cl}(X) \cong \text{CaCl}(X). \]

As $X$ is integral, $\mathcal{K}^*$ is just the constant sheaf $K^*$. Given a Cartier divisor $\{(U_i, f_i)\}$, for a given prime Weil divisor $Y$, choose $i$ such that $U_i \cap Y \neq \emptyset$. Then in fact $U_i \cap Y$ is dense in $Y$, and $v_Y$ is contained in $U_i$. Thus we can calculate $v_Y(f_i)$, and make this the coefficient of $Y$. This sum is finite, as $X$ is noetherian. This construction can be carried through even when $X$ is only normal. In fact:

\[ \Rightarrow \text{Remark 6.11.2: If } X \text{ is only a normal scheme (not locally factorial), then we can define a subgroup } \text{LPDiv}(X) \text{ of } \text{Div}(X) \text{ consisting of the locally principal Weil divisors, those } D \in \text{Div}(X) \text{ which can be covered by open sets such that } D|_U \text{ is principal for all } U. \text{ Then the above construction gives an isomorphism } \text{CaDiv}(X) \cong \text{LPDiv}(X). \text{ Moreover, as principal Weil divisors are locally principal, this descends to class groups, and } \text{CaCl}(X) \text{ is isomorphic to the subgroup of } \text{Cl}(X) \text{ consisting of locally principal divisors.} \]

\[ \Rightarrow \text{Example 6.11.3: To give a Weil divisor which is not Cartier, we revisit example 6.5.2, the quadric cone } xz = y^2. \text{ We just need to see that } Y \text{ is not locally principal. That is, we should see that } (y, z) \text{ is not principal in the ring } k[x, y, z]/(xy - z^2)[S^{-1}], \text{ where } S \text{ is the multiplicative set of elements which are nonzero at } (0, 0, 0). \text{ This is true! Thus } \text{CaCl} = 0 \text{ while } \text{Cl} = \mathbb{Z}_2. \]

\[ ^6 \text{A ring all of whose localisations are UFDs is called locally factorial. For a one-dimensional local ring, being a UFD is the same as being regular.} \]
Invertible sheaves

\(\Rightarrow\) Given a Cartier divisor \(D = (U_i, f_i)\), we obtain an invertible sheaf \(\mathcal{L}(D)\) as the sub-\(\mathcal{O}_X\)-module of \(\mathcal{K}\) generated by \(f_i^{-1}\) over \(U_i\).

\(\Rightarrow\) To describe the map \(\mathcal{L}\), note that we have defined:

\[\mathcal{L}(D)(U) := \{ f \in \mathcal{K} : (f) + D \geq 0 \text{ on } U \}.\]

**Proposition 6.13:** \(D \mapsto \mathcal{L}(D)\) gives a bijection between Cartier divisors and invertible subsheaves of \(\mathcal{K}\). Moreover \(D_1 \sim D_2\) iff \(\mathcal{L}(D_1) \cong \mathcal{L}(D_2)\).

**Corollary 6.14:** \(D \mapsto \mathcal{L}(D)\) gives an injective map \(\text{CaCl}(X) \hookrightarrow \text{Pic}(X)\).

**Proposition 6.15:** \(\text{CaCl}(X) \longrightarrow \text{Pic}(X)\) is an isomorphism when \(X\) is integral.

**Corollary 6.16:** If \(X\) is a noetherian, integral, separated locally factorial scheme, then there is a natural isomorphism \(\text{CaCl}(X) \cong \text{Pic}(X)\).

**Corollary 6.17:** \(\text{Pic}(\mathbb{P}^n_\mathbb{K}) \cong \mathbb{Z}\).

\(\Rightarrow\) An effective Cartier divisor is one of the form \((U_i, f_i)\) with \(f_i \in \Gamma(U_i, \mathcal{O}_{U_i})\). Then we can define the associated subscheme of codimension one, as the subscheme defined by the sheaf of ideals locally generated by \(f_i\).

\(\Rightarrow\) This gives a bijective correspondence between effective Cartier divisors and locally principal closed subschemes. Moreover:

**Proposition 6.18:** If \(D\) is an effective Cartier divisor, and \(Y\) is the associated locally principal closed subscheme, then \(\mathcal{I}_Y \cong \mathcal{L}(-D)\).

Hartshorne II.7 — Projective Morphisms

\(\Rightarrow\) **Theorem 7.1:** Suppose \(A\) is a ring and \(X\) is a scheme over \(A\). Then it is the same to give an \(A\)-morphism \(\varphi : X \rightarrow \mathbb{P}^n_A\) as to give an invertible sheaf \(\mathcal{L}\) on \(X\) generated by global sections \(s_0, \ldots, s_n\).

*Proof.* Given \(\varphi\), let \(\mathcal{L} = \varphi^*\mathcal{O}(1)\), and let \(s_i = \varphi^*(x_i)\), where \(x_i \in \mathcal{O}(1)\). On the other hand, let \(X_i = \{ P \in X \mid (s_i)_P \notin \mathfrak{m}_P \mathcal{L}_P \}\). This is an open subset of \(X\) (which is not the support of \(s_i\)). Define a morphism \(X_i \rightarrow U_i\) via a map \(A \left[ \frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i} \right] \rightarrow \Gamma(X_i, \mathcal{O}_{X_i})\) sending \(\frac{r}{x_i}\) to the ratio of \(s_r\) to \(s_i\) in each stalk. \(\square\)

**Example 7.1.1: Automorphisms of \(\mathbb{P}^n_\mathbb{K}\).** Every automorphism of \(\mathbb{P}^n_\mathbb{K}\) is \(\text{PGL}\). Suppose that \(\varphi\) is an automorphism. Then \(\varphi^*(\mathcal{O}(1))\) generates \(\text{Pic}(\mathbb{P}^n)\), so is \(\mathcal{O}(\pm 1)\). But it must have a nonzero section, so that \(\varphi^*(\mathcal{O}(1)) \cong \mathcal{O}(1)\). But now \(\varphi\) is determined by the global sections \(\varphi^*(x_i)\), which must a basis of the \((n+1)\)-dimensional vector space \(\Gamma(\mathbb{P}^n_\mathbb{K}, \mathcal{O}(1))\).

**Proposition 7.2:** If \(\varphi\) is the morphism as in 7.1, then \(\varphi\) is a closed immersion iff

1. each open set \(X_i\) is affine, and
2. the maps \(A \left[ \frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i} \right] \rightarrow \Gamma(X_i, \mathcal{O}_{X_i})\) are surjective.

**Proposition 7.3:** If \(A = k = \overline{k}\) is an algebraically closed field, and \(X\) is projective over \(k\). Let \(V \subset \Gamma(X, \mathcal{L})\) be the subspace spanned by the \(s_i\). Then \(\varphi\) is a closed immersion iff

1. elements of \(V\) separate points: for all distinct closed points \(P, Q\), there’s an \(s \in V\) such that \(s \in \mathfrak{m}_P \mathcal{L}_P\) but \(s \notin \mathfrak{m}_Q \mathcal{L}_Q\) or vice versa, and

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elements of $V$ separate tangent vectors: for each closed point $P$, the set $\{s \in V \mid s_P \in m_P.L_P\}$ spans the cotangent space $m_P.L_P/m_P^2.L_P$.

An invertible sheaf $\mathcal{L}$ on a noetherian scheme is said to be ample if for all coherent sheaves $\mathcal{F}$ there is an integer $n_0$ such that for $n \geq n_0$, $\mathcal{F} \otimes \mathcal{L}^n$ is g.b.g.s. Note that ‘ample’ is absolute, while ‘very ample’ is relative.

Proposition 7.5: For $\mathcal{L}$ an invertible sheaf on a noetherian scheme, $\mathcal{L}$ is ample iff all positive tensor powers of $\mathcal{L}$ are ample iff some positive tensor power of $\mathcal{L}$ is ample.

Theorem 7.6: Let $X$ be a scheme of finite type over a noetherian ring $A$, and $\mathcal{L}$ an invertible sheaf. The $\mathcal{L}$ is ample iff some power of $\mathcal{L}$ is very ample over Spec $A$.

Example 7.6.2: The invertible sheaf of type $(a,b)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ is very ample when $a,b > 0$, using two uple embeddings. It is not generated by global sections when either of $a$ or $b$ are negative, so the ample sheaves coincide with the very ample sheaves.

Given an invertible sheaf $\mathcal{L}$ on $X$, and a section $s \in \Gamma(X,\mathcal{L})$, define an effective divisor $D = (s)_0$, the divisor of zeros as follows. Over an open set where $\mathcal{L}$ is trivial, choose an isomorphism $\varphi : \mathcal{L}|_U \rightarrow O_U$, and take the effective Cartier divisor $\{(U, \varphi(s))\}$.

Proposition 7.7: Let $X$ be a nonsingular projective variety over an algebraically closed field $k$ (so that Weil and Cartier divisors are equivalent). For any divisor $D_0$ there is a bijection:

$$\mathbb{P}(\Gamma(X, \mathcal{L})) \rightarrow \{\text{effective divisors } D \mid D \sim D_0\} =: |D_0|$$

The sets $|D_0|$ are called complete linear systems on $X$. A linear system $\mathfrak{d}$ is a linear subspace of a complete linear system. Its dimension is its dimension as a linear projective variety.

No two points on an elliptic curve are linearly equivalent: It is the same to show that an intersection $\ell \cap C$ is only linearly equivalent to other such intersections. [UMMM.. There’s a SES:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(1) \xrightarrow{x} \mathcal{O}_{\mathbb{P}^2}(1) \rightarrow \mathcal{O} \rightarrow 0$$

Now pullback is exact, so that it can be shown that there’s a surjection.

$$H^0(X, \mathcal{O}_{\mathbb{P}^2}(1)) \rightarrow H^0(C, \mathcal{O}_{\mathbb{P}^2}(1)|_C).$$

But $\ell \cap C$ is the divisor arising from the section of $H^0(C, \mathcal{O}_{\mathbb{P}^2}(1)|_C)$ given by the restriction of some linear form on $\mathbb{P}^2$ to $C$. So we are done.

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