FIRST MIDTERM
MATH 18.02, MIT, AUTUMN 12

You have 50 minutes. This test is closed book, closed notes, no calculators.

There are 5 problems, and the total number of points is 100. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:____________________

Signature:____________________

Student ID #:____________________

Recitation instructor:____________________

Recitation Number+Time:____________________

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

1. (20pts) Let $P = (1, 2, 3)$, $Q = (-1, 2, 1)$ and $R = (1, 1, -3)$.
(i) What is the cosine of the angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$?

Solution:

$$\overrightarrow{PQ} = \langle -2, 0, -2 \rangle \qquad \text{and} \qquad \overrightarrow{PR} = \langle 0, -1, -6 \rangle.$$

Hence

$$\cos\theta = \frac{\langle -2, 0, -2 \rangle \cdot \langle 0, -1, -6 \rangle}{|\langle -2, 0, -2 \rangle||\langle 0, -1, -6 \rangle|} = \frac{12}{\sqrt{8}\sqrt{37}} = \frac{6}{\sqrt{2}\sqrt{37}}.$$

(ii) If $\vec{r}(t) = \langle -3\cos 2t, 3\sin 2t, 8t \rangle$, then what is the speed at time t?

Solution:

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \langle 6\sin 2t, 6\cos 2t, 8 \rangle = 2\langle 3\sin 2t, 3\cos 2t, 4 \rangle.$$

Speed is the magnitude of the velocity,

$$2(3^2 \sin^2 2t + 3^2 \cos^2 2t + 4^2)^{1/2} = 2(3^2 + 4^2)^{1/2} = 10.$$

2. (20pts) (i) Let

$$A = \begin{pmatrix} -3 & 3 & 3 \\ -3 & 4 & 3 \\ -3 & 3 & 4 \end{pmatrix}$$

then $\det(A) = -3$ and

$$A^{-1} = \begin{pmatrix} -7/3 & 1 & 1 \\ -1 & a & b \\ -1 & 0 & 1 \end{pmatrix}.$$

Find a and b.

Solution:

We have $AA^{-1} = I_3$. Comparing entries in the first row second column, we have

$$-3 + 3a = 0 \qquad \text{so that} \qquad a = 1$$

and comparing entries in the first row third column, we have

$$-3 + 3b + 3 = 0 \qquad \text{so that} \qquad b = 0.$$

(ii) Solve the system $A\vec{x} = \vec{b}$, where

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \text{and} \qquad \vec{b} = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}.$$

Solution:

$$\vec{x} = A^{-1}\vec{b} = \begin{pmatrix} -7/3 & 1 & 1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}.$$

(iii) In the matrix A, replace the entry -3 in the lower-left corner by c. Find a value of c for which the resulting matrix M is not invertible. For this value of c the system $M\vec{x} = \vec{0}$ has other solutions than the obvious one $\vec{x} = \vec{0}$: find such a solution by using vector operations.

Solution: M is invertible if and only if $\det M \neq 0$.

$$\begin{vmatrix} -3 & 3 & 3 \\ -3 & 4 & 3 \\ c & 3 & 4 \end{vmatrix} = -3 \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} - 3 \begin{vmatrix} -3 & 3 \\ c & 4 \end{vmatrix} + 3 \begin{vmatrix} -3 & 4 \\ c & 3 \end{vmatrix} = -21 + 9c + 36 - 12c - 27 = -12 - 3c.$$

So $c = -4$. We are looking for a vector orthogonal to all three rows. Take the cross product of the first two rows,

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -3 & 3 & 3 \\ -3 & 4 & 3 \end{vmatrix} = \hat{\imath} \begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} -3 & 3 \\ -3 & 3 \end{vmatrix} + \hat{k} \begin{vmatrix} -3 & 3 \\ -3 & 4 \end{vmatrix} = -3\hat{\imath} - 3\hat{k}.$$

3. (20pts) (i) (6 points) Find the area of the triangle whose vertices are $P_0 = (1, 1, 1)$, $P_1 = (1, 2, 3)$ and $P_2 = (-1, -1, 2)$.

Solution:
Let $\vec{v} = \overrightarrow{P_0P_1} = \langle 0, 1, 2\rangle$ and $\vec{w} = \overrightarrow{P_0P_2}\langle -2, -2, 1\rangle$. Then

$$\vec{v}\times\vec{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 2 \\ -2 & -2 & 1 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 0 & 2 \\ -2 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 0 & 1 \\ -2 & -2 \end{vmatrix} = 5\hat{\imath} - 4\hat{\jmath} + 2\hat{k}.$$

The area of the triangle is half the magnitude of the cross product

$$\frac{1}{2}(5^2 + 4^2 + 2^2)^{1/2} = \frac{1}{2}\sqrt{45} = \frac{3}{2}\sqrt{5}.$$

(ii) (6 points) Find the equation of the plane containing these points.

Solution:
Let $P = \langle x, y, z\rangle$. Then $\overrightarrow{P_0P} = \langle x-1, y-1, z-1\rangle$ is orthogonal to $\vec{n} = \vec{v}\times\vec{w} = 5\hat{\imath} - 4\hat{\jmath} + 2\hat{k}$. Therefore

$0 = \overrightarrow{P_0P}\cdot\vec{n} = \langle x-1, y-1, z-1\rangle\cdot\langle 5, -4, 2\rangle = 5(x-1) - 4(y-1) + 2(z-1).$

Rearranging, we get

$$5x - 4y + 2z = 3.$$

(iii) (8 points) Find the point of intersection of this plane and the line through the point $(-1, 2, -1)$, parallel to the vector $\vec{v} = \langle 3, 2, 1\rangle$.

Solution:
The parametric form of the line is

$$\vec{r}(t) = \langle -1, 2, -1\rangle + t\langle 3, 2, 1\rangle = \langle -1+3t, 2+2t, -1+t\rangle.$$

Plug this into the equation for the plane and solve for t,

$$3 = 5(-1+3t) - 4(2+2t) + 2(-1+t) = 9t - 15.$$

Hence $t = 2$ and the point we are looking for is $(5, 6, 1)$.

4. (20pts) (i) (8 points) Show that the line $\vec{r}(t) = \langle 1-t, 2+3t, 2+t\rangle$ and the plane $2x+y-z=4$ are parallel.

Solution:
The line is parallel to $\vec{v} = \langle -1, 3, 1\rangle$. A normal vector to the plane is $\vec{n} = \langle 2, 1, -1\rangle$. We have

$$\vec{v}\cdot\vec{n} = \langle -1, 3, 1\rangle \cdot \langle 2, 1, -1\rangle = -2+3-1 = 0,$$

so that the vector $\vec{v}$ is indeed parallel to the plane.

(ii) (12 points) Find the distance between the line $\vec{r}(t) = \langle 1-t, 2+3t, 2+t\rangle$ and the plane $2x+y-z=4$.

Solution:
Set $t=0$ to get a point $P=(1,2,2)$ of the line. If Q is the point on the plane which lies on the line through P and parallel to $\vec{n}$, then $|\overrightarrow{PQ}|$ is the distance between the the plane and the line.

$$\vec{Q}(t) = \langle 1, 2, 2\rangle + t\langle 2, 1, -1\rangle = \langle 1+2t, 2+t, 2-t\rangle,$$

is the parametric form for the line through P parallel to $\vec{n}$. This meets the plane when

$$2(1+2t)+(2+t)-(2-t) = 4 \qquad \text{so that} \qquad 2+6t=4,$$

that is $t=1/3$. Therefore $Q=(5/3, 7/3, 5/3)$ and $\overrightarrow{PQ} = \langle 2/3, 1/3, -1/3\rangle$. It follows that the distance is

$$\frac{1}{3}|\langle 2, 1, -1\rangle| = \frac{1}{3}(4+1+1)^{1/2} = \frac{\sqrt{6}}{3}.$$

5. (20pts) (i) A wheel of radius a is rolling along the ground, the x-axis, in the xy-plane. The wheel is rotating at 2 radians per second clockwise. At time $t = 0$, the bottom of the wheel is at the origin. Find the position vector $\vec{r}(t)$ of the point on the rim which is at the top of the wheel at time $t = 0$.

Solution:
Let A be the point where the wheel touches the ground, B be the centre of the wheel and P be the position of the point on the rim. We have

$$\vec{A} = \langle 2at, 0 \rangle, \qquad \overrightarrow{AB} = \langle 0, a \rangle \qquad \text{and} \qquad \overrightarrow{BP} = \langle a \sin 2t, a \cos 2t \rangle.$$

So

$$\vec{r}(t) = \vec{A} + \overrightarrow{AB} + \overrightarrow{BP} = \langle 2at + a \sin 2t, a + a \cos 2t \rangle.$$

(ii) What is the speed of this point at time t?

Solution:

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \langle 2a + 2a \cos 2t, -2a \sin 2t \rangle.$$

Speed is the magnitude of the velocity,

$$\begin{aligned} 2a\left((1 + \cos 2t)^2 + \sin^2 2t\right)^{1/2} &= 2a\left(1 + 2\cos 2t + \cos^2 2t + \sin^2 2t\right)^{1/2} \\ &= 2a(2 + 2\cos 2t)^{1/2} \\ &= 2a(4\cos^2 t)^{1/2} \\ &= 4a|\cos t|. \end{aligned}$$

(iii) How far does this point move in one revolution?

The distance travelled is the integral of the speed,

$$s = 2\int_0^{\pi/2} 4a \cos t \, \mathrm{d}t = 8a.$$