FIRST PRACTICE MIDTERM
MATH 18.703, MIT, SPRING 13

You have 80 minutes. This test is closed book, closed notes, no calculators.

There are 7 problems, and the total number of points is 100. Show all your work. *Please make your work as clear and easy to follow as possible.* Points will be awarded on the basis of neatness, the use of complete sentences and the correct presentation of a logical argument.

Name:______________________________

Signature:______________________________

Student ID #:______________________________

Problem	Points	Score
1	15	
2	15	
3	20	
4	10	
5	10	
6	15	
7	10	
Presentation	5	
Total	100	

1. (15pts) Give the definition of a normal subgroup.

Solution:
Let G be a group and let H be a subgroup. We say that H is normal in G, if for every $g \in G$,
$$gHg^{-1} \subset H.$$

(ii) Give the definition of a homomorphism.

Solution:
A function
$$\phi \colon G \longrightarrow G',$$
between two groups is said to be a homomorphism if for every x and $y \in G$,
$$\phi(xy) = \phi(x)\phi(y).$$

(iii) Give the definition of A_n, the alternating group.

Solution:
The alternating group A_n is the subgroup of S_n consisting of all even permutations.

2. (15pts) (i) Exhibit a proper normal subgroup V of A_4. To which group is V isomorphic to?

Solution:

$$V = \{\, e, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\, \}.$$

V is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(ii) Give the left cosets of V inside A_4.

Solution:

$$\begin{aligned} [e] &= V \\ [(1,2,3)] &= \{(1,2,3), (1,3,4), (2,4,3), (1,4,2))\} \\ [(1,3,2)] &= \{(1,3,2), (2,3,4), (1,2,4), (1,4,3)\} \end{aligned}$$

(iii) To which group is A_4/V isomorphic to?

Solution:
As it has order 3, it must be isomorphic to $\mathbb{Z}_3$.

3. (20pts) Let G be a group and let H be a subgroup. Prove that the following are equivalent.

(1) H is normal in G.
(2) For every $g \in G$, $gHg^{-1} = H$.
(3) For every $a \in G$, $aH = Ha$.
(4) The set of left cosets is equal to the set of right cosets.

Solution: Suppose that H is normal in G. Then for all $a \in G$,

$$aHa^{-1} \subset H.$$

Taking $a = g$ and $a = g^{-1}$ we get

$$gHg^{-1} \subset H \qquad \text{and} \qquad g^{-1}Hg \subset H.$$

Multiplying the second inclusion on the left by g and on the right by g^{-1} we get,

$$H \subset gHg^{-1}.$$

Hence (2) holds. Now suppose that (2) holds. Multiplying

$$aHa^{-1} = H,$$

on the right by a, we get

$$aH = Ha.$$

Hence (3) holds. Now suppose that (3) holds. Then (4) certainly holds. Finally suppose (4) holds. Pick $g \in G$. Then $g \in gH$ and $g \in Hg$. As the set of left cosets equals the set of right cosets, it follows that $gH = Hg$. Multiplying on the right by g^{-1} we get

$$gHg^{-1} = H.$$

As g is arbitrary, it follows that H is normal in G. Hence (1). Thus the four conditions are certainly equivalent.

4. (10pts) Let G be a group and let N be a normal subgroup. Prove that G/N is abelian iff N contains the commutator of every pair of elements of G.

Solution:
Suppose that G/N is abelian. Let a and $b \in G$ and set $x = aN$ and $y = bN$. As G/N is abelian, $xy = yx$, so that $abN = baN$ and $ab = ban$, for some $n \in N$. But then

$$n = a^{-1}b^{-1}ab \in N,$$

so that N contains the commutator of a and b.
Now suppose that N contains the commutator of any pair of elements of G. Pick x and $y \in G/N$. Then $x = aN$ and $y = bN$. We have

$$\begin{aligned} yx &= (bN)(aN) \\ &= baN \\ &= ba(a^{-1}b^{-1}ab)N \\ &= abN \\ &= xy. \end{aligned}$$

Thus G/N is abelian.

5. (10pts) Let H and K be two normal subgroups of a group G, whose intersection is the trivial subgroup. Prove that every element of H commutes with every element of K.

Solution:
Let $h \in H$ and $k \in K$ and let $a = hkh^{-1}k^{-1}$. As K is normal, $hkh^{-1} \in K$, so that $a = (hkh^{-1})k^{-1} \in K$. On the other hand, as H is normal $kh^{-1}k^{-1} \in H$ and so $a = h(kh^{-1}k^{-1}) \in H$. Thus $a \in H \cap K$ and so $a = e$. Thus $hk = kh$ and so h and k commute. But then H and K commute.

6. (15pts) (i) State the Sylow Theorems.

Solution: Let G be a group of order of order n and let p be a prime dividing n.
Then the number of Sylow p-subgroups is equal to one modulo p, divides n and any two Sylow p-subgroups are conjugate.

(ii) Prove that if G is a group of order pq, where p and q are distinct primes, then G is not simple.

Solution:
Let n_q be the number of Sylow q-subgroups. Then $n_q = 1$ or $n_q \geq q+1$ and n_q divides n. Therefore n_q divides p so that $n_q = 1$. But then there is a unique subgroup Q of order q and so Q is normal in G.

7. (10pts) Let G be a group and let N be a normal subgroup. Show that there is a natural bijection between the set of subgroups H of G which contain N and the set of subgroups of the quotient group G/N. Show that this bijection preserves normality, so that normal subgroups of G which contain N correspond to normal subgroups of G/N.

Solution:
There is a natural group homomorphism

$$u\colon G \longrightarrow G/N \qquad \text{given by} \qquad g \longrightarrow gN.$$

Let H be a subgroup of G which contains N. Then N is normal in H and $H/N = u(H)$, which is a group with the induced law of multiplication, so that is a subgroup of G/N.
Conversely, if $H' \subset G/N$ is a subgroup of G/N let $H = u^{-1}(H')$. If h_1 and $h_2 \in H$ then $u(h_1)$ and $u(h_2) \in H'$. But then

$$u(h_1h_2) = u(h_1)u(h_2) \in H',$$

as H' is a subgroup so that $h_1h_2 \in H$ and H is closed under products. If $h \in H$ then $u(h) \in H'$ and so

$$u(h^{-1}) = (u(h))^{-1} \in H',$$

so that $h^{-1} \in H$. Therefore H is a subgroup of G and H contains N. It is clear that $u(H) = H'$ so that we have a bijection between the set of subgroups H of G which contain N and the set of subgroups of G/N.
Now suppose that H is normal in G. We check H/N is normal in G. Let $gN \in G/N$ and $hN \in H/N$. Then $ghg^{-1} \in H$ as H is normal and we have

$$\begin{aligned}(gN)(hN)(gN)^{-1} &= (gN)(hN)(g^{-1}N)\\ &= (ghg^{-1})N \in H/N.\end{aligned}$$

Thus H/N is normal.
Now suppose H' is normal. Suppose that $h \in H = u^{-1}(H')$ and $g \in G$. Then

$$u(ghg^{-1}) = u(g)u(h)u(g)^{-1} \in H',$$

as H' is normal. But then $ghg^{-1} \in H$ and H is normal.