PRACTICE FINAL B
MATH 18.02, MIT, AUTUMN 12

You have three hours. This test is closed book, closed notes, no calculators.

There are 16 problems, and the total number of points is 240. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:______________________________

Signature:______________________________

Student ID #:______________________________

Recitation instructor:______________________________

Recitation Number+Time:______________________________

Problem	Points	Score
1	15	
2	10	
3	10	
4	10	
5	15	
6	20	
7	15	
8	15	
9	15	
10	15	
11	15	
12	15	
13	15	
14	15	
15	20	
16	20	
Total	240	

1. (15pts) (i) Let $A = (1, 2, 3)$, $B = (4, -1, 4)$ and $C = (2, 4, 6)$. Find the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Solution: $\overrightarrow{AB} = \langle 3, -3, 1\rangle$ and $\overrightarrow{AC} = \langle 1, 2, 3\rangle$. Then

$$\cos\theta = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} = \frac{\langle 3, -3, 1\rangle \cdot \langle 1, 2, 3\rangle}{|\langle 3, -3, 1\rangle||\langle 1, 2, 3\rangle|} = 0.$$

So $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are orthogonal.

(ii) Let $P = (0, 1, 1)$, $Q = (2, 1, 0)$ and $R = (1, 3, 2)$. Find the cross product of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

Solution: $\overrightarrow{PQ} = \langle 2, 0, -1\rangle$ and $\overrightarrow{PR} = \langle 1, 2, 1\rangle$.

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 0 & -1 \\ 1 & 2 & 1 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} = 2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}.$$

(iii) Find the equation of the plane containing P, Q and R.

Solution:
This plane is orthogonal to $\vec{n} = \langle 2, -3, 4\rangle$ and contains the point $P = (0, 1, 1)$. So

$\langle x, y-1, z-1\rangle\cdot\langle 2, -3, 4\rangle = 0$ so that $2x-3(y-1)+4(z-1) = 0$.

Rearranging $2x - 3y + 4z = 1$.

2. (10pts) At what point does the line through $(1, 0, -1)$ and $(2, 1, 2)$ intersect the plane $x + y - z = 3$?

Solution: The line through these points is given parametrically by

$$\vec{r}(t) = \langle 1, 0, -1\rangle + t\langle 1, 1, 3\rangle = \langle 1 + t, t, -1 + 3t\rangle.$$

This meets the plane when

$$(1 + t) + t + 1 - 3t = 3 \qquad \text{that is} \qquad t = -1.$$

The point is then $(0, -1, -4)$.

3. (10pts) Give parametric equations for the line given as the intersection of the two planes $2x - y - 4z = 0$ and $5x - 2z = 1$.

Solution:
We need to find two points on the line. Intersect with a third plane. Try the plane given by $x = 1$. We get $z = 2$ and $y = -6$. So $(1, -6, 2)$ lies on the line. Now try the plane $x = -1$. We get $z = -3$ and $y = 10$. So $(-1, 10, -3)$ lies on the line.
The line is given as

$$\vec{r}(t) = \langle 1, -6, 2\rangle + t\langle -2, 16, -5\rangle = \langle 1 - 2t, -6 + 16t, 2 - 5t\rangle.$$

4. (10pts) A moon M revolves around a planet P in a circular orbit of radius one in the xy-plane, so that in one year it completes two revolutions. Meanwhile the planet revolves around a star O in a circular orbit of radius five, one revolution per year. The star is always at the origin and at time $t = 0$ the planet and the moon are on the x-axis, the planet to the right of the sun and the moon to the right of the planet. Find the position of the moon as a function of the number of years t.

Solution: We are given that

$$\overrightarrow{PM} = \langle \cos 2t, \sin 2t \rangle \qquad \text{and} \qquad \vec{P} = \langle 5\cos t, 5\sin t \rangle.$$

So

$$\vec{M} = \vec{P} + \overrightarrow{PM} = \langle \cos 2t, \sin 2t \rangle + \langle 5\cos t, 5\sin t \rangle = \langle \cos 2t + 5\cos t, \sin 2t + 5\sin t \rangle.$$

So the position of the moon at time t is $(\cos 2t + 5\cos t, \sin 2t + 5\sin t)$.

5. (15pts) Let S be the surface defined by the equation

$$z = x^2y + xy^2 - 3y^2.$$

(i) Find the tangent plane to S at the point $P = (2, 1, 3)$.

Solution: Let

$$f(x, y) = x^2y + xy^2 - 3y^2.$$

We have

$$f_x = 2xy + y^2 \qquad \text{and} \qquad f_y = x^2 + y^2 - 6y.$$

The equation of the tangent plane is

$$(z - 3) = f_x(x - 2) + f_y(y - 1) = 5(x - 2) + 2(y - 1).$$

(ii) Give a formula approximating the change Δz in z if x and y change by small amounts Δx and Δy.

Solution:

$$\Delta z \approx 5\Delta x + 2\Delta y.$$

(iii) Approximate the value of z at the point $(x, y) = (2.01, 1.01)$.

Solution:

$$z = f(2.01, 1.01) = f(2, 1) + \Delta z \approx 3 + 7(.01) = 3.07.$$

6. (20pts) A rectangular box lies in the first quadrant. One vertex is at the origin and the diagonally opposite vertex P is on the plane $2x+y+z=2$. We want the coordinates of the point P which maximises the volume of the box.

(i) Show that this lead to maximising the function

$$f(x,y) = xy(2-2x-y).$$

Find the critical points of $f(x,y)$.

Solution: The volume is

$$xyz = xy(2-2x-y) = 2xy - 2x^2y - xy^2.$$

To find the critical points, set the partials equal to zero:

$$f_x = 2y - 4xy - y^2 = y(2-4x-y) = 0 \qquad \text{and} \qquad f_y = 2x - 2x^2 - 2xy = 2x(1-x-y) = 0.$$

If $xy \neq 0$ then $4x+y=2$ and $x+y=1$. The critical points are

$$(0,0) \qquad (0,2) \qquad (1,0) \qquad \text{and} \qquad (1/3, 2/3).$$

(ii) Determine the type of the critical point in the first quadrant.

Solution: Use 2nd derivative test:

$$f_{xx} = -4y \qquad f_{xy} = 2-4x-2y \qquad \text{and} \qquad f_{yy} = -2x.$$

So at $(1/3, 2/3)$

$$A = -\frac{8}{3} \qquad B = -\frac{2}{3} \qquad \text{and} \qquad C = -\frac{2}{3}.$$

Hence $AC - B^2 > 0$. As $A < 0$ we have a maximum.

(iii) Now solve this problem using the method of Lagrange multipliers.

Solution: We introduce a new variable λ and find x, y, z and λ such that:

$$\begin{aligned} yz &= 2\lambda \\ xz &= \lambda \\ xy &= \lambda \\ 2x+y+z &= 2. \end{aligned}$$

We are assuming $x>0$, $y>0$ and $z>0$. Dividing the second equation by the third we have $y=z$. Dividing the first equation by the second we have $y=2x$. Hence $y+y+y=2$ and so $y=2/3$, $x=1/3$ and $z=2/3$. This is a maximum since

- it is the only critical point, and
- as we approach the boundary the volume tends to zero.

7. (15pts) Find the point on the surface

$$z^2 = xy + x + 1$$

closest to the origin, using the method of Lagrange multipliers.

Solution: We minimise the square of the distance to the origin:

$$\text{minimise} \quad x^2 + y^2 + z^2 \qquad \text{subject to} \qquad z^2 - xy - x = 1.$$

To use the method of Lagrange multipliers, we introduce another variable λ and find x, y, z and λ such that $\nabla f = \lambda \nabla g$:

$$\begin{aligned} 2x &= -\lambda(y+1) \\ 2y &= -\lambda x \\ 2z &= \lambda 2z \\ z^2 - xy - x &= 1. \end{aligned}$$

Either $z = 0$ or $\lambda = 1$. Suppose $z = 0$. Then $x(y+1) = -1$. If we multiply the first equation by x we get $2x^2 = \lambda$. If we multiply the second equation by $y+1$ we get $2y(y+1) = \lambda = 2x^2$ and so $y(y+1) = x^2$. Multiply both sides by x to get $y = -x^3$. Finally add one to both sides and multiply by x to get $x^4 - x + 1 = 0$. We check that $g(x) = x^4 - x + 1$ has no real roots. $g'(x) = 4x^3 - 1$, so g has a minimum where $x^3 = 1/4$. But $g(1/4) > 0$ so $g(x)$ has no real roots. Therefore $z = 0$ is impossible.

If $\lambda = 1$ then $2y = -x$ and $2x = -y - 1$. Hence $4x + 2y = -2$, so that $3x = -2$, $x = -2/3$, $y = 1/3$, and $z = 1/3$.

8. (15pts) Let $w(x, y, z) = x^4 + 2xy^2 - z^3$.
(i) Find the equation of the tangent plane to the surface $w = 2$ at $(1, 1, 1)$.

Solution:

$$\nabla w = \langle 4x^3 + 2y^2, 4xy, -3z^2 \rangle \qquad \text{so that} \qquad (\nabla w)_{(1,1,1)} = \langle 6, 4, -3 \rangle.$$

The tangent plane has normal vector $\vec{n} = \langle 6, 4, -3 \rangle$ and contains $(1, 1, 1)$:

$$\langle x-1, y-1, z-1 \rangle \cdot \langle 6, 4, -3 \rangle = 0 \qquad \text{so that} \qquad 6(x-1)+4(y-1)-3(z-1) = 0$$

Rearranging, we have $6x + 4y - 3z = 7$.

(ii) Assume that x, y and z are constrained by the equation $w(x, y, z) = 2$. Find the value of

$$\left(\frac{\partial x}{\partial z}\right)_y$$

at $(1, 1, 1)$.

Solution: We use the method of differentials:

$$0 = \mathrm{d}w = w_x \,\mathrm{d}x + w_y \,\mathrm{d}y + w_z \,\mathrm{d}z = 6 \,\mathrm{d}x + 4 \,\mathrm{d}y - 3 \,\mathrm{d}z.$$

We solve for $\mathrm{d}x$:

$$\mathrm{d}x = -\frac{2}{3} \,\mathrm{d}y + \frac{1}{2} \,\mathrm{d}z.$$

Therefore

$$\left(\frac{\partial x}{\partial z}\right)_y = \frac{1}{2}.$$

9. (15pts) Let R be the plane triangle with vertices $(0,0)$, $(1,-1)$ and $(1,1)$. Set up an iterated integral which gives the average distance of a point from the origin,
(i) in rectangular coordinates

Solution: The area of R is 1. So the average distance is

$$\iint_R r \, \mathrm{d}A = \int_0^1 \int_{-x}^{x} \sqrt{x^2+y^2} \, \mathrm{d}y \, \mathrm{d}x.$$

(ii) in polar coordinates.

Solution:

$$\iint_R r \, \mathrm{d}A = \int_{-\pi/4}^{\pi/4} \int_0^{\sec\theta} r^2 \, \mathrm{d}r \, \mathrm{d}\theta.$$

10. (15pts) Let C_1 be the line segment from $(0,0)$ to $(1,0)$, C_2 the arc of the unit circle running from $(1,0)$ to $(0,1)$ and let C_3 be the line segment $(0,1)$ to $(0,0)$. Let C be the simple closed curve formed by C_1, C_2 and C_3 and let

$$\vec{F} = x^3\hat{\imath} + x^2y\hat{\jmath}.$$

Calculate

$$\oint_C \vec{F}\cdot \mathrm{d}\vec{r},$$

(i) directly.

Solution:

Parametrise C_1 by $\vec{r}(t) = \langle t, 0\rangle$, $0 \le t \le 1$, so that $\vec{F} = \langle t^3, 0\rangle$ and $\mathrm{d}\vec{r} = \langle 1, 0\rangle$. Then

$$\oint_{C_1} \vec{F}\cdot \mathrm{d}\vec{r} = \int_0^1 t^3\,\mathrm{d}t = \left[\frac{t^4}{4}\right]_0^1 = \frac{1}{4}.$$

Parametrise C_2 by $\vec{r}(t) = \langle \cos t, \sin t\rangle$, so that $\vec{F} = \langle \cos^3 t, \cos^2 t \sin t\rangle$ and $\mathrm{d}\vec{r} = \langle -\sin t, \cos t\rangle$. Then

$$\oint_{C_2} \vec{F}\cdot \mathrm{d}\vec{r} = \int_0^1 0\,\mathrm{d}t = 0.$$

Parametrise C_2 by $\vec{r}(t) = \langle 0, 1-t\rangle$, $0 \le t \le 1$, so that $\vec{F} = \langle 0, 0\rangle$ and $\mathrm{d}\vec{r} = \langle 0, -1\rangle$. Then

$$\oint_{C_3} \vec{F}\cdot \mathrm{d}\vec{r} = \int_0^1 0\,\mathrm{d}t = 0.$$

Therefore

$$\oint_C \vec{F}\cdot \mathrm{d}\vec{r} = \oint_{C_1} \vec{F}\cdot \mathrm{d}\vec{r} + \oint_{C_2} \vec{F}\cdot \mathrm{d}\vec{r} + \oint_{C_3} \vec{F}\cdot \mathrm{d}\vec{r} = \frac{1}{4}.$$

(ii) using Green's theorem.

Solution: Let R be the region bounded by C. Note that $\operatorname{curl}\vec{F} = 2xy$.

$$\oint_C \vec{F}\cdot \mathrm{d}\vec{r} = \iint_R \operatorname{curl}\vec{F}\,\mathrm{d}A = \int_0^{\pi/2}\int_0^1 2r^3\cos\theta\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta.$$

The inner integral is

$$\int_0^1 2r^3\cos\theta\sin\theta\,\mathrm{d}r = \left[\frac{r^4}{2}\cos\theta\sin\theta\right]_0^1 = \frac{1}{2}\cos\theta\sin\theta.$$

The outer integral is

$$\int_0^{\pi/2} \frac{1}{2}\cos\theta\sin\theta\,\mathrm{d}\theta = \left[\frac{1}{4}\sin^2\theta\right]_0^{\pi/2} = \frac{1}{4}.$$

11. (15pts) (i) Calculate the flux of $\vec{F} = x\hat{\imath}$ out of each side, S_1, S_2, S_3 and S_4 of the square $-1 \le x \le 1$, and $-1 \le y \le 1$. Label the sides so that S_1 and S_3 are horizontal, S_1 below S_3, and S_2 and S_4 are vertical, S_2 to the right of S_4.

Solution:
As $\vec{F}$ represents motion parallel to the x-axis, the flux out of the two horizontal sides, S_1 and S_3 is zero. Along S_2, $\hat{n} = \hat{\imath}$ and $\vec{F} = \hat{\imath}$, so that

$$\int_{S_2} \vec{F} \cdot \hat{n}\, \mathrm{d}s = \int_{S_2} 1\, \mathrm{d}s = 2,$$

since S_2 has length 2. Along S_4, $\hat{n} = -\hat{\imath}$ and $\vec{F} = -\hat{\imath}$, so that

$$\int_{S_2} \vec{F} \cdot \hat{n}\, \mathrm{d}s = \int_{S_2} 1\, \mathrm{d}s = 2,$$

since S_4 has length 2.

(ii) Explain why the total flux out of any square of sidelength 2 is the same, regardless of its location or how its sides are tilted.

Solution:
Let C be the sum of the sides of the square and let S be the interior of the square. Green's theorem in normal form says

$$\oint_C \vec{F} \cdot \hat{n}\, \mathrm{d}s = \iint_S \operatorname{div} \vec{F}\, \mathrm{d}A.$$

But the divergence $\operatorname{div} \vec{F} = 1$ is constant and the integral on the right is nothing but the area of S, which is always 4, regardless of the position of the square.
On the other hand, the integral on the left is the total flux.

12. (15pts) Find the area of the region R bounded by the curves $xy = 2$, $xy = 5$, $y = x^2$ and $y = 4x^2$.

Solution: Let $u = xy$ and $v = y/x^2$. Then

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} y & x \\ -2\frac{y}{x^3} & \frac{1}{x^2} \end{vmatrix} = \frac{3y}{x^2} = 3v.$$

As $J > 0$ over R we have

$$\mathrm{d}u\,\mathrm{d}u = 3v\mathrm{d}x\,\mathrm{d}y.$$

So the area of R is

$$\iint_R 1\,\mathrm{d}A = \iint_R 1\,\mathrm{d}x\,\mathrm{d}y = \int_1^4 \int_2^5 \frac{1}{3v}\,\mathrm{d}u\,\mathrm{d}v.$$

The inner integral is

$$\int_2^5 \frac{1}{3v}\,\mathrm{d}u = \left[\frac{u}{3v}\right]_2^5 = \frac{1}{v}.$$

The outer integral is

$$\int_1^4 \frac{1}{v}\,\mathrm{d}v = \Big[\ln v\Big]_1^4 = \ln 4.$$

13. (15pts) Let

$$\vec{F} = (y - z)\hat{\imath} + (x + y)\hat{\jmath} + (1 - x)\hat{k}$$

(i) Find a potential function f for $\vec{F}$.

Solution: We solve the three pdes

$$f_x = y - z \qquad f_y = x + y \qquad \text{and} \qquad f_z = 1 - x.$$

If we integrate the first pde with respect to x we get

$$f(x, y, z) = yx - zx + g(y, z),$$

where $g(y, z)$ is an arbitrary function of y and z. Plugging this into the second pde we get

$$x + g_y = x + y \qquad \text{so that} \qquad g_y = y.$$

Integrating this equation with respect to y we get

$$g(y, z) = \frac{y^2}{2} + h(z),$$

where $h(z)$ is an arbitrary function of z. So $f(x, y, z) = yx - zx + y^2/2 + h(z)$. Plugging this into the third pde we get

$$-x + h_z = 1 - x \qquad \text{so that} \qquad h_z = 1.$$

Therefore $h(z) = z + c$ and

$$f(x, y, z) = yx - zx + \frac{y^2}{2} + z$$

is a potential function.

(ii) Let C be the parametric curve

$$x = 3\cos^3 t \qquad y = 3\sin^3 t \qquad z = t \qquad \text{for} \qquad 0 \leq t \leq 2\pi.$$

Find

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r}.$$

Solution: By the fundamental theorem of calculus

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_C \nabla f \cdot \mathrm{d}\vec{r} = f(3, 0, 2\pi) - f(3, 0, 0) = -6\pi + 2\pi - -6\pi = 2\pi.$$

14. (15pts) Let D be the portion of the solid sphere

$$x^2 + y^2 + z^2 < 1,$$

lying above the plane

$$z = \frac{\sqrt{2}}{2}.$$

The surface bounding D consists of two parts, a curved part S and flat part T. Orient both surfaces so that the normal vector points upwards. Let

$$\vec{F} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}.$$

(i) Calculate the flux of $\vec{F}$ across S.

Solution: We use spherical coordinates:

$$\mathrm{d}\vec{S} = \langle x, y, z\rangle \sin\phi \, \mathrm{d}\phi \, \mathrm{d}\theta.$$

So the flux of $\vec{F}$ across S is

$$\iint_S \vec{F} \cdot \mathrm{d}\vec{S} = \int_0^{2\pi} \int_0^{\pi/4} (x^2 + y^2 + z^2) \sin\phi \, \mathrm{d}\phi \, \mathrm{d}\theta.$$

As $x^2 + y^2 + z^2 = 1$ on S, the inner integral is

$$\int_0^{\pi/4} \sin\phi \, \mathrm{d}\phi = \Big[-\cos\phi \Big]_0^{\pi/4} = 1 - \frac{1}{\sqrt{2}}.$$

The outer integral is

$$\int_0^{2\pi} \left(1 - \frac{\sqrt{2}}{2}\right) \mathrm{d}\theta = 2\pi - \sqrt{2}\pi.$$

(ii) Calculate the flux of $\vec{F}$ across T.

Solution: We project T down to the xy-plane. We get a circle of radius $1/\sqrt{2}$ centred at the origin. We have

$$\mathrm{d}\vec{S} = \hat{k} \, \mathrm{d}A.$$

So the flux of $\vec{F}$ across T is

$$\iint_T \vec{F} \cdot \mathrm{d}\vec{S} = \iint_R \frac{\sqrt{2}}{2} \, \mathrm{d}A = \frac{\sqrt{2}\pi}{4},$$

as the area of R is $\pi/2$.

(iii) Find the volume of D using the divergence theorem.

Solution: As $\operatorname{div} \vec{F} = 3$, the divergence theorem says

$$\oiint_S \vec{F} \cdot \mathrm{d}\vec{S} - \oiint_T \vec{F} \cdot \mathrm{d}\vec{S} = \oiint_{S-T} \vec{F} \cdot \mathrm{d}\vec{S} = \iiint_D \operatorname{div} \vec{F} \, \mathrm{d}V = \iiint_D 3 \, \mathrm{d}V = 3V,$$

where V is the volume of D. So the volume of D is

$$\frac{\pi}{12}\left(8 - 5\sqrt{2}\right).$$

15. (20pts) Calculate the flux of

$$\vec{F} = x\hat{\imath} + y\hat{\jmath} + (1 - 2z)\hat{k}$$

out of the solid bounded by the xy-plane and the paraboloid $z = 4 - x^2 - y^2$.

(i) directly,

Solution: The paraboloid is the graph of the function $f(x, y) = 4 - x^2 - y^2$ over the disk R of radius 2 in the xy-plane.

$$\mathrm{d}\vec{S} = \langle -f_x, -f_y, 1\rangle \,\mathrm{d}x\,\mathrm{d}y = \langle 2x, 2y, 1\rangle \,\mathrm{d}x\,\mathrm{d}y$$

The flux of $\vec{F}$ out of the curved part of the paraboloid S_1 is

$$\iint_{S_1} \vec{F}\cdot\mathrm{d}\vec{S} = \iint_R 2x^2+2y^2+1-8+2x^2+2y^2 \,\mathrm{d}x\,\mathrm{d}y = \int_0^{2\pi}\int_0^2 4r^3-7r\,\mathrm{d}r\,\mathrm{d}\theta.$$

The inner integral is

$$\int_0^2 4r^3 - 7r\,\mathrm{d}r = \left[r^4 - \frac{7}{2}r^2\right]_0^2 = 16 - 14 = 2.$$

So the flux across S_1 is 4π. For the flat bit $S_2 = R$, $\hat{n} = -\hat{k}$ and $\vec{F}\cdot\hat{n} = 1$. So the integral is -4π. The total flux is zero.

(ii) using the divergence theorem.

Solution:
$\operatorname{div}\vec{F} = 1 + 1 - 2 = 0$. So by the divergence theorem we have

$$\oint_S \vec{F}\cdot\mathrm{d}\vec{S} = \iiint_D \operatorname{div}\vec{F}\,\mathrm{d}V = 0.$$

16. (20pts) Let $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$ and let S be the surface of the hemisphere

$$x^2 + y^2 + (z-1)^2 = 1 \qquad \text{and} \qquad z \geq 1,$$

oriented updwards.

(i) Calculate the flux of $\vec{F}$ across S.

Solution: S is defined implicitly by a single function, $g(x,y,z) = x^2 + y^2 + (z-1)^2 = 1$.

$$\vec{N} = \nabla g = \langle 2x, 2y, 2(z-1) \rangle \qquad \text{and} \qquad \vec{N} \cdot \hat{k} = 2(z-1).$$

If we project onto the xy-plane, we get

$$\mathrm{d}\vec{S} = \frac{\vec{N}}{\vec{N} \cdot \hat{k}} \,\mathrm{d}x\,\mathrm{d}y = \frac{1}{(z-1)} \langle x, y, z-1 \rangle \,\mathrm{d}x\,\mathrm{d}y.$$

Therefore

$$\iint_S \vec{F} \cdot \mathrm{d}\vec{S} = \iint_S 0 \,\mathrm{d}x\,\mathrm{d}y = 0.$$

(ii) Find the curl of $\vec{F}$.

Solution:

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = 2\hat{k}.$$

(iii) Calculate the flux of $\operatorname{curl} \vec{F}$ across S using Stokes' theorem.

Solution: Let C be the unit circle in the plane $z = 1$ centred at $(0,0,1)$.

$$\iint_S \vec{F} \cdot \mathrm{d}\vec{S} = \oint_C \operatorname{curl} \vec{F} \cdot \mathrm{d}\vec{r} = 0,$$

as the work done going around C is zero, as $\operatorname{curl} \vec{F}$ is vertical.