FIRST PRACTICE MIDTERM A
MATH 18.02, MIT, AUTUMN 12

You have 50 minutes. This test is closed book, closed notes, no calculators.

There are 5 problems, and the total number of points is 90. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:________________________

Signature:________________________

Student ID #:________________________

Recitation instructor:________________________

Recitation Number+Time:________________________

Problem	Points	Score
1	20	
2	20	
3	20	
4	15	
5	15	
Total	90	

1. (20pts) Let $P = (1, -2, 4)$, $Q = (1, -2, 1)$ and $R = (2, 1, 1)$.
(i) What is the distance between Q and R?

Solution:

$$\overrightarrow{QR} = \langle 1, 3, 0 \rangle.$$

The length of $\overrightarrow{QR}$ is

$$(1 + 9)^{1/2} = \sqrt{10}.$$

(ii) What is the area of the triangle with vertices P, Q and R?

Solution: We want half the magnitude of the cross product of $\overrightarrow{PQ}$ and $\overrightarrow{QR}$.

$$\overrightarrow{PQ} = \langle 0, 0, -3 \rangle.$$

The cross product is

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 3 & 0 \\ 0 & 0 & -3 \end{vmatrix} = \hat{\imath} \begin{vmatrix} 3 & 0 \\ 0 & -3 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} 1 & 0 \\ 0 & -3 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 3 \\ 0 & 0 \end{vmatrix} = -9\hat{\imath} + 3\hat{\jmath}.$$

The magnitude is

$$3(3^2 + 1)^{1/2} = 3\sqrt{10}.$$

So the area of the triangle is

$$\frac{3}{2}\sqrt{10}.$$

2. (20pts) (i) Find the determinant of

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Solution:

$$\det A = \begin{vmatrix} -1 & 0 & -1 \\ 0 & 2 & -1 \\ 1 & 1 & 1 \end{vmatrix} = -\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + -\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} = -3 + 2 = -1.$$

(ii) Find the inverse of A.

Solution:
We first calculate the matrix of minors.

$$\begin{pmatrix} 3 & 1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & -2 \end{pmatrix}$$

Next we calculate the matrix of cofactors, by flipping the signs

$$\begin{pmatrix} 3 & -1 & -2 \\ -1 & 0 & 1 \\ 2 & -1 & -2 \end{pmatrix}$$

Next we calculate the adjoint matrix, by taking the transpose matrix.

$$\begin{pmatrix} 3 & -1 & 2 \\ -1 & 0 & -1 \\ -2 & 1 & -2 \end{pmatrix}$$

Finally we divide by the determinant to get the inverse matrix,

$$A^{-1} = \begin{pmatrix} -3 & 1 & -2 \\ 1 & 0 & 1 \\ 2 & -1 & 2 \end{pmatrix}.$$

3. (20pts) A ladybug is climbing on a Volkswagen Bug (=VW). In its starting position, the surface of the VW is represented by the unit semicircle $x^2+y^2=1$, $y \geq 0$ in the xy-plane. The road is represented as the x-axis. At time $t=0$ the ladybug starts at the front bumper, $(1,0)$, and walks counterclockwise around the VW at unit speed relative to the VW. At the same time the VW moves to the right at speed 10.
(i) Find the parametric formula for the trajectory of the ladybug, and find its position when it reaches the rear bumper. (At $t=0$, the rear bumper is at $(-1,0)$.)

Solution:
Let P be the position of the ladybug and let A be the position of the centre of the VW. We have $\vec{A} = \langle 10t, 0\rangle$. The angle the ladybug makes with the x-axis is t radians (since we have a circle of radius 1). So

$$\overrightarrow{AP} = \langle \cos t, \sin t\rangle.$$

Hence

$$\vec{P} = \vec{A} + \overrightarrow{AP} = \langle 10t, 0\rangle + \langle \cos t, \sin t\rangle = \langle \cos t + 10t, \sin t\rangle.$$

The ladybug reaches the back bumper at time $t=\pi$. Its position is then

$$(10\pi - 1, 0).$$

(ii) Compute the speed of the bug, and find where it is largest and smallest. (*Hint: It is easier to work with the square of the speed*).
Solution:
The velocity vector is

$$\vec{v} = \frac{d\vec{P}}{dt} = \langle 10 - \sin t, \cos t\rangle.$$

Speed is the magnitude of the velocity,

$$|\langle 10 - \sin t, \cos t\rangle| = ((10-\sin t)^2 + \cos^2 t)^{1/2} = \sqrt{(101 - 20\sin t)}.$$

The square of the speed is $101 - 20\sin t$. Derivative is $-20\cos t$. This is zero when $t = \pi/2$ and $3\pi/2$. The speed at these points are $\sqrt{81} = 9$ (minimum) and $\sqrt{121} = 11$. But the solution $t = 3\pi/2$ makes no sense (this corresponds to a point under the road). So the maximum is achieved at one of the endpoints; in fact both $t=0$ and $t=\pi$ give the maximum speed $\sqrt{101}$.

4. (15pts) Find the equation of the plane containing the points $P = (1, 1, 1)$, $Q = (2, 1, 3)$ and parallel to the vector $\vec{w} = \langle -1, 2, 3 \rangle$.

Solution:
Two vectors in the plane are

$$\vec{v} = \overrightarrow{PQ} = \langle 1, 0, 2 \rangle \qquad \text{and} \qquad \vec{w} = \langle -1, 2, 3 \rangle.$$

A normal to the plane is given by the cross product:

$$\vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & 2 \\ -1 & 2 & 3 \end{vmatrix} = \hat{\imath} \begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} = -4\hat{\imath} - 5\hat{\jmath} + 2\hat{k}.$$

If $R = (x, y, z)$, then $\overrightarrow{PR} = \langle x-1, y-1, z-1 \rangle$ and the equation of the plane is

$$0 = \overrightarrow{PR} \cdot \vec{n} = \langle x-1, y-1, z-1 \rangle \cdot \langle -4, -5, 2 \rangle = -4(x-1) - 5(y-1) + 2(z-1).$$

Rearranging,

$$4x + 5y - 2z = 7,$$

is the equation of the plane.

5. (15pts) A ladder of length a rests against a wall. Suppose that the bottom of the wall is at the origin and that the ladder is in the first quadrant. The bottom of the ladder moves away from the wall along the floor as the top of the ladder slides down the wall.
Express the position of the centre of the ladder P in terms of the angle θ the ladder makes with the floor.

Solution:
Let A be the position of the bottom of the ladder and let B be the position of the top of the ladder. Then

$$\vec{A} = \langle a\cos\theta, 0\rangle \qquad \text{and} \qquad \vec{B} = \langle 0, a\sin\theta\rangle.$$

We have

$$\vec{P} = \frac{1}{2}(\vec{A} + \vec{B}).$$

So

$$\vec{P} = \frac{1}{2}\langle a\cos\theta, a\sin\theta\rangle.$$