SECOND PRACTICE MIDTERM
MATH 18.703, MIT, SPRING 13

You have 80 minutes. This test is closed book, closed notes, no calculators.

There are 6 problems, and the total number of points is 100. Show all your work. *Please make your work as clear and easy to follow as possible.* Points will be awarded on the basis of neatness, the use of complete sentences and the correct presentation of a logical argument.

Name:______________________________

Signature:______________________________

Student ID #:______________________________

Problem	Points	Score
1	15	
2	15	
3	15	
4	15	
5	20	
6	15	
Presentation	5	
Total	100	

1. (15pts) (i) Give the definition of an irreducible element of an integral domain.

Solution: Let R be an integral domain. We say that $a \in R$ is irreducible if whenever $a = bc$ then either b or c is a unit.

(ii) Give the definition of a prime element of an integral domain.

Solution: Let R be an integral domain. We say that $p \in R$ is prime if the ideal $\langle p \rangle$ is a prime ideal, not equal to either $\{0\}$ or R.

(iii) Give the definition of a principal ideal domain.

Solution: Let R be an integral domain. We say that R is a principal ideal domain if every ideal is principal.

2. (15pts) (i) State the Sylow Theorems.

Solution: Let G be a group of order of order n and let p be a prime dividing n.
Then the number of Sylow p-subgroups is equal to one modulo p, divides n and any two Sylow p-subgroups are conjugate.

(ii) Prove that there is no simple group of order 120.

Solution: $120 = 2^3 \cdot 3 \cdot 5$. Then the number n_5 of Sylow 5-subgroups is congruent to one modulo 5 and divides $2^3 \cdot 3$. By the first condition

$$n_5 = 1, 6, 11, 16, 21$$

so that either $n_5 = 1$ or $n_5 = 6$. If $n_5 = 1$ then there is a unique Sylow 5-subgroup P and this subgroup must be normal. If $n_5 = 6$ then there is a nontrivial action of G on the set of Sylow 5-subgroups and so there is a representation

$$\rho\colon G \longrightarrow S_6.$$

If ρ is not injective then the kernel is a normal subgroup. If ρ is injective then G is isomorphic to a subgroup of S_6. Consider $H = A_6 \cap G \subset G$. If G is not contained in A_6 then H is a subgroup of index 2, in which case it is normal. So we may assume that G is a subgroup of A_6 of index 3. The action of A_6 on the left cosets G in A_6 defines a non-trivial representation

$$\rho\colon A_6 \longrightarrow S_3,$$

whose kernel is a non-trivial normal subgroup of A_6, which contradicts the fact that A_6 is simple.

3. (15pts) Let R be an integral domain and let I be an ideal. Show that R/I is a field iff I is a maximal ideal.

Solution: We first show that if R is an integral domain then R is a field if and only if $\{0\}$ and R are the only ideals.
Suppose that R is a field. If $I \neq \{0\}$ is an ideal then pick $a \neq 0 \in I$. As R is a field we may find $b \in R$ such that $ba = 1$. But then $1 \in I$ so that $I = R$. Thus $\{0\}$ and R are the only ideals.
Now suppose that $\{0\}$ and R are the only ideals. Pick $a \neq 0 \in R$. Then $I = \langle a \rangle$ is a non-zero ideal. It follows that $I = R$ and so $1 \in I$. But then we may find $b \in R$ such that $1 = ba$. Thus a has an inverse and R is a field.
Now suppose that I is an ideal. Then there is a correspondence between ideals $J \triangleleft R$ which contain I and ideals I' in the quotient ring R/I; given J let $I' = J/I$ be the image and given I', let J be the inverse image, under the natural map $R \longrightarrow R/I$.
Then R/I is a field if and only if $I' = \{0\}$ and $I' = R/I$ are the only ideals if and only if $J = I$ and $J = R$ are the only ideals containing I if and only if I is maximal.

4. (15pts) Let R be a ring.
(i) If
$$I_1 \subset I_2 \subset I_3 \subset \cdots \subset I_n \subset \cdots ,$$
is an ascending sequence of ideals then the union I is an ideal.

Solution: Suppose that a and $b \in I$. Then $a \in I_m$ and $b \in I_n$ some m and n. Let $k = \max(m, n)$. Then a and $b \in I_k$ so that $a + b \in I_k \subset I$. Thus I is closed under addition.
Now suppose that $a \in I$ and $r \in R$. Then $a \in I_m$ some m and $ra \in I_m \subset I$, so that I is an ideal.

(ii) Show that if R is a PID then every ascending chain condition of ideals eventually stabilises.

Solution: By assumption $I = \langle a \rangle$ for some $a \in I$. But then $a \in I_m$, some m. Let $n \geq m$. Then
$$\langle a \rangle \subset I_m \subset I_n \subset I = \langle a \rangle.$$
It follows that $I_n = \langle a \rangle$, so that the sequence above stabilises.

5. (20pts) Let R be a ring and let I and J be two ideals.
(i) Show that the intersection $I \cap J$ is an ideal.

Solution: Suppose that a and $b \in I \cap J$. Then $a, b \in I$ so that $a+b \in I$. Similarly $a + b \in J$. But then $a + b \in I \cap J$ so that $I \cap J$ is closed under addition.
Now suppose that $a \in I \cap J$ and $r \in R$. Then $a \in I$ and so $ra \in I$. Similarly $ra \in J$. But then $ra \in I \cap J$ and so $I \cap J$ is an ideal.

(ii) Is the union $I \cup J$ an ideal?

Solution: No. Let $R = \mathbb{Z}$, $I = \langle 2 \rangle$ and $J = \langle 3 \rangle$, the multiples of 2 and 3. Then $2 \in I \subset I \cup J$ and $3 \in J \subset I \cup J$ but $5 = 2 + 3 \notin I$ and $5 = 2 + 3 \notin J$ so that $I \cup J$ is not closed under addition.

6. (15pts) Let R be a principal ideal domain and let a and b be two non-zero elements of R. Show that the gcd d of a and b exists and prove that there are elements r and s of R such that

$$d = ra + sb.$$

Solution:
As R is a PID, there is an element d of R such that

$$\langle a, b \rangle = \langle d \rangle.$$

As $a \in \langle a, b \rangle = \langle d \rangle$, $d|a$. Similarly $d|b$. Suppose that $d'|a$ and $d'|b$. Then $a \in \langle d' \rangle$ and $b \in \langle d' \rangle$, so that

$$\langle d \rangle \langle a, b \rangle \subset \langle d' \rangle.$$

But then $d'|d$ so that d is the gcd.
Since $d \in \langle d \rangle = \langle a, b \rangle$, there are elements r and s of R such that

$$d = ra + sb.$$