FOURTH PRACTICE MIDTERM B
MATH 18.02, MIT, AUTUMN 12

You have 50 minutes. This test is closed book, closed notes, no calculators.

There are 5 problems, and the total number of points is 100. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:______________________________

Signature:______________________________

Student ID #:______________________________

Recitation instructor:______________________________

Recitation Number+Time:______________________________

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

1. (20pts) Set up a triple integral in cylindrical coordinates for the mass of the region of space bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 4$. Assume the density $\delta = 3x$.

Solution:

$$\iiint_R 3x \, \mathrm{d}V = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 3r^2 \cos\theta \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta.$$

2. (20pts) Set up an iterated integral, in both cylindrical and spherical coordinates, giving the average distance from the origin to the portion of the unit cylinder $x^2 + y^2 < 1$ which lies between $z = 0$ and $z = 1$.

Solution: Let D be the given solid with volume V.

$$\bar{\rho} = \frac{1}{V} \iiint_D \rho \, \mathrm{d}V.$$

Clearly $V = \pi$. In cylindrical coordinates we have

$$\bar{\rho} = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \int_0^1 r(r^2 + z^2)^{1/2} \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta.$$

In spherical coordinates we have to split the into two pieces:

$$\bar{\rho} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sec\phi} \rho^3 \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta + \frac{1}{\pi} \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{\csc\phi} \rho^3 \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta.$$

3. (20pts) A solid D has the shape of a right circular cone, with axis along the z-axis, and a flat base. The base radius and the height are a. Set up an integral in spherical coordinates which gives the gravitational attraction on a unit mass placed at the vertex. Assume the density δ is one.

Solution: Put the vertex at the origin. If the force is

$$\vec{F} = \langle F_x, F_y, F_z \rangle,$$

then $F_x = F_y = 0$ by symmetry.

$$F_z = \iiint_D \frac{Gz}{\rho^3}\,\mathrm{d}V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{a\sec\phi} G\cos\phi\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta.$$

4. (20pts) Let S be the surface formed by the part of the paraboloid $z = 1 - x^2 - y^2$ lying above the xy-plane. Orient S so that the normal vector is pointing upwards. Let $\vec{F} = x\hat{\imath} + y\hat{\jmath} + 2(1-z)\hat{k}$.
(i) Find the flux of $\vec{F}$ across S directly.

Solution: We have
$$\mathrm{d}\vec{S} = \langle 2x, 2y, 1\rangle \,\mathrm{d}x\,\mathrm{d}y.$$
So the flux is
$$\iint_S \vec{F}\cdot\mathrm{d}\vec{S} = \iint_{x^2+y^2\leq 1} \langle x, y, 2(x^2+y^2)\rangle\cdot\langle 2x, 2y, 1\rangle \,\mathrm{d}x\,\mathrm{d}y = \int_0^{2\pi}\int_0^1 4r^3\,\mathrm{d}r\,\mathrm{d}\theta.$$
The inner integral is
$$\int_0^1 4r^3\,\mathrm{d}r = \Big[r^4\Big]_0^1 = 1.$$
The outer integral is
$$\int_0^{2\pi} 1\,\mathrm{d}\theta = 2\pi.$$

(ii) By computing the flux across a simpler surface and using the divergence theorem.

Solution: Let S' be the disk $x^2+y^2 \leq 1$, $z = 0$, oriented upwards. By the divergence theorem,
$$\oiint_{S-S'} \vec{F}\cdot\mathrm{d}\vec{S} = \iiint_D \operatorname{div}\vec{F}\,\mathrm{d}V = \iiint_D 0\,\mathrm{d}V = 0.$$
So
$$\iint_S \vec{F}\cdot\mathrm{d}\vec{S} = \iint_{S'} \vec{F}\cdot\mathrm{d}\vec{S}.$$
Now $\hat{n} = \hat{k}$ so that $\vec{F}\cdot\hat{n} = 2$. Therefore the integral is 2π.

5. (20pts) Let

$$\vec{F} = (y+z)\hat{\imath} - x\hat{\jmath} + (7x+5)\hat{k},$$

be a vector field and let S be the part of the surface $z = 9 - x^2 - y^2$ that lies above the xy-plane. Orient S by using the outward normal vector. Find the outward flux of $\vec{F}$ across S.

Solution: Let S' be the surface $x^2 + y^2 < 9$, $z = 0$, oriented upwards. By the divergence theorem

$$\oiint_{S-S'} \vec{F} \cdot \mathrm{d}\vec{S} = \iiint_V \operatorname{div} \vec{F} \, \mathrm{d}V = \iiint_V 0 \, \mathrm{d}V = 0.$$

So

$$\iint_S \vec{F} \cdot \mathrm{d}\vec{S} = \iint_{S'} \vec{F} \cdot \mathrm{d}\vec{S}.$$

The unit normal to S' is $\hat{k}$. So

$$\vec{F} \cdot \hat{k} = 7x + 5.$$

We have

$$\iint_{S'} \vec{F} \cdot \mathrm{d}\vec{S} = \iint_{S'} 7x + 5 \, \mathrm{d}A = \iint_{S'} 5 \, \mathrm{d}A = 45\pi,$$

since x is skew-symmetric about the y-axis.