HOMEWORK #8, DUE THURSDAY APRIL 18TH

1. Herstein, Chapter 4, §3, 1.
2. Herstein, Chapter 4, §3, 2.
3. Let I and J be ideals of a ring R.
(i) Show that $I \cap J$ is an ideal of R.
(ii) Show that

$$I + J = \{\, i + j \,|\, i \in I, j \in J \,\}$$
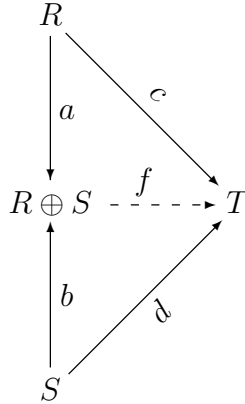
is an ideal of R.
(iii) Let IJ be the set of all finite sums of elements of the form ij, where $i \in I$ and $j \in J$. Show that IJ is an ideal of R.
4. Herstein, Chapter 4, §3, 9.
5. Let R be the set of all rational numbers whose denominator (when in reduced form) is not divisible by p, p a fixed prime.
(i) Show that R is a subring of the rational numbers.
(ii) Let I be the subset of R of rational numbers whose numerator is divisible by p. Show that I is an ideal of R and $R/I \simeq \mathbb{Z}_p$.
6. Herstein, Chapter 4, §3, 16.
7. Herstein, Chapter 4, §3, 18.
8. Herstein, Chapter 4, §3, 19.
9. Herstein, Chapter 4, §3, 20.
10. Herstein, Chapter 4, §3, 22 & 23: Let $\mathbb{Z}$ be the ring of integers, and m, n two coprime integers, I_m the multiples of m and I_n the multiples of n.
(i) What is $I_m \cap I_n$?
(ii) Show that there is an injective homomorphism from $\mathbb{Z}/I_{mn}$ to $\mathbb{Z}/I_m \oplus \mathbb{Z}/I_n$.
(iii) Show that $\mathbb{Z}_{mn} \simeq \mathbb{Z}_m \oplus \mathbb{Z}_n$.
11. Herstein, Chapter 4, §4, 1.
12. **Challenge Problem:** Herstein, Chapter 4, §3, 26.
13. **Challenge Problem:** Herstein, Chapter 4, §3, 27.
Some parenthetical remarks
For some parts of questions 7–10, you may find it useful to use the fact that $R \oplus S$ is both a direct product and a **direct sum**. We have already seen the definition of the direct product. The direct sum is the same as the direct product except that the arrows are reversed. That is, the direct sum $R \oplus S$ of R and S satisfies the following universal property:

There are ring homomorphisms, $a\colon R \longrightarrow R \oplus S$ and $b\colon S \longrightarrow R \oplus S$ such that given any pair of ring homomorphisms $c\colon R \longrightarrow T$ and $d\colon S \longrightarrow T$ there is a unique ring homomorphism $f\colon R \oplus S \longrightarrow T$ such that the following diagram commutes,

$$
\begin{array}{ccc}
R & & \\
\Big\downarrow a & \searrow^{c} & \\
R \oplus S & \xrightarrow{\;\;f\;\;} & T \\
\Big\uparrow b & \nearrow_{d} & \\
S & &
\end{array}
$$

(The reader is invited to prove that $R \oplus S$ does indeed satisfy the given universal property.)

Note that it is very unusual for the same object to be both the direct product and the direct sum (for rings we focus on the fact that $R \oplus S$ is the direct sum; if we wanted to focus on the fact that it is the direct product, we would use the notation $R \times S$). Consider for example what happens in the category of sets.