FOURTH MIDTERM
MATH 18.02, MIT, AUTUMN 12

You have 50 minutes. This test is closed book, closed notes, no calculators.

There are 5 problems, and the total number of points is 100. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:______________________________

Signature:______________________________

Student ID #:______________________________

Recitation instructor:______________________________

Recitation Number+Time:______________________________

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

1. (20pts) Let R be the region in space which lies above the xy-plane and below the paraboloid $z = 1 - x^2 - y^2$. Calculate the moment of inertia about the z-axis; assume the density $\delta = 1$.

Solution:

$$I_z = \iiint_R x^2 + y^2 \,\mathrm{d}V = \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} r^3 \,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta.$$

The inner integral is

$$\int_0^{1-r^2} r^3 \,\mathrm{d}z = \Big[r^3 z \Big]_0^{1-r^2} = r^3(1-r^2).$$

The middle integral is

$$\int_0^1 r^3 - r^5 \,\mathrm{d}r = \left[\frac{r^4}{4} - \frac{r^6}{6} \right]_0^1 = \frac{1}{12}.$$

The outer integral is

$$\int_0^{2\pi} \frac{1}{12} \,\mathrm{d}\theta = \frac{\pi}{6}.$$

2. (20pts) (i) A solid sphere of radius a is placed above the xy-plane so it is tangent at the origin and so the z-axis is a diameter. Give its equation in spherical coordinates.

Solution:
Using symmetry in θ, we might as well assume that $\theta = 0$ and we may work in the xz-plane. So we have a circle, centred at $(0, a)$ of radius a;

$$x^2 + (z - a)^2 = a^2 \qquad \text{so that} \qquad x^2 + z^2 = 2za.$$

As $x^2 + z^2 = \rho^2$ and $z = \rho \cos \phi$, the equation of the sphere is

$$\rho = 2a \cos \phi.$$

(ii) Give the equation of the horizontal plane $z = a$ in spherical coordinates.

Solution:

$$\rho \cos \phi = a.$$

(iii) Set up a triple integral in spherical coordinates which gives the volume of the portion of the sphere S lying above the plane $z = a$.

Solution: Let's figure out the limits for ϕ. ϕ starts at zero. We go down all the way to a point on the plane $\rho \cos \phi = a$ and on the sphere $\rho = 2a \cos \phi$. So $2 \cos^2 \phi = 1$, that is, $\phi = \pi/4$ (one can also check this by drawing a picture).

$$\iiint_S 1 \, \mathrm{d}V = \int_0^{2\pi} \int_0^{\pi/4} \int_{a \sec \phi}^{2a \cos \phi} \rho^2 \sin \phi \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta.$$

3. (20pts) Let D be the disk described by $x^2 + y^2 \leq 4$ and $z = 2$. Let T be the right circular cone formed by joining every point of D to the origin, so that $(0, 0, 0)$ is the vertex of the cone. Assume that T has constant density $\delta = 1$. Set up an iterated integral that gives the magnitude of the gravitational force acting on a unit mass at the origin.

Solution: If

$$\vec{F} = \langle F_x, F_y, F_z \rangle,$$

is the force due to gravity then $F_x = F_y = 0$ by symmetry. We have

$$|\vec{F}| = F_z = \iiint_T \frac{Gz}{\rho^3} \,\mathrm{d}V = G \int_0^{2\pi} \int_0^{\pi/4} \int_0^{2\sec\phi} \cos\phi \sin\phi \,\mathrm{d}\rho \,\mathrm{d}\phi \,\mathrm{d}\theta.$$

4. (20pts) Find the flux of the vector field

$$\vec{F} = y^4\hat{\imath} - x^3\hat{\jmath} + z\hat{k},$$

coming out of the unit sphere centred at the origin.

Solution: We apply the divergence theorem. Let S be the surface of the unit sphere, oriented outwards and let V the solid enclosed by S. Then the flux coming out of the sphere is

$$\oint\!\!\!\!\oint_S \vec{F} \cdot \mathrm{d}\vec{S} = \iiint_V \operatorname{div} \vec{F} \, \mathrm{d}V = \iiint_V 1 \, \mathrm{d}V = \frac{4\pi}{3}.$$

5. (20pts) Let

$$\vec{F} = (y+z)\hat{\imath} - x\hat{\jmath} + (7x+5)\hat{k},$$

be a vector field and let S be the part of the surface $z = 9 - x^2 - y^2$ that lies above the xy-plane. Orient S by using the outward normal vector. Find the outward flux of $\vec{F}$ across S.

Solution: Let S' be the surface $x^2 + y^2 < 9$, $z = 0$, oriented upwards. By the divergence theorem

$$\oint\!\!\!\oint_{S-S'} \vec{F} \cdot \mathrm{d}\vec{S} = \iiint_V \operatorname{div} \vec{F} \, \mathrm{d}V = \iiint_V 0 \, \mathrm{d}V = 0.$$

So

$$\iint_S \vec{F} \cdot \mathrm{d}\vec{S} = \iint_{S'} \vec{F} \cdot \mathrm{d}\vec{S}.$$

The unit normal to S' is $\hat{k}$. So

$$\vec{F} \cdot \hat{k} = 7x + 5.$$

We have

$$\iint_{S'} \vec{F} \cdot \mathrm{d}\vec{S} = \iint_{S'} 7x + 5 \, \mathrm{d}A = \iint_{S'} 5 \, \mathrm{d}A = 45\pi,$$

since x is skew-symmetric about the y-axis.