MODEL ANSWERS TO HWK #5

1. There are two cosets. The first coset is $[1] = N$, the second is the coset containing -1, which is the set of all negative real numbers.
$[1] \cdot [1] = [1]$, $[1] \cdot [-1] = [-1] \cdot [1] = [-1]$ and $[-1] \cdot [-1] = [1]$.
2. Let $a \in \mathbb{R}$. Then $[a] = \{a, -a\}$. Thus any coset contains two elements, exactly one of which is a positive real number. Given a and b positive, $[a][b] = [ab]$. Define a homomorphism

$$\phi \colon G \longrightarrow \mathbb{R}^+,$$

by sending a to $|a|$. The kernel is $N = \{1, -1\}$. By the first Isomorphism Theorem, $G/N \simeq \mathbb{R}^+$.
3. Consider the canonical homomorphism

$$u \colon G \longrightarrow G/N.$$

Then $M = u^{-1}(\overline{M})$. As the kernel of u is N, it follows that M contains N, as $\overline{M}$ contains the identity of G/N.
To show that M is a subgroup of G, it suffices to prove that it is closed under products and inverses. Suppose that a and b are in M. Then $u(a)$ and $u(b)$ are in $\overline{M}$. Then $u(ab) = u(a)u(b) \in \overline{M}$ as $\overline{M}$ is closed under products. Thus $ab \in M$ and M is closed under products.
Similarly $u(a^{-1}) = u(a)^{-1} \in \overline{M}$ as $\overline{M}$ is closed under inverses. Thus $a^{-1} \in M$ and M is closed under inverses.
Thus M is a subgroup of G.
(ii) Suppose that $\overline{M}$ is normal in G/N. Pick $g \in G$. We want to prove $gMg^{-1} \subset M$. Pick $a \in M$. Then

$$u(gag^{-1}) = u(g)u(a)u(g)^{-1}.$$

As $\overline{M}$ is normal in G/N, it follows that $u(g)u(a)u(g)^{-1} \in \overline{M}$. But then $gag^{-1} \in M$.
4. As G is cyclic, G is generated by a single element a. But then G/N is generated by $u(a) = aN$.
5. Pick two elements of G/N. As G/N is the set of left cosets in G, these two elements have the form aN and bN. It follows that

$$\begin{aligned}(aN)(bN) &= abN \\ &= baN \\ &= (bN)(aN),\end{aligned}$$

where we use the fact that G is abelian to deduce $ab = ba$. But then G/N is abelian.
6. Suppose that G/Z is cyclic. Then there is an element a of G such that aZ generates G/Z. Hence every left coset has the form $a^i Z$, for some i. Pick two elements x and y of G. Then $xZ = a^i Z$ and $yZ = a^j Z$, for some i and j, so that $x = a^i z_1$ and $y = a^j z_2$, where z_1 and $z_2 \in Z$. We have

$$\begin{aligned} xy &= (a^i z_1)(a^j z_2) \\ &= a^i a^j z_1 z_2 \\ &= a^{i+j} z_1 z_2, \end{aligned}$$

where we used the fact that z_1 is in the centre. Similarly $yx = a^{i+j} z_1 z_2$. Thus $xy = yx$ and G is abelian.
7. Suppose that G/N is abelian. Pick a and $b \in G$. Then

$$\begin{aligned} abN &= aNbN \\ &= bNaN \\ &= baN, \end{aligned}$$

so that $abN = baN$ and $ab = ban$, for some $n \in N$. It follows that $a^{-1}b^{-1}ab = n \in N$, for every a and b.
Now suppose that $n = a^{-1}b^{-1}ab \in N$, for every a and b. We have

$$ban = ba(a^{-1}b^{-1}ab) = ab,$$

and so

$$\begin{aligned} aNbN &= abN \\ &= baN \\ &= bNaN, \end{aligned}$$

and G/N is abelian.
8. We want to use the First Isomorphism Theorem. Define a homomorphism

$$\phi \colon G \longrightarrow \mathbb{R}$$

by sending f to $\phi(f) = f(1/4)$. Suppose that f and $g \in G$. Then

$$\begin{aligned} \phi(f+g) &= (f+g)(1/4) \\ &= f(1/4) + g(1/4) \\ &= \phi(f) + \phi(g). \end{aligned}$$

Thus ϕ is a homorphism. ϕ is clearly surjective. For example, given a real number a, let f be the constant function $f(x) = a$. Then $\phi(f) = f(1/4) = a$.

The kernel of ϕ consists of all functions that vanish at $1/4$, that is, N. Thus by the First Isomorphism Theorem, $G/N \simeq \mathbb{R}$.
9. We first prove (i) and (iii). Define a homomorphism

$$\phi\colon G \longrightarrow G_2,$$

by sending $g = (g_1, g_2)$ to g_2. Suppose that $g = (g_1, g_2)$ and $h = (h_1, h_2)$ are in G. Then

$$\begin{aligned}\phi(gh) &= \phi(g_1h_1, g_2h_2)\\ &= g_2h_2\\ &= \phi(g_1, g_2)\phi(h_1, h_2)\\ &= \phi(g)\phi(h).\end{aligned}$$

Thus ϕ is a homomorphism. ϕ is clearly surjective as given $g_2 \in G$, $\phi(e_1, g_2) = g_2$.
Suppose that $(g_1, g_2) \in \operatorname{Ker}\phi$. Then $g_2 = e_2$. Thus $N = \operatorname{Ker}\phi$. Hence (i). (iii) follows from the First Isomorphism Theorem.
To prove (ii), define a homomorphism

$$f\colon N \longrightarrow G_1$$

by sending (g_1, e_2) to g_1. This is clearly an isomorphism.
10. By definition the order of a is the order of the subgroup $H = \langle a\rangle$ and the order of aN is the order of the subgroup $H' = \langle aN\rangle$. Now it is clear that H' is the image of H under the canonical homomorphism

$$u\colon G \longrightarrow G/N.$$

So it suffices to prove that if we have a surjective homomorphism

$$\phi\colon H \longrightarrow H'$$

then the order of H' divides the order of H. But by the first isomorphism Theorem,

$$H' \simeq H/H'',$$

where H'' is the kernel of ϕ. Thus the order of H' is the index of H'' in H, the number of left cosets of H'' in H, which by Lagrange divides the order of H.
11. Let $\phi\colon G_1 \times G_2 \longrightarrow G_2 \times G_1$ be the homomorphism that sends (g_1, g_2) to (g_2, g_1). This is clearly a bijection. We check that it is a

homomorphism. Suppose that (g_1, g_2) and $(h_1, h_2) \in G_1 \times G_2$. Then

$$\begin{aligned}\phi((g_1, g_2)(h_1, h_2)) &= \phi(g_1 h_1, g_2 h_2)\\ &= (g_2 h_2, g_1 h_1)\\ &= (g_2, g_1)(h_2, h_1)\\ &= \phi(g_1, g_2)\phi(h_1, h_2).\end{aligned}$$

Thus ϕ is an isomorphism.
Alternatively, we could use the universal property of the product. Both $G_1 \times G_2$ and $G_2 \times G_1$ satisfy the universal properties of a product and so they must be isomorphic, by uniqueness.
12. Let $h \in H$ and $k \in K$ and let $a = hkh^{-1}k^{-1}$. As K is normal, $hkh^{-1} \in K$, so that $a = (hkh^{-1})k^{-1} \in K$. On the other hand, as H is normal $kh^{-1}k^{-1} \in H$ and so $a = h(kh^{-1}k^{-1}) \in H$. Thus $a \in H \cap K$ and so $a = e$. Thus $hk = kh$ and so h and k commute. But then H and K commute.
13. Suppose that G is isomorphic to $H' \times K'$. Then we might as well assume that $G = H' \times K'$. In this case take $H = H' \times \{f\}$ and $K' = \{e\} \times K$, where e is the identity of H' and f is the identity of K'. In this case we already proved in question 9 that H and K are normal in G and (i) holds. Suppose that $(a, b) \in H \cap K$. Then $a = e$ and $b = f$ so that $(a, b) = (e, f)$ is the identity of G. Hence (ii) holds. Suppose that $(h', k') \in G$, where $h' \in H'$ and $k' \in K'$. Then $(h', k') = (h', f)(e, k') = hk$ where $h = (h', f) \in H$ and $k = (e, k') \in K$. Thus $(h', k') \in H \vee K$ and $G = H \vee K$. Hence (iii).
Now suppose that (i-iii) hold. Since H and K generate G, every element of G is a product of elements of H and K. As H and K are normal in G and (ii) holds, we proved in question 10 that the elements of H commute with the elements of K. Thus it is easy to prove that HK is closed under products and inverses and it follows that every element of G is of the form hk so that $G = HK$.
Define a homomorphism

$$\phi \colon G \longrightarrow H \times K,$$

by sending $g = hk$ to (h, k). Suppose that $h_1 k_1 = h_2 k_2$. Then $h_2^{-1} h_1 = k_2 k_1^{-1} \in H \cap K$. Thus $h_2^{-1} h_1 = k_2 k_1^{-1} = e$, the identity of G. Thus $h_1 = h_2$ and $k_1 = k_2$. Thus ϕ is well-defined.
The composition of ϕ with the two projection maps are the two identities, and these are both homomorphisms. By the universal property of a product, it follows that ϕ is a homomorphism.
ϕ is clearly surjective, and it is injective, as the kernel is clearly trivial.

Thus ϕ is an isomorphism and G is isomorphic to $H \times K$. But $H \times K$ is clearly isomorphic to $H' \times K'$ and so we are done.

14. Let X be the set of elements of $\mathbb{Z}/3\mathbb{Z}$. Put a binary operation $\star$ on X be setting

$$[a] \star [b] = [a - b].$$

It is clear that this operation is well-defined. The element [0] acts as identity and

$$[a] \star [a] = [a - a] = [0],$$

so that every element is its own inverse. However set $c = [3] = [0]$, $b = [2]$ and $a = [1]$. Then

$$[c]-([b]-[a]) = [3]-[1] = [2] \qquad \text{and} \qquad ([c]-[b])-[a] = [2]-[1] = [1] \neq [2]$$

Thus $\star$ is not associative.