7. Functions of more than one variable

Most functions in nature depend on more than one variable. Pressure of a fixed amount of gas depends on the temperature and the volume; increase the temperature and pressure goes up; increase the volume and the pressure goes down.

To understand a function of one variable, $f(x)$, look at its graph, $y = f(x)$. This is a curve in the plane.

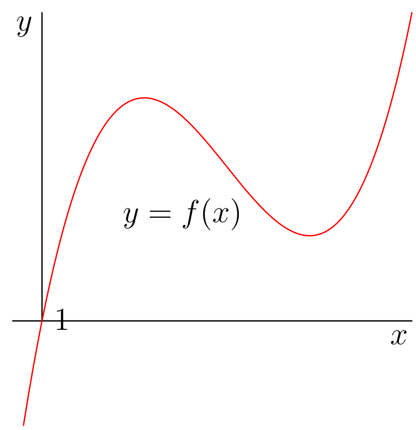

Figure 1. Graph of a function of one variable

To understand a function of two variables, $f(x, y)$, look at its graph $z = f(x, y)$. This is a surface in $\mathbb{R}^3$.

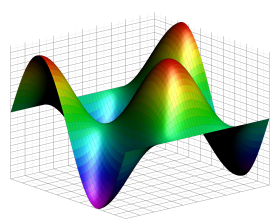

Figure 2. Graph of a function of two variables

Let's do a couple of examples. $f(x, y) = -x$. The graph is $z = -x$. What does this surface look like in $\mathbb{R}^3$? Well, $x + z = 0$ is the equation of a plane. Normal vector $\vec{n} = \langle 1, 0, 1 \rangle$ and it passes through the origin.

One way to get a picture is to slice by coordinate planes. If we slice by $y = 0$, we get $z = -x$, a line of slope -1 in the xz-plane. In fact if we slice by any coordinate plane $y = a$, a a constant, we get the same line $z = -x$. If we slice by $x = 0$, we get $z = 0$, a horizontal line in the

yz-plane. If we slice by $x = 1$, we get $z = -1$, a different horizontal line.

How about $f(x, y) = 1 - x^2 - y^2$? If we slice by $y = 0$, we get $z = 1 - x^2$, an upside down parabola. If we slice by $y = 1$, we get $z = -x^2$, another upside down parabola. Similarly if we slice by $y = a$, we get the parabola, $z = -x^2 - a^2$. By symmetry in x and y, we get the same picture if we slice by $x = a$.

How about if we fix z? Then $x^2 + y^2 = 1 - z$. So we only get a non-empty slice, if we take $z \leq 1$. If $z = 0$, we get the circle $x^2 + y^2 = 1$. If we increase z, we get circles of smaller radii. If we decrease z they get bigger.

In fact the graph is a paraboloid.

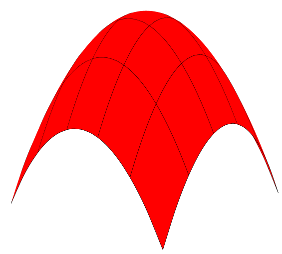

FIGURE 3. Paraboloid

One way to get a picture of the graph is to look at the contour lines. These are lines in the xy-plane of constant height. Formally, they are the solutions to the equation

$$f(x, y) = c,$$

where c is fixed. The contour lines of $f(x, y) = 1 - x^2 - y^2$ are concentric circles centred at the origin:

What does

$$z = \sqrt{x^2 + y^2},$$

look like? Well the contour lines are circles, so it looks like a paraboloid. But if we cut by coordinate planes, we get a different picture. If we take the plane $y = 0$, we get $z = \sqrt{x^2}$, or what comes to the same thing $z = |x|$. The graph of this look like a V. In fact $z = \sqrt{x^2 + y^2}$ is the graph of a cone.

It is not hard to see that $z = x^2 + y^2$ is another paraboloid. It is the same story as $z = 1 - x^2 - y^2$. The contour lines are the circles $x^2 + y^2 = c$. Cutting by coordinate hyperplanes, we get parabolas, but this time the right way up, so that the graph of $z = x^2 + y^2$ is a paraboloid the other way up to $z = 1 - x^2 - y^2$.

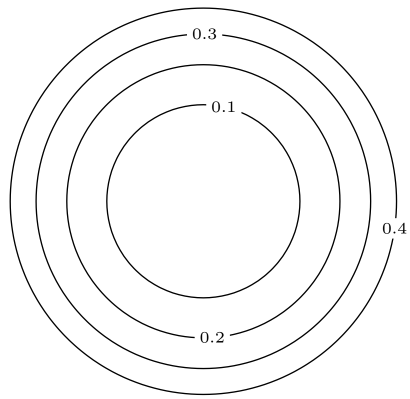

FIGURE 4. Contour lines of paraboloid

What does

$$z = y^2 - x^2,$$

look like? Well the contour lines are hyperbolae:

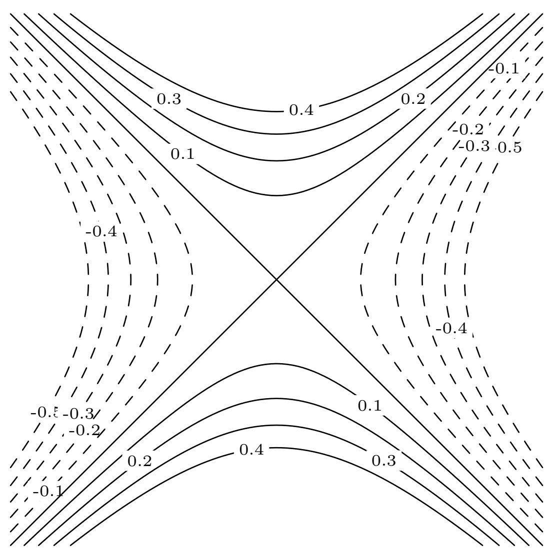

FIGURE 5. Contour lines for $y^2 - x^2$

How about if we take cross sections? Fix $x = a$, we get parabolas $z = y^2 - a^2$. Fix $y = a$, we get upside down parabolas $z = a^2 - x^2$.

The graph of this function is called a saddle point:

One way to understand a function of one variable is to differentiate. The derivative is the slope of the tangent line.

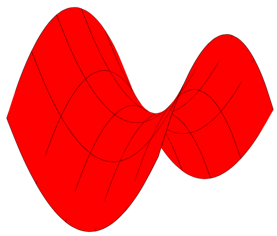

FIGURE 6. Saddle point

If we have a function of two variables, there are two obvious derivatives. We could fix y and vary x, to get a partial derivative

$$f_x(x_0, y_0) = \left.\frac{\partial f}{\partial x}\right|_{x=x_0, y=y_0} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}.$$

Similarly, we can fix x and vary y.

$$f_y(x_0, y_0) = \left.\frac{\partial f}{\partial y}\right|_{x=x_0, y=y_0} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}.$$

f_x is the slope of the tangent line if you cut by the plane $y = y_0$; f_y is the slope of the tangent line to if you cut by the plane $x = x_0$.

It is straightforward to calculate partial derivatives. Let $f(x, y) = x^2 y - \sin(x + y^2)$.

$$f_x = 2xy - \cos(x + y^2) \qquad \text{and} \qquad f_y = x^2 - 2y\cos(x + y^2).$$

$$\frac{\partial(\ln(x \cos y))}{\partial x} = \cos y \frac{1}{x \cos y} = \frac{1}{x},$$

and

$$\frac{\partial(\ln(x \cos y))}{\partial y} = -x \sin y \frac{1}{x \cos y} = -\tan y.$$

We can use partial derivatives to estimate the change in f, if we change x and y by a small amount.

$$\Delta f \approx f_x \Delta x + f_y \Delta y.$$

In fact, we can calculate the tangent plane at a point (x_0, y_0, z_0), where $z_0 = f(x_0, y_0)$. One way to calculate the tangent plane is to use the approximation formula,

$$(\dagger) \qquad z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

In fact the approximation formula works by approximating Δf by using linear approximation. The tangent plane is the best linear approximation to the function f.

The tangent plane is the plane which should contain the tangent line to any curve in the graph. You can get two curves easily, either by fixing y and varying x or by fixing x and varying y. These are the curves you get by cutting by either the plane $y = y_0$ or the plane $x = x_0$. The tangent line to the first curve is

$$z - z_0 = f_x(x_0, y_0)(x - x_0),$$

and the tangent line to the second curve is

$$z - z_0 = f_y(x_0, y_0)(y - y_0).$$

Visibly (†) contains both tangent lines.