SECOND PRACTICE MIDTERM B
MATH 18.02, MIT, AUTUMN 12

You have 50 minutes. This test is closed book, closed notes, no calculators.

There are 6 problems, and the total number of points is 100. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:______________________________

Signature:______________________________

Student ID #:______________________________

Recitation instructor:______________________________

Recitation Number+Time:______________________________

Problem	Points	Score
1	15	
2	20	
3	20	
4	15	
5	15	
6	15	
Total	100	

1. (15pts) Two sides of a triangle are a and b and θ is the angle between them. The third side is c.
(i) Write down an approximation for Δc in terms of a, b, c, θ, Δa, Δb and $\Delta\theta$.

Solution: We have
$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$
So
$$2c\Delta c \approx (2a - 2b\cos\theta)\Delta a + (2b - 2a\cos\theta)\Delta b + 2ab\sin\theta\Delta\theta.$$

(ii) If $a = 1$, $b = 2$, $\theta = \pi/3$, is c more sensitive to changes in Δa or Δb?

Solution:
$$2a - 2b\cos\theta = 0 \qquad \text{and} \qquad (2b - 2a\cos\theta) = 3.$$
Δc is more sensitive to changes in b.

2. (20pts) A particle is moving in the xy-plane so that relative to the origin it is rotating clockwise at a rate of 3 radians per second while its distance to the origin is increasing at a rate of 7 metres per second. At a time when the particle is at $(5, -12)$, what is

$$\frac{dx}{dt}?$$

Solution: We have

$$\frac{d\theta}{dt} = -3 \qquad \text{and} \qquad \frac{dr}{dt} = 7,$$

so that

$$\mathrm{d}\theta = -3\mathrm{d}t \qquad \text{and} \qquad \mathrm{d}r = 7\mathrm{d}t.$$

At the point in question, $r = 13$, $\cos\theta = 5/13$ and $\sin\theta = -12/13$. Now

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta, \end{aligned}$$

so that

$$\begin{aligned} \mathrm{d}x &= \cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta \\ &= (7\cos\theta + 3r\sin\theta)\,\mathrm{d}t \\ &= \left(\frac{35}{13} - 36\right)\,\mathrm{d}t. \end{aligned}$$

Thus

$$\frac{dx}{dt} = \frac{35}{13} - 36$$

metres per second.

3. (20pts) Find the global maximum and minimum of

$$f(x, y) = x^2 - xy + y^2 + 4$$

over the triangle with sides $x = 0$, $y = 4$ and $y = x$.

Solution:

First find the critical points. $f_x = 2x - y$ and $f_y = -x + 2y$. Set these equal to zero. $y = 2x$ and $x = 2y$. This has the unique solution $x = y = 0$. 2nd derivative test, $A = f_{xx} = 2$, $B = f_{xy} = -1$ and $C = f_{yy} = 2$. $AC - B^2 = 4 - 1 > 0$ so we have a local maximum or minimum. $A > 0$ so we have a local minimum. So the global minimum is at $(0, 0)$. The value at this point is 4.

The maximum occurs at the boundary. When $x = 0$, we have $y^2 + 4$. This is largest at $y = 4$, when it is 20. When $y = 4$, we have $x^2 - 4x + 20$. This has a local minimum; at the other endpoint it is also equal to 20. When $x = y$, we have $x^2 + 4$. Again the maximum is at $(4, 4)$. So the maximum is 20 (which occurs at both $(0, 4)$ and $(4, 4)$).

4. (15pts) Let $u = x^2 - y^2$, $v = 2xy$ and $w = w(u, v)$.
(i) Express the partial derivatives w_x and w_y in terms of w_u and w_v (and x and y).

Solution:

$$\mathrm{d}u = 2x\,\mathrm{d}x - 2y\,\mathrm{d}y \qquad \text{and} \qquad \mathrm{d}v = 2y\,\mathrm{d}x + 2x\,\mathrm{d}y.$$

So

$$\begin{aligned}\mathrm{d}w &= w_u\,\mathrm{d}u + w_v\,\mathrm{d}v \\ &= w_u(2x\,\mathrm{d}x - 2y\,\mathrm{d}y) + w_v(2y\,\mathrm{d}x + 2x\,\mathrm{d}y) \\ &= (2xw_u + 2yw_v)\,\mathrm{d}x + (-2yw_u + 2xw_v)\,\mathrm{d}y.\end{aligned}$$

It follows that

$$w_x = 2xw_u + 2yw_v \qquad \text{and} \qquad w_y = -2yw_u + 2xw_v.$$

(ii) Assume that w solves the differential equation $uw_u + vw_v = 1$. Find $xw_x + yw_y$.

Solution: We have

$$1 = uw_u + vw_v = (x^2 - y^2)w_u + 2xyw_v.$$

So

$$\begin{aligned}xw_x + yw_y &= x(2xw_u + 2yw_v) + y(-2yw_u + 2xw_v) \\ &= 2x^2w_u + 4xyw_v - 2y^2w_u \\ &= 2(x^2 - y^2)w_u + 4xyw_v \\ &= 2.\end{aligned}$$

5. (15pts) (i) Write down the equations for the Lagrange multiplier for the point of the surface

$$x^4 + y^4 + z^4 + yz + xz + xy = 6$$

with the largest x-value.
We want to

$$\text{maximise } x \qquad \text{subject to} \qquad x^4 + y^4 + z^4 + yz + xz + xy = 6.$$

Solution:

$$\begin{aligned} 1 &= \lambda(4x^3 + z + y) \\ 0 &= \lambda(4y^3 + x + z) \\ 0 &= \lambda(4z^3 + y + z). \end{aligned}$$

(ii) If the point with the largest x-value is (x_0, y_0, z_0) find the equation of the tangent plane at this point.

Solution:
We know that $(\nabla g)_{(x_0,y_0,z_0)}$ is a normal to the tangent plane. By definition of λ, $(\nabla g)_{(x_0,y_0,z_0)}$ is parallel to $(\nabla f)_{(x_0,y_0,z_0)} = \langle 1, 0, 0 \rangle$. So the equation of the tangent plane is

$$x - x_0 = 0 \qquad \text{that is} \qquad x = x_0.$$

6. (15pts) Suppose that $x^2y + xz^2 = 1$ and let $w = x^3y$. Express

$$\left(\frac{\partial w}{\partial z}\right)_y,$$

as a function of x, y and z and determine its value when $(x, y, z) = (1, 1, 2)$.

Solution: We have

$$\mathrm{d}w = 3x^2y\,\mathrm{d}x + x^3\,\mathrm{d}y.$$

Our goal is to eliminate $\mathrm{d}x$. We have

$$(2xy + z^2)\,\mathrm{d}x + x^2\,\mathrm{d}y + 2xz\,\mathrm{d}z = 0.$$

So

$$\mathrm{d}x = \frac{-1}{2xy + z^2}(x^2\,\mathrm{d}y + 2xz\,\mathrm{d}z).$$

So

$$\mathrm{d}w = \frac{-3x^2y}{2xy + z^2}(x^2\,\mathrm{d}y + 2xz\,\mathrm{d}z) + x^3\,\mathrm{d}y = \left(\frac{-3x^4y}{2xy + z^2} + x^3\right)\mathrm{d}y + \left(\frac{-6x^3yz}{2xy + z^2}\right)\mathrm{d}z.$$

It follows that

$$\left(\frac{\partial w}{\partial z}\right)_y = \frac{-6x^3yz}{2xy + z^2}.$$

When $(x, y, z) = (1, 1, 2)$, we have

$$\left(\frac{\partial w}{\partial z}\right)_y = -2.$$