THIRD MIDTERM
MATH 18.02, MIT, AUTUMN 12

You have 50 minutes. This test is closed book, closed notes, no calculators.

There are 6 problems, and the total number of points is 100. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:________________________________

Signature:________________________________

Student ID #:________________________________

Recitation instructor:__________________________

Recitation Number+Time:________________________

Problem	Points	Score
1	10	
2	15	
3	20	
4	15	
5	20	
6	20	
Total	100	

1. (10pts) (i) Change the order of integration in

$$\int_0^1 \int_x^1 \frac{x}{1+y^3}\,\mathrm{d}y\,\mathrm{d}x.$$

Solution: First we determine the region of integration. It is the triangle R with sides $x = 0$, $y = 1$ and $y = x$.

$$\int_0^1 \int_x^1 \frac{x}{1+y^3}\,\mathrm{d}y\,\mathrm{d}x = \iint_R \frac{x}{1+y^3}\,\mathrm{d}A = \int_0^1 \int_0^y \frac{x}{1+y^3}\,\mathrm{d}x\,\mathrm{d}y.$$

(ii) Evaluate the integral.

Solution: The inner integral is

$$\int_0^y \frac{x}{1+y^3}\,\mathrm{d}x = \left[\frac{x^2}{2(1+y^3)}\right]_0^y = \frac{y^2}{2(1+y^3)}.$$

So the outer integral is

$$\int_0^1 \frac{y^2}{2(1+y^3)}\,\mathrm{d}y = \left[\frac{1}{6}\ln(1+y^3)\right]_0^1 = \frac{1}{6}\ln 2.$$

2. (15pts) (i) Find the mass of the upper half of the annulus $1 < x^2 + y^2 < 9$ $(y \geq 0)$ with density

$$\delta = \frac{y}{x^2 + y^2}.$$

Solution:

$$M = \iint_R \frac{y}{x^2 + y^2}\,\mathrm{d}A = \int_0^\pi \int_1^3 \frac{r^2 \sin\theta}{r^2}\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^\pi \int_1^3 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta.$$

The inner integral is

$$\int_1^3 \sin\theta\,\mathrm{d}r = \Big[r\sin\theta\Big]_1^3 = 2\sin\theta.$$

So the outer integral is

$$\int_0^\pi 2\sin\theta\,\mathrm{d}\theta = \Big[-2\cos\theta\Big]_0^\pi = 4.$$

(ii) Express the x-coordinate of the centre of mass, $\bar{x}$, as an iterated integral. Explain why $\bar{x} = 0$.

Solution:

$$\bar{x} = \frac{1}{M}\iint_R \frac{xy}{x^2 + y^2}\,\mathrm{d}A = \frac{1}{M}\int_0^\pi \int_1^3 r\cos\theta\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta.$$

Both the region and the density are symmetric about the y-axis, and so $\bar{x} = 0$.

3. (20pts) (i) Show that

$$\vec{F} = (3x^2 - 6y^2)\hat{\imath} + (-12xy + 4y)\hat{\jmath}.$$

is conservative.

Solution: We have $M = 3x^2 - 6y^2$ and $N = -12xy + 4y$. So

$$M_y = -12y \qquad \text{and} \qquad N_x = -12y.$$

As these are equal, $\vec{F}$ is conservative.

(ii) Find a potential function $f(x, y)$ for $\vec{F}$.

Solution: We want to find $f(x, y)$ such that

$$f_x = 3x^2 - 6y^2 \qquad \text{and} \qquad f_y = -12xy + 4y.$$

If we integrate the first equation with respect to x, we get

$$f(x, y) = x^3 - 6xy^2 + g(y),$$

where $g(y)$ is determined by the second equation.

$$-12xy + \frac{dg}{dy} = -12xy + 4y \qquad \text{so that} \qquad \frac{dg}{dy} = 4y.$$

Integrating with respect to y, we get $g(y) = 2y^2 + c$. So

$$f(x, y) = x^3 - 6xy^2 + 2y^2,$$

is a potential function.

(iii) Let C be the curve $x = 1 + y^5(1 - y)^5$, $0 \leq y \leq 1$, oriented so we start at the $(1, 0)$. Calculate

$$\int \vec{F} \cdot \mathrm{d}\vec{r}.$$

Solution:

$$\int \vec{F} \cdot \mathrm{d}\vec{r} = \int \nabla f \cdot \mathrm{d}\vec{r} = f(1, 1) - f(1, 0) = (1 - 6 + 2) - 1 = -4,$$

by the fundamental theorem of calculus for line integrals.

4. (15pts) (i) For $\vec{F} = xy\hat{\imath} + y\hat{\jmath}$ find

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r},$$

over the curve $y = x^2$ from $(0,0)$ to $(1,1)$.

Solution: Parametrise C by $x = t$, $y = t^2$, $0 \leq t \leq 1$. Then

$$\vec{F} = \langle t^3, t^2 \rangle \qquad \text{and} \qquad \mathrm{d}\vec{r} = \langle 1, 2t \rangle \, \mathrm{d}t.$$

So

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_0^1 \langle t^3, t^2 \rangle \cdot \langle 1, 2t \rangle \, \mathrm{d}t = \int_0^1 3t^3 \, \mathrm{d}t = \left[\frac{3}{4} t^4 \right]_0^1 = \frac{3}{4}.$$

(ii) Let C be a simple closed curve going counterclockwise around a region R. Let $M = M(x,y)$. Express

$$\oint_C M \, \mathrm{d}x,$$

as a double integral over R.

Solution: By Green's theorem,

$$\oint_C M \, \mathrm{d}x = \iint_R -M_y \, \mathrm{d}A.$$

(iii) Find M such that

$$\oint_C M \, \mathrm{d}x,$$

is the mass of R, when the density $\delta = (x+y)^2$.

Solution: We want M such that

$$-M_y = (x+y)^2.$$

So

$$M = -\frac{(x+y)^3}{3},$$

will do.

5. (20pts) Consider the rectangle R with vertices $(0, 0)$, $(1, 0)$, $(1, 4)$ and $(0, 4)$. Let $C = C_1 + C_2 + C_3 + C_4$ be the boundary of R, starting at $(0, 0)$ and going around counterclockwise, so that C_1 and C_3 are horizontal and C_2 and C_4 are vertical. Let

$$\vec{F} = (xy + \sin x \cos y)\hat{\imath} - (\cos x \sin y)\hat{\jmath}.$$

(i) Find the flux of $\vec{F}$ across C.

Solution:

$$\oint_C \vec{F}\cdot\hat{n}\,\mathrm{d}s = \iint_R \operatorname{div}\vec{F}\,\mathrm{d}A = \iint_R y+\cos x\cos y-\cos x\cos y\,\mathrm{d}A = \iint_R y\,\mathrm{d}A.$$

The last integral, divided by the area, is the y-coordinate $\bar{y}$ of the centre of mass. But the area of R is 4 and $\bar{y} = 2$, so that the total flux is 8.

(ii) Is the total flux out of R across C_1, C_2 and C_3 more than or less than across C?

Solution:

The difference is the total flux across C_4. C_4 is vertical, so the normal vector $\hat{n} = -\hat{\imath}$. $x = 0$ along C_4, so the first component of $\vec{F}$ is 0. So the flux across C_4 is zero and in fact the flux across C is precisely the flux across C_1, C_2 and C_3.

6. (20pts) Compute the area of the region in the plane bounded by the curves $xy = 1$, $xy = 3$, $xy^3 = 2$ and $xy^3 = 4$.

Solution: Let R be the region. We want

$$\text{area}(R) = \iint_R 1 \, \mathrm{d}A.$$

Let $u = xy$ and $v = xy^3$. Then

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} y & x \\ y^3 & 3xy^2 \end{vmatrix} = 2xy^3.$$

As this is positive over the whole region, we have

$$\mathrm{d}u \, \mathrm{d}v = 2xy^3 \mathrm{d}x \, \mathrm{d}y = 2v \mathrm{d}x \, \mathrm{d}y.$$

It follows that

$$\text{area}(R) = \int_2^4 \int_1^3 \frac{1}{2v} \, \mathrm{d}u \, \mathrm{d}v.$$

The inner integral is

$$\int_1^3 \frac{1}{2v} \, \mathrm{d}u = \frac{1}{v}.$$

The outer integral is

$$\int_2^4 \frac{1}{v} \, \mathrm{d}v = \Big[\ln v \Big]_2^4 = \ln 4 - \ln 2 = \ln 2.$$