FIRST PRACTICE MIDTERM B
MATH 18.02, MIT, AUTUMN 12

You have 50 minutes. This test is closed book, closed notes, no calculators.

There are 5 problems, and the total number of points is 100. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:____________________________

Signature:____________________________

Student ID #:____________________________

Recitation instructor:____________________________

Recitation Number+Time:____________________________

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

1. (20pts) The unit cube lies in the first octant in $\mathbb{R}^3$, so that one vertex is at the origin. Let Q be the vertex diagonally opposite the origin and let R be the midpoint of a face not containing the origin.
(i) Express $\vec{Q}$ and $\vec{R}$ in terms of $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ (there are three choices for R; pick one).

Solution:

$$\vec{Q} = \hat{\imath} + \hat{\jmath} + \hat{k} \qquad \text{and} \qquad \vec{R} = \hat{\imath} + 1/2\hat{\jmath} + 1/2\hat{k}.$$

(ii) Find the cosine of the angle between $\vec{Q}$ and $\vec{R}$.

Solution:

$$\cos\theta = \frac{\langle 1,1,1\rangle \cdot \langle 1, 1/2, 1/2\rangle}{|\langle 1,1,1\rangle||\langle 1,1/2,1/2\rangle|} = \frac{2}{\sqrt{3}\sqrt{3/2}} = \frac{2\sqrt{2}}{3}.$$

2. (20pts) (i) Let

$$A = \begin{pmatrix} 1 & 1 & -3 \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{pmatrix}$$

then $\det(A) = -2$ and

$$A^{-1} = \begin{pmatrix} -4 & a & b \\ 2 & -1/2 & 3/2 \\ -1 & 1/2 & -1/2 \end{pmatrix}.$$

Find a and b.

Solution:

We know that $AA^{-1} = I_3$. Comparing entries in the first row second column we get

$$a - 1/2 - 3/2 = 0 \qquad \text{so that} \qquad a = 2.$$

Comparing entries in the first row third column we get

$$b + 3/2 + 3/2 = 0 \qquad \text{so that} \qquad b = -3.$$

(ii) Solve the system $A\vec{x} = \vec{b}$, where

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \text{and} \qquad \vec{b} = \begin{pmatrix} 4 \\ 3 \\ -3 \end{pmatrix}.$$

Solution:

$$\vec{x} = A^{-1}\vec{b} = \begin{pmatrix} -4 & 2 & -3 \\ 2 & -1/2 & 3/2 \\ -1 & 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

(iii) In the matrix A, replace the entry -3 in the upper-right corner by c. Find a value of c for which the resulting matrix M is not invertible. For this value of c the system $M\vec{x} = \vec{0}$ has other solutions than the obvious one $\vec{x} = \vec{0}$: find such a solution by using vector operations.

Solution: M invertible if and only if $\det M \neq 0$;

$$0 = \begin{vmatrix} 1 & 1 & c \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ -1 & 4 \end{vmatrix} + c \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = 8 - 4 + 2c.$$

$c = -2$. Cross product is a solution of homogeneous,

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{vmatrix} = \hat{\imath} \begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} 1 & 0 \\ -1 & 4 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = 8\hat{\imath} - 4\hat{\jmath} + 2\hat{k}.$$

3. (20pts) Find the equation of the plane containing the point $P_0 = (-1, 1, 1)$ and the line given as the intersection of the two planes

$$\begin{aligned} 2x - y + z &= -1 \\ x + y + z &= 3. \end{aligned}$$

Solution:

Find two points on the line. Intersect the line with the plane $x = 0$ and $x = -2$. If $x = 0$, we have

$$\begin{aligned} -y + z &= -1 \\ y + z &= 3. \end{aligned}$$

Adding we get $2z = 2$, so that $z = 1$. But then $y = 2$. $P_1 = (0, 2, 1)$ is a point on the line. If $x = -2$ we have

$$\begin{aligned} -y + z &= 3 \\ y + z &= 5. \end{aligned}$$

Adding we get $2z = 8$, so that $z = 4$. But then $y = 1$. Two points on the plane are $P_1 = (0, 2, 1)$ and $P_2 = (-2, 1, 4)$.

$$\vec{v} = \overrightarrow{P_0P_1} = \langle 1, 1, 0\rangle \qquad \text{and} \qquad \vec{w} = \overrightarrow{P_0P_2} = \langle -1, 0, 3\rangle$$

are two vectors parallel to the plane. The cross product $\vec{v} \times \vec{w}$ is a normal vector to the plane,

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 0 \\ -1 & 0 & 3 \end{vmatrix} = \hat{\imath} \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} 1 & 0 \\ -1 & 3 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} = 3\hat{\imath} - 3\hat{\jmath} + \hat{k}.$$

Hence $\vec{n} = \langle 3, -3, 1\rangle$ is a vector normal to the plane.

$$\langle x+1, y-1, z-1\rangle \cdot \langle 3, -3, 1\rangle = 0 \qquad \text{so that} \qquad 3(x+1) - 3(y-1) + (z-1) = 0.$$

Rearranging, we get

$$3x - 3y + z = -5.$$

4. (20pts)

(i) Find the area of the triangle with vertices $P_0 = (1,-1,2)$, $P_1 = (2,1,-3)$ and $P_2 = (3,1,-1)$.

Solution:

Let $\vec{v} = \overrightarrow{P_0P_1} = \langle 1,2,-5\rangle$ and $\vec{w} = \langle 2,2,-3\rangle$. Then

$$\vec{v}\times\vec{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & -5 \\ 2 & 2 & -3 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 2 & -5 \\ 2 & -3 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 1 & -5 \\ 2 & -3 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = 4\hat{\imath} - 7\hat{\jmath} - 2\hat{k}.$$

The area of the triangle is half the magnitude of the cross product

$$\frac{1}{2}(4^2+7^2+2^2)^{1/2} = \frac{1}{2}\sqrt{69}.$$

(ii) Find the equation of the plane containing these points.

Solution:

Let $P = \langle x,y,z\rangle$. Then $\overrightarrow{P_0P} = \langle x-1, y+1, z-2\rangle$ is orthogonal to $\vec{n} = \vec{v}\times\vec{w} = 4\hat{\imath} - 7\hat{\jmath} - 2\hat{k}$. Therefore

$0 = \overrightarrow{P_0P}\cdot\vec{n} = \langle x-1, y+1, z-2\rangle\cdot\langle 4,-7,-2\rangle = 4(x-1)-7(y+1)-2(z-2).$

Rearranging, we get

$$4x - 7y - 2z = 7.$$

(iii) What is the shortest distance between the plane and the point $(1,2,3)$.

Solution:

The line through $(1,2,3)$ and parallel to $\vec{n} = \langle 4,-7,-2\rangle$ intersects the plane at the closest point Q. This line is

$$\vec{r}(t) = \langle 1,2,3\rangle + t\langle 4,-7,-2\rangle = \langle 1+4t, 2-7t, 3-2t\rangle.$$

It lies on the plane when

$$4(1+4t) - 7(2-7t) - 2(3-2t) = 7 \qquad \text{so that} \qquad 69t - 16 = 7.$$

Thus $t = \frac{1}{3}$ and the point $Q = (7/3, -1/3, 7/3)$. The distance is then

$$|\overrightarrow{PQ}| = |\langle 4/3, -7/3, -2/3\rangle = \frac{1}{3}(4^2+7^2+2^2)^{1/2} = \frac{1}{3}\sqrt{69}.$$

5. (20pts)
(i) Let $\vec{r}(t)$ be the position vector of a particle in $\mathbb{R}^3$. Give a formula for

$$\frac{d(\vec{r}\cdot\vec{r})}{dt}$$

in vector coordinates.

Solution:

$$\frac{d(\vec{r}\cdot\vec{r})}{dt} = 2\vec{r}\cdot\frac{d\vec{r}}{dt} = 2\vec{r}\cdot\vec{v}.$$

(ii) Show that if $\vec{r}$ has constant length, then $\vec{r}$ and the velocity vector $\vec{v}$ are orthogonal.

Solution:
If $\vec{r}$ has constant length, then $\vec{r}\cdot\vec{r}$ is constant and the first derivative is zero. But then $\vec{r}\cdot\vec{v} = 0$ and so $\vec{r}$ and $\vec{v}$ are orthogonal.

(iii) Let $\vec{a}$ be the acceleration: still assuming that $\vec{r}$ has constant length, and using vector differentiation, express the quantity $\vec{r}\cdot\vec{a}$ in terms of the velocity vector only.

Solution:
Let us differentiate both sides of the equation

$$\vec{r}\cdot\vec{v} = 0.$$

We have

$$0 = \frac{d(\vec{r}\cdot\vec{v})}{dt} = \vec{r}\cdot\frac{d\vec{v}}{dt} + \frac{d\vec{r}}{dt}\cdot\vec{v}. = \vec{v}\cdot\vec{v} + \vec{r}\cdot\vec{a}.$$

It follows that

$$\vec{r}\cdot\vec{a} = -\vec{v}\cdot\vec{v}.$$