SECOND PRACTICE MIDTERM A
MATH 18.02, MIT, AUTUMN 12

You have 50 minutes. This test is closed book, closed notes, no calculators.

There are 6 problems, and the total number of points is 100. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:______________________________

Signature:______________________________

Student ID #:______________________________

Recitation instructor:______________________________

Recitation Number+Time:______________________________

Problem	Points	Score
1	20	
2	10	
3	20	
4	20	
5	15	
6	15	
Total	100	

1. (20pts) Let p be the point on the curve $x^2 + x^3 - y^2 = 3.1$ which is closest to $(2, 3)$. Use the gradient to estimate the coordinates of p.

Solution:
Let $f(x, y) = x^2 + x^3 - y^2$. As $f(2, 3) = 3$, we want to choose the point closest to $(2, 3)$ so that $\Delta f = 0.1$. So we want to move in the direction of greatest increase in f.

$$\nabla f = \langle 2x + 3x^2, -2y \rangle,$$

and so at $(x_0, y_0) = (2, 3)$ we have $\nabla f = \langle 16, -6 \rangle = 2\langle 8, -3 \rangle$. The magnitude of the gradient is $2\sqrt{73}$. So

$$\hat{u} = \frac{1}{\sqrt{73}}\langle 8, -3 \rangle$$

is the direction of the gradient. This is the direction of maximal change, the greatest increase in f. If we move along $\hat{u}$ we get a change of $2\sqrt{73}$. To get a change of 0.1 we need to move a distance of $\frac{1}{20\sqrt{73}}$, which gives a displacement of

$$\frac{1}{20\sqrt{73}}\hat{u} = \frac{1}{20 \cdot 73}\langle 8, -3 \rangle.$$

So we want the point

$$(2 + \frac{2}{5 \cdot 73}, 3 - \frac{3}{20 \cdot 73}).$$

2. (10pts) Find the equation of the tangent plane to the surface

$$x^2 + 3y^2 + 2z^2 = 12 \qquad \text{at the point} \qquad (1, -1, 2).$$

Solution: Let $f(x, y, z) = x^2 + 3y^2 + 2z^2$. Then we are at point of the level surface of f. $\nabla f = \langle 2x, 6y, 4z\rangle$. At the point $(1, -1, 2)$, $\nabla f = \langle 2, -6, 8\rangle$. So $\vec{n} = \langle 1, -3, 4\rangle$ is a normal vector to the tangent plane.

$\langle x-1, y+1, z-2\rangle \cdot \langle 1, -3, 4\rangle = 0 \qquad$ so that $\qquad (x-1)-3(y+1)+4(z-2) = 0.$

Rearranging, we get

$$x - 3y + 4z = 12.$$

3. (20pts) (i) Find the critical points of

$$w = f(x, y) = 5x^2 - 2xy + 2y^2 - 8x - 2y + 7,$$

and determine their type.

Solution: $f_x = 10x - 2y - 8$, $f_y = -2x + 4y - 2$. Set these equal to zero to find the critical points

$$\begin{aligned} 5x - y &= 4 \\ -x + 2y &= 1. \end{aligned}$$

Add twice the first equation to the second equation, $9x = 9$, so $x = 1$. But then $y = 1$. $(x_0, y_0) = (1, 1)$ is the only critical point. Apply the 2nd derivative test to determine type. $A = f_{xx} = 10$, $B = f_{xy} = -2$, $C = f_{yy} = 4$. $AC - B^2 = 40 - 4 = 36 > 0$. $A > 0$ so we have a local minimum.

(ii) Find where $f(x, y)$ is smallest in the first quadrant, $x \geq 0$ and $y \geq 0$. Justify your answer.

Solution:
$(x_0, y_0) = (1, 1)$ is a local minimum, $f(1, 1) = 2$. There are no other critical points, so this must be a global minimum.
Or one can analyse what happens on the boundary. If $x = 0$ then we have $2y^2 - 2y + 7$, which has a minimum at $y = 1/2$. $f(0, 1/2) = 13/2 > 2$, always larger on the y-axis. If $y = 0$ then we have $5x^2 - 8x + 7$, which has a minimum at $x = 4/5$. $f(4/5, 0) = 19/5 > 2$, always larger on the x-axis. If either x or y goes to infinity, w goes to infinity. Hence $(1, 1)$ is the point where w is smallest.

4. (20pts) Using Lagrange multipliers, find the points on the ellipse $x^2 + 2y^2 = 1$ where the function $f(x, y) = xy$ has a maximum and a minimum.
(i) Write down the equations satisfied by the Lagrange multiplier.

Solution:

$$\begin{aligned} y &= \lambda 2x \\ x &= \lambda 4y \\ x^2 + 2y^2 &= 1. \end{aligned}$$

(ii) Solve these equations and find the global maximum and minimum.

Solution:

$$\begin{aligned} xy &= \lambda 2x^2 \\ xy &= \lambda 4y^2 \end{aligned}$$

So $2\lambda x^2 = 4\lambda y^2$. If $\lambda = 0$ then $x = y = 0$, impossible. Otherwise, $x^2 = 2y^2$, so that $x = \pm\sqrt{2}y$. In this case $4y^2 = 1$, so that $y = \pm\frac{1}{2}$. So the maximum is $\sqrt{2}/4$, which occurs at $(\sqrt{2}/2, 1/2)$ and $(-\sqrt{2}/2, -1/2)$ and the minimum $-4/\sqrt{2}$, which occurs at $(\sqrt{2}/2, -1/2)$ and $(-\sqrt{2}/2, 1/2)$.

5. (15pts) Given that the variables w, x, y and z satisfy $w = xyz$ and $w^2 + z^2 = 13$, find

$$\left(\frac{\partial w}{\partial x}\right)_y,$$

when $w = 3$, $x = 3$, $y = 1/2$ and $z = 2$.

Solution:

$$\mathrm{d}w = yz\,\mathrm{d}x + xz\,\mathrm{d}y + xy\,\mathrm{d}z = \mathrm{d}x + 6\,\mathrm{d}y + 3/2\,\mathrm{d}z.$$

We are thinking of w as a function of x and y. Our goal is to eliminate $\mathrm{d}z$. We have

$$2w\,\mathrm{d}w + 2z\,\mathrm{d}z = 0 \qquad \text{so that} \qquad 3\,\mathrm{d}w + 2\,\mathrm{d}z = 0.$$

So

$$\mathrm{d}w = \mathrm{d}x + 6\,\mathrm{d}y - 9/4\,\mathrm{d}w.$$

Hence

$$13/4\mathrm{d}w = \mathrm{d}x + 6\,\mathrm{d}y,$$

and so

$$\left(\frac{\partial w}{\partial x}\right)_y = \frac{4}{13}.$$

6. (15pts) The two surfaces $x^4 - y^3 + z^2 = 1$ and $x^2y^2 + 3z^4 = 4$ intersect along a curve for which y is a function of x. Find

$$\frac{dy}{dx} \qquad \text{at} \qquad (x_0, y_0, z_0) = (1, 1, 1).$$

Solution: We use the method of differentials.

$4x^3\,\mathrm{d}x - 3y^2\,\mathrm{d}y + 2z\,\mathrm{d}z = 0 \qquad \text{and} \qquad 2xy^2\,\mathrm{d}x + 2x^2y\,\mathrm{d}y + 12z^3\,\mathrm{d}z = 0.$

At the point $(x_0, y_0, z_0) = (1, 1, 1)$, we have

$$4\,\mathrm{d}x - 3\,\mathrm{d}y + 2\,\mathrm{d}z = 0 \qquad \text{and} \qquad 2\,\mathrm{d}x + 2\,\mathrm{d}y + 12\,\mathrm{d}z = 0.$$

From the first equation we have

$$\mathrm{d}z = -2\,\mathrm{d}x + \frac{3}{2}\,\mathrm{d}y.$$

Plugging this into the second equation we have,

$$0 = \mathrm{d}x + \mathrm{d}y + 6\left(-2\,\mathrm{d}x + \frac{3}{2}\,\mathrm{d}y\right) = -11\,\mathrm{d}x + 10\,\mathrm{d}y.$$

So

$$\frac{dy}{dx} = \frac{11}{10}.$$