11. The Alternating Groups

Consider the group S_3. Then this group contains a normal subgroup, generated by a 3-cycle.

Now the elements of S_3 come in three types. The identity, the product of zero transpositions, the transpositions, the product of one transposition, and the three cycles, products of two transpositions. Then the normal subgroup above, consists of all permutations that can be represented as a product of an even number of transpositions.

In general there is no canonical way to represent a permutation as a product of transpositions. But we might hope that the pattern above continues to hold in every permutation group.

Definition 11.1. *Let $\sigma \in S_n$ be a permutation.*

We say that σ is **even** *if it can be represented as a product of an even number of transpositions. We say that σ is* **odd** *if it can be represented as a product of an odd number of transpositions.*

The following result is much trickier to prove than it looks.

Lemma 11.2. *Let $\sigma \in S_n$ be a permutation.*

Then σ is not both an even and an odd permutation.

There is no entirely satisfactory proof of (11.2). Here is perhaps the simplest.

Definition 11.3. *Let $x_1, x_2, \dots, x_n$ be indeterminates and set*

$$f(x_1, x_2, \dots, x_n) = \prod_{i<j}(x_i - x_j).$$

For example, if $n = 3$, then

$$f(x_1, x_2, x_3) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3).$$

Definition 11.4. *Given a permutation $\sigma \in S_n$, let*

$$g = \sigma^*(f) = \prod_{i<j}(x_{\sigma(i)} - x_{\sigma(j)}).$$

Suppose that $\sigma = (1, 2) \in S_3$. Then

$$g = \sigma^*(f) = (x_2-x_1)(x_2-x_3)(x_1-x_3) = -(x_1-x_2)(x_1-x_3)(x_2-x_3) = -f.$$

The following Lemma is the key part of the proof of (11.2).

Lemma 11.5. *Let σ and τ be two permutations and let $\rho = \sigma\tau$. Then*

(1) $\sigma^*(f) = \pm f$.
(2) $\rho^*(f) = \sigma^*(\tau^*(f))$.
(3) $\sigma^*(f) = -f$, whenever σ is a transposition.

Proof. g is clearly a product of terms of the form $x_i - x_j$ or $x_j - x_i$, where $i < j$. Thus $g = \pm f$. Hence (1).

$$\begin{aligned}\sigma^*(\tau^*(f)) &= \sigma^*(\prod_{i<j}(x_{\tau(i)} - x_{\tau(j)}) \\ &= \prod_{i<j}(x_{(\sigma(\tau(i))} - x_{\sigma(\tau(j))})) \\ &= \prod_{i<j}(x_{\rho(i)} - x_{\rho(j)}) \\ &= \rho^*(f).\end{aligned}$$

Hence (2).

Suppose that $\sigma = (a, b)$, where $a < b$. Then the only terms of f affected by σ are the ones that involve either x_a or x_b. There are three cases:

- $i \neq a$ and $j = b$
- $i = a$ and $j \neq b$
- $i = a$ and $j = b$.

Suppose $i \neq a$ and $j = b$. If $i < a$, then $x_i - x_b$ is sent to $x_i - x_a$ and there is no change of sign. If $a < i < b$ then $x_a - x_i$ is sent to $x_b - x_i = -(x_i - x_b)$. Thus there is a change in sign. If $i > b$ then $x_a - x_i$ is sent $x_b - x_i$ and there is no change in sign.

Similarly if $i = a$ and $j \neq b$. If $j < a$ or $j > b$ there is no change in sign. If $a < j < b$ there is a change in sign. But then the first two cases contribute in total an even number of signs changes (in fact, there will be exactly $(a - b - 1) + (a - b - 1) = 2(a - b - 1)$ sign changes).

Finally we need to consider the case $i = a$ and $j = b$. In this case $x_a - x_b$ gets replaced by $x_b - x_a$ and there is a change in sign. Hence (3). □

Proof. Suppose that σ is a product of an even number of transpositions. Then by (2) and (3) of (11.5), $\sigma^*(f) = f$. Similarly if $\sigma^*(f)$ is a product of an odd number of transpositions, then $\sigma^*(f) = -f$. Thus σ cannot be both even and odd. □

Definition-Lemma 11.6. *There is a surjective homomorphism*

$$\phi\colon S_n \longrightarrow \mathbb{Z}_2$$

The kernel consists of the even transpositions, and is called the **alternating group** A_n.

Proof. The map sends an even transposition to 1 and an odd transposition to -1. (2) of (11.5) implies that this map is a homomorphism. □

Note that half of the elements of S_n are even, so that the alternating group A_n contains $\frac{n!}{2}$. One of the most important properties of the alternating group is,

Theorem 11.7. *Suppose that* $n \neq 4$.

The only normal subgroup of S_n *is* A_n. *Moreover* A_n *is simple, that is,* A_n *has no proper normal subgroups.*

If $n = 4$ then we have already seen that

$$\{e, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}.$$

is a normal subgroup of S_4. In fact it is also a normal subgroup of A_4, so that A_4 is not simple.