MODEL ANSWERS TO HWK #3

1. (a)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 2 & 1 & 3 & 6 \end{pmatrix}.$$

The cycle decomposition is

$$(1,4)(2,5,3)$$

and so the order is 6.

(b)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 4 & 5 \end{pmatrix}.$$

The cycle decomposition is

$$(1,3,2)$$

and so the order is 3.

(c)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 2 & 5 \end{pmatrix}.$$

The cycle decomposition is

$$(2,4)$$

and so the order is 2.

2. As σ and τ are cycles, we may find integers $a_1, a_2, \dots, a_k$ and $b_1, b_2, \dots, b_l$ such that $\sigma = (a_1, a_2, \dots, a_k)$ and $\tau = (b_1, b_2, \dots, b_l)$. To say that σ and τ are disjoint cycles is equivalent to saying that the two sets $S = \{a_1, a_2, \dots, a_k\}$ and $T = \{b_1, b_2, \dots, b_l\}$ are disjoint.

We want to prove that

$$\sigma\tau = \tau\sigma.$$

As both sides of this equation are permutations of the first n natural numbers, it suffices to show that they have the same effect on any integer $1 \leq j \leq n$.

If j is not in $S \cup T$, then there is nothing to prove; both sides clearly fix j. Otherwise $j \in S \cup T$. By symmetry we may asume $j \in S$. As S and T are disjoint, it follows that $j \notin T$.

As $j \in S$, $j = a_i$, some i. Then $\sigma(a_i) = a_{i+1}$, where we take $i+1$ modulo k (that is, we adopt the convention that $a_{k+1} = a_1$). In this case $a_{i+1} \in S$ so $a_{i+1} \notin T$ as well. Thus both sides send $j = a_i$ to a_{i+1}.

Thus both sides have the same effect on j, regardless of j and so

$$\sigma\tau = \tau\sigma.$$

3. (i)

$$(1, 3, 4, 2)(5, 7, 9),$$

which has order 12.
(ii)

$$(1, 7)(2, 6)(3, 5),$$

which has order 2.
(iii)

$$(1, 6)(2, 5)(3, 7),$$

which has order 2.
4.

$$(2, 4, 1)(3, 5, 7, 6) = (2, 1)(2, 4)(3, 6)(3, 7)(3, 5).$$

The order is 12.
(f)

$$(1, 4, 2, 5, 3) = (1, 3)(1, 5)(1, 2)(1, 4).$$

The order is 5.
5. The conjugate is $(2, 7, 5, 3)(1, 6, 4)$. The order of σ is 12 and the order of τ is three.
6. There are quite a few possibilities for τ. One obvious one is

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 2 & 5 & 4 & 7 & 6 \end{pmatrix}.$$

7. Let $H = \langle (1, 2)(1, 2, 3, \ldots, n) \rangle$. We want to show that H is the whole of S_n. As the transpositions generate S_n, it suffices to prove that every transposition is in H.
Now the idea is that it is very hard to compute products in S_n, but it is easy to compute conjugates. So instead of using the fact that H is closed under products and inverses, let us use the fact that it is closed under taking conjugates (clear, as a conjugate is a product of elements of H and their inverses).
Since conjugation preserves cycle type, we start with the transposition $\sigma = (1, 2)$ (in fact this is the only place to start).
To warm up, consider conjugating σ with $\tau = (1, 2, 3, \ldots, n)$. The conjugate is $(2, 3)$. Thus H must contain $(2, 3)$.
Given that H contains $(2, 3)$ it must contain the conjugate of $(2, 3)$ by τ, which is $(3, 4)$ (or what comes to the same thing, H must contain the conjugate of $(1, 2)$ by τ^2).

Continuing in this way, it is clear that H (by an easy induction in fact) must contain every transposition of the form $(i, i+1)$ and of course the last one, $(n, 1) = (1, n)$.

There are now two ways to show that $H = S_n$.

The first is to complete the proof that H must contain every transposition. First, let us try to show that H contains every transposition of the form $(1, i)$. For example, to get $(1, 3)$, start with $(1, 2)$ and conjugate it by $(2, 3)$. Suppose, by way of induction, that H contains $(1, i)$. Then H must contain the conjugate of $(1, i)$ by $(i, i + 1)$ which is $(1, i + 1)$. Thus by induction H contains every transposition of the form $(1, i)$.

Now we are almost home. First note that H must contain every transposition of the form $(2, j)$. Indeed $(2, j)$ is the conjugate of $(1, j)$ by the transposition $(1, 2)$.

Now consider an arbitrary transposition (i, j). This is the conjugate of $(1, 2)$ by the element $(1, i)(2, j)$. Thus H contains every transposition and so $H = S_n$. This completes the first proof.

For the second proof we consider the second proof that the transpositions generate S_n. We need to show that we can put any deck of cards, in arbitrary order, into the standard order by only switching adjacent cards. By induction on i, we may assume that the first $i - 1$ cards are in the correct order. We may suppose that the ith card is in position $j > i$ (if it is in position i there is nothing to do). We need to check that we can put card i in the jth position into the ith position, without moving the first $i - 1$ cards. If we switch cards j and $j - 1$, then the ith card is now in the $j - 1$th position. Therefore switching cards cards j and $j - 1$, then cards $j - 1$ and $j - 2$, ..., and then cards $i + 1$ and i, card i now occupies the ith position and we haven't moved the first $i - 1$ cards (note though that the card that was in the ith position is not now in the jth position; it is in position $i + 1$).

This completes the induction on i. Thus one can undo any permutation just using the transpositions $(i, i + 1)$ which is to say that these transpositions generate S_n.