FOURTH PRACTICE MIDTERM A
MATH 18.02, MIT, AUTUMN 12

You have 50 minutes. This test is closed book, closed notes, no calculators.

There are 5 problems, and the total number of points is 100. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:____________________________

Signature:____________________________

Student ID #:____________________________

Recitation instructor:____________________________

Recitation Number+Time:____________________________

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

1. (20pts) Let D be the domain in the first octant cut off by the plane $3x + 2y + z = 1$. Assume the density $\delta = z$.
(a) Set up an iterated integral in rectangular coordinates for the total mass of D.

Solution:
$$M = \iiint_D z \, \mathrm{d}V = \int_0^{1/3} \int_0^{1/2-3x/2} \int_0^{1-2y-3x} z \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x.$$

(b) Evaluate only the inner integral

Solution:
$$\int_0^{1-2y-3x} z \, \mathrm{d}z = \left[\frac{z^2}{2}\right]_0^{1-2y-3x} = \frac{(1-2y-3x)^2}{2}.$$

2. (20pts) A solid hemisphere has radius a and density 1 and it is placed so its flat side is on the xy-plane, with the centre at $(0,0)$. Set up and evaluate a triple integral in spherical coordinates which gives its gravitational attraction on a unit point mass at the origin $(0,0,0)$.

Solution: Let
$$\vec{F} = \langle F_x, F_y, F_z \rangle.$$
By symmetry, $F_x = F_y = 0$.
$$F_z = \iiint_R \frac{Gz}{\rho^3}\,\mathrm{d}V = G\int_0^{2\pi}\int_0^{\pi/2}\int_0^a \cos\phi\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta.$$
The inner integral is
$$\int_0^a \cos\phi\sin\phi\,\mathrm{d}\rho = a\cos\phi\sin\phi.$$
The middle integral is
$$\int_0^{\pi/2} a\cos\phi\sin\phi\,\mathrm{d}\phi = \left[\frac{a}{2}\sin^2\phi\right]_0^{\pi/2} = \frac{a}{2}.$$
The outer integral is
$$\int_0^{2\pi}\frac{a}{2}\,\mathrm{d}\theta = \pi a.$$
So the answer is
$$\vec{F} = \langle 0, 0, G\pi a\rangle.$$

3. (20pts) Consider the surface S given by the equation

$$z = (x^2 + y^2 + z^2)^2.$$

(a) Show that S lies in the upper half space $z \geq 0$.

Solution: The RHS is never less than zero, as it is a square. So the LHS is at least zero. But then $z \geq 0$.

(b) Write out the equation for this surface in spherical coordinates.

Solution: We have

$$z = \rho \cos \phi \qquad \text{and} \qquad \rho^2 = x^2 + y^2 + z^2.$$

So

$$\rho \cos \phi = \rho^4.$$

Dividing through by ρ, we have

$$\rho^3 = \cos \phi.$$

(c) Write down an iterated integral for the volume of the region inside S.

Solution:

$$\iiint_V 1 \, \mathrm{d}V = \int_0^{2\pi} \int_0^{\pi/2} \int_0^{(\cos \phi)^{1/3}} \rho^2 \sin \phi \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta.$$

4. (20pts) Let S be the portion of the ellipsoid $x^2 + y^2 + 3z^2 = 1$ that lies above the xy-plane. Let

$$\vec{F} = (x + y^3)\hat{\imath} + (2y - e^x)\hat{\jmath} - (3z + 1)\hat{k}.$$

Compute the flux of $\vec{F}$ through S (orient S upwards).

Solution: We use the divergence theorem. Let S' be the unit disk $x^2 + y^2 \leq 1$ in the xy-plane, oriented upwards.

$$\oiint_{S-S'} \vec{F} \cdot \mathrm{d}\vec{S} = \iiint_D \operatorname{div} \vec{F} \, \mathrm{d}V = \iiint_D 0 \, \mathrm{d}V = 0.$$

So

$$\iint_S \vec{F} \cdot \mathrm{d}\vec{S} = \iint_{S'} \vec{F} \cdot \mathrm{d}\vec{S}.$$

But $\hat{n} = \hat{k}$ and $z = 0$ on S' so that $\vec{F} \cdot \hat{n} = -1$. Thus the flux is $-\pi$.

5. (20pts) Let S be the part of the surface $z = xy$ where $x^2 + y^2 < 1$. Compute the flux of

$$\vec{F} = y\hat{\imath} + x\hat{\jmath} + z\hat{k},$$

upward across S.

Solution: Let R be the region $x^2 + y^2 < 1$, $z = 0$. Let $f(x, y) = xy$, so that S is the graph of f lying over R. Then

$$\mathrm{d}\vec{S} = \langle -f_x, -f_y, 1\rangle \,\mathrm{d}x\,\mathrm{d}y = \langle -y, -x, 1\rangle \,\mathrm{d}x\,\mathrm{d}y.$$

Therefore

$$\vec{F} \cdot \mathrm{d}\vec{S} = \langle y, x, z\rangle \cdot \langle -y, -x, 1\rangle \,\mathrm{d}x\,\mathrm{d}y = (-y^2 - x^2 + xy)\,\mathrm{d}x\,\mathrm{d}y.$$

The flux of $\vec{F}$ across S is therefore

$$\iint_S \vec{F} \cdot \mathrm{d}\vec{S} = \int_0^{2\pi} \int_0^1 r^3 \cos\theta \sin\theta - r^3 \,\mathrm{d}r\,\mathrm{d}\theta$$

The inner integral is

$$\int_0^1 r^3 \cos\theta \sin\theta - r^3 \,\mathrm{d}r = \frac{1}{4}\Big[r^4(\cos\theta \sin\theta - 1)\Big]_0^1 = \frac{1}{4}(\cos\theta \sin\theta - 1).$$

The outer integral is

$$\int_0^{2\pi} \frac{1}{4}(\cos\theta \sin\theta - 1)\,\mathrm{d}\theta = \frac{1}{4}\left[\frac{1}{2}\sin^2\theta - \theta\right]_0^{2\pi} = -\frac{\pi}{2}.$$