THIRD PRACTICE MIDTERM B
MATH 18.02, MIT, AUTUMN 12

You have 50 minutes. This test is closed book, closed notes, no calculators.

There are 6 problems, and the total number of points is 100. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:________________________________

Signature:______________________________

Student ID #:____________________________

Recitation instructor:______________________

Recitation Number+Time:___________________

Problem	Points	Score
1	15	
2	15	
3	10	
4	20	
5	20	
6	20	
Total	100	

1. (15pts) Let $(\bar{x}, \bar{y})$ be the centre of mass of the triangle with vertices at $(-2,0)$, $(0,1)$ and $(2,0)$ (assume uniform density $\delta = 1$).
(i) Express $\bar{y}$ in terms of an integral.

Solution:

$$\bar{y} = \frac{1}{A}\iint_R y \,\mathrm{d}A = \frac{1}{2}\int_0^1 \int_{-2+2y}^{2-2y} y \,\mathrm{d}x \,\mathrm{d}y.$$

(ii) Find $\bar{x}$.

Solution: $\bar{x} = 0$ by symmetry.

2. (15pts) Find the moment of inertia about the origin of the half disk $x^2 + y^2 < a^2$, $x > 0$, where the density $\delta = x^2$.

Solution: Let R be the given region. We have

$$\iint_R x^2(x^2 + y^2)\,\mathrm{d}A = \int_{-\pi/2}^{\pi/2} \int_0^a r^5 \cos^2\theta \,\mathrm{d}r\,\mathrm{d}\theta.$$

The inner integral is

$$\int_0^a r^5 \cos^2\theta \,\mathrm{d}r = \left[\frac{r^6}{6}\cos^2\theta\right]_0^a = \frac{a^6}{6}\cos^2\theta.$$

The outer integral is

$$\int_{-\pi/2}^{\pi/2} \frac{a^6}{6}\cos^2\theta \,\mathrm{d}\theta = \frac{a^6}{12}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = \frac{\pi a^6}{12}.$$

3. (10pts) For which value of a is the vector field

$$\vec{F} = \left(axy - \frac{1}{x}\right)\hat{\imath} + \left(x^2 - \frac{1}{y}\right)\hat{\jmath}$$

a gradient vector field?

Solution: We have

$$M = axy - \frac{1}{x} \qquad \text{and} \qquad N = x^2 - \frac{1}{y}.$$

$\vec{F}$ is a gradient vector field if and only if

$$ax = M_y = N_x = 2x.$$

So $\vec{F}$ is a gradient vector field if and only if $a = 2$.

4. (20pts) Let $\vec{F} = 2y\hat{\imath} - x\hat{\jmath}$. Let C be the curve $y = x^2$ starting at $(0, 0)$ and ending at $(1, 1)$.
(a) Compute the work done on a particle that moves along C.

Solution: Let $x(t) = t$ and $y(t) = t^2$, $0 \le t \le 1$. Then

$$\vec{F} = \langle 2t^2, -t \rangle \qquad \text{and} \qquad \mathrm{d}\vec{r} = \langle 1, 2t \rangle \, \mathrm{d}t.$$

The work done is

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_0^1 \langle 2t^2, -t \rangle \cdot \langle 1, 2t \rangle \, \mathrm{d}t = 0.$$

(b) Compute the flux of $\vec{F}$ across C.

Solution: Let $x(t) = t$ and $y(t) = t^2$, $0 \le t \le 1$. Then

$$\vec{F} = \langle 2t^2, -t \rangle \qquad \mathrm{d}x = \mathrm{d}t \qquad \text{and} \qquad \mathrm{d}y = 2t \, \mathrm{d}t.$$

The flux is

$$\int_C \vec{F} \cdot \hat{n} \, \mathrm{d}s = \int_C x \, \mathrm{d}x + 2y \, \mathrm{d}y = \int_0^1 (t + 4t^3) \, \mathrm{d}t = \Big[t^2/2 + t^4 \Big]_0^1 = \frac{3}{2}.$$

5. (20pts) (i) Express the work done by the force field

$$\vec{F} = (5x + 3y)\hat{\imath} + (1 + \cos y)\hat{\jmath}$$

on a particle going counterclockwise once around the unit circle centred at the origin in the form

$$\int_a^b f(t)\,\mathrm{d}t.$$

Solution: Parametrise the circle by $x(t) = \cos t$, $y(t) = \sin t$, $0 \le t \le 2\pi$. In this case

$$\vec{F} = \langle 5\cos t + 3\sin t, 1 + \cos(\sin t)\rangle \qquad \text{and} \qquad \mathrm{d}\vec{r} = \langle -\sin t, \cos t\rangle\,\mathrm{d}t.$$

Hence

$$\oint_C \vec{F}\cdot \mathrm{d}\vec{r} = \int_0^{2\pi} -5\cos t\sin t - 3\sin^2 t + \cos t + \cos t\cos(\sin t)\,\mathrm{d}t.$$

(ii) Evaluate the work done by using Green's theorem.

Solution:

$$\oint_C \vec{F}\cdot \mathrm{d}\vec{r} = \iint_R -3\,\mathrm{d}A = \iint_R -3\,\mathrm{d}A = -3\pi.$$

6. (20pts) Consider the region R enclosed by the x-axis, $x = 1$ and $y = x^3$. Let $\vec{F} = (1 + y^2)\hat{\jmath}$.
(i) Use the normal form of Green's theorem to find the flux of $\vec{F}$ out of R.

Solution:

$$\oint_C \vec{F} \cdot \hat{n}\,\mathrm{d}s = \iint_R \operatorname{div} F\,\mathrm{d}A = \int_0^1 \int_0^{x^3} 2y\,\mathrm{d}y\,\mathrm{d}x.$$

The inner integral is

$$\int_0^{x^3} 2y\,\mathrm{d}y = \Big[y^2\Big]_0^{x^3} = x^6.$$

So the outer integral is

$$\int_0^1 x^6\,\mathrm{d}x = \left[\frac{x^7}{7}\right]_0^1 = \frac{1}{7}.$$

(ii) Find the flux across the horizontal side C_1 of R and the vertical side C_2 of R.

Solution: Parametrise C_1 by $x(t) = t$, $y(t) = 0$, $0 \leq t \leq 1$. Then

$$\vec{F} = \hat{\jmath} \qquad \text{and} \qquad \hat{n} = -\hat{\jmath}.$$

So

$$\int_{C_1} \vec{F} \cdot \hat{n}\,\mathrm{d}s = \int_0^1 -1\mathrm{d}t = -1.$$

Along C_2, $\vec{F}$ is parallel to $\hat{\jmath}$ and $\hat{n}$ is parallel to $\hat{\imath}$, so the flux across C_2 is zero.

(iii) Find the flux across the third side C_3.

Solution:

$$\frac{1}{7} = \oint_C \vec{F} \cdot \hat{n}\,\mathrm{d}s = \int_{C_1} \vec{F} \cdot \hat{n}\,\mathrm{d}s + \int_{C_2} \vec{F} \cdot \hat{n}\,\mathrm{d}s + \int_{C_3} \vec{F} \cdot \hat{n}\,\mathrm{d}s.$$

So the flux across C_3 is

$$\frac{1}{7} + 1 = \frac{8}{7}.$$