5. Parametric curves

We have already seen that one way to represent lines in $\mathbb{R}^3$ is to think of them as being the intersection of two planes. Another approach is to parametrise the line.

Pick two points $Q_0 = (1, -2, 4)$ and $Q_1 = (3, -1, 3)$ and consider the line which contains both points. Imagine a particle traveling along the line at constant speed, which is at Q_0 at time $t = 0$ and at Q_1 at time $t = 1$. In general the position vector of the particle at time t is

$$\vec{Q}(t) = \vec{Q}_0 + t\overrightarrow{Q_0Q_1} = \langle 1, -2, 4\rangle + t\langle 2, 1, -1\rangle = \langle 1 + 2t, -2 + t, 4 - t\rangle.$$

In other words, if $\vec{Q}(t) = \langle x(t), y(t), z(t)\rangle$, then

$$\begin{aligned} x(t) &= 1 + 2t \\ y(t) &= -2 + t \\ z(t) &= 4 - t. \end{aligned}$$

Note that the velocity velocity $\vec{v}$ of the particle is $\overrightarrow{Q_0Q_1} = \langle 2, 1, -1\rangle$. Indeed it is traveling with constant velocity and this is how far the particle moves in unit time. Note that $\vec{v}$ is parallel to the line (or points in the direction of the line).

Question 5.1. *What are the positions of Q_0 and Q_1 relative to the plane $2x - y - z = 3$?*

Well, plug in the coordinates of both points into the equation of the plane. The first point gives $2 + 2 - 4 = 0 < 3$ and the second point gives $6 + 1 - 3 = 4 > 3$. Note that every point is contained in a plane parallel to the plane $2x - y - z = 3$ (think of a stack of pancakes, an infinite stack of pancakes). Q_0 is contained in the plane $2x - y - z = 0$ and Q_1 is contained in the plane $2x - y - z = 4$. So the points are opposite sides of the plane.

It follows that the particle is on the plane at some time t between 0 and 1, so that the line meets the plane.

To find the point of intersection of the plane with the line, plug in $\vec{Q}(t)$ into the equation of the plane and solve for t,

$$3 = 2(1 + 2t) - (-2 + t) - (4 - t) = 4t \qquad \text{and so} \qquad t = \frac{3}{4},$$

which is indeed between 0 and 1. The point is

$$(\frac{5}{2}, -\frac{5}{4}, \frac{13}{4}) = \frac{1}{4}(10, -5, 13).$$

Suppose we tried the same trick with a line parallel to this plane. What would happen? Well, if the line misses the plane, we couldn't solve for t. So we would get an equation of the form

$$a = 3,$$

where a is a constant, not equal to 3. If the line is contained in the plane, then we would get the equation

$$3 = 3,$$

which is valid for any t.

Suppose we are given a line as the intersection of two planes,

$$2x - y + z = 3 \qquad \text{and} \qquad x + 3y - z = 1.$$

How can we find a parametric form of the line? There are two methods.

One is to find two points on this line. Pick another plane and intersect with these two planes. It is convenient to pick the plane $x = 0$. The two equations above reduce to

$$\begin{aligned} -y + z &= 3 \\ 3y - z &= 1. \end{aligned}$$

Adding we get $2y = 4$, so that $y = 2$. This gives $z = 5$. So one point is $Q_0 = (0, 2, 5)$. Now let's pick the plane $x = 1$. The two equations above reduce to

$$\begin{aligned} -y + z &= 1 \\ 3y - z &= 0. \end{aligned}$$

Adding we get $2y = 1$, so that $y = 1/2$. This gives $z = 3/2$. So the other point is $Q_1 = (1, 1/2, 3/2)$.

The line is given parametrically as

$$\vec{Q}(t) = \vec{Q}_0 + t\overrightarrow{Q_0Q_1} = \langle 0, 2, 5\rangle + t\langle 1, -3/2, -7/2\rangle = \langle t, 2-3t/2, 5-7t/2\rangle.$$

Another method is to use the cross product to find the direction of the line. A normal vector to the first plane is $\vec{n}_1 = \langle 2, -1, 1\rangle$ and a normal vector to the second plane is $\vec{n}_2 = \langle 1, 3, -1\rangle$. The line lies in both planes so its direction is orthogonal to both planes. In other words the line is parallel to the cross product:

$$\vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{vmatrix} = \hat{\imath}\begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} = -2\hat{\imath} + 3\hat{\jmath} + 7\hat{k}.$$

Together with the point Q_0 this gives us another way to parametrise the lines

$$\vec{P}(t) = \vec{Q}_0 + t\vec{v} = \langle 0, 2, 5\rangle + t\langle -2, 3, 7\rangle = \langle -2, 2 + 3t, 5 + 7t\rangle.$$

Notice that this is the same line with a different parametrisation.

Question 5.2. *How are the lines*

$$\vec{P}(t) = \langle 1+2t, -2+t, 2+5t \rangle \qquad \text{and} \qquad \vec{Q}(t) = \langle -2+t, -6+3t, -4+t \rangle$$

related?

These two lines intersect. So they are neither skew nor parallel. The key point is to use two different parameters. We want to know if we can find s and t such that

$$\langle 1 + 2s, -2 + s, 2 + 5s \rangle = \langle -2 + t, -6 + 3t, -4 + t \rangle.$$

This gives us three simultaneous linear equations for s and t,

$$\begin{aligned} 1 + 2s &= -2 + t \\ -2 + s &= -6 + 3t \\ 2 + 5s &= -4 + t. \end{aligned}$$

With a little bit of work, one can check that $s = -1$ and $t = 1$ is a solution. So the lines intersect.

Note that we can parametrise a lot more curves than just lines. Consider the example of a cycloid. Here we have a wheel rolling along the ground, and we keep track of a point on the rim of the wheel. What sort of curve does this point trace out?

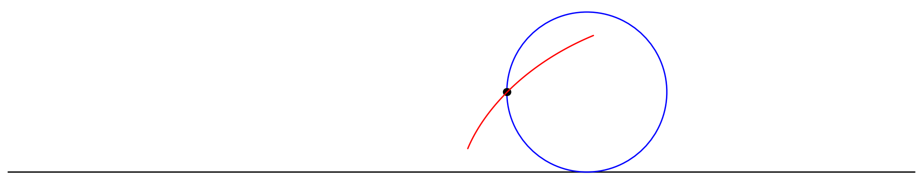

FIGURE 1. A rolling stone

Let's suppose that the wheel has radius a. We will parametrise the motion using the angle θ which the wheel has turned since the start. Then the centre of the wheel has moved a distance of $a\theta$. Let's suppose that the point on the rim starts at $(0, 0)$, so that the centre of the wheel starts at $(0, a)$.

Call P the point on the rim, A the point of contact of the wheel with the floor and B the centre of the wheel.

Then

$$\vec{P} = \vec{A} + \overrightarrow{AB} + \overrightarrow{BP}.$$

Now

$$\vec{A} = \langle a\theta, 0 \rangle \qquad \text{and} \qquad \overrightarrow{AB} = \langle 0, a \rangle,$$

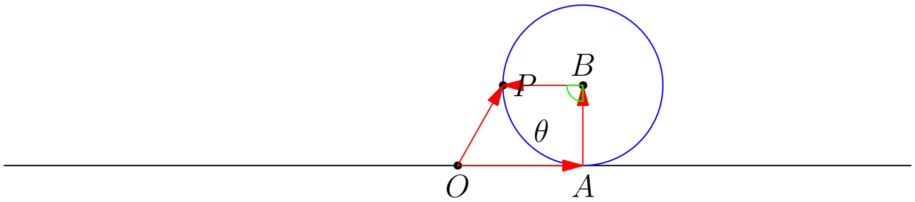

FIGURE 2. Labels

since the centre of the wheel is always directly above the point of contact with the floor. Now the length of $\overrightarrow{BP}$ is a and the angle θ is the angle from the $-y$-axis.

$$\overrightarrow{BP} = \langle -a\sin\theta, -a\cos\theta\rangle.$$

Putting all of this together,

$$\vec{P} = \langle a(\theta - \sin\theta), a(1-\cos\theta)\rangle.$$

Question 5.3. *What is happening when the marked point is touching the floor?*

Use Taylor series approximation. To simplify the computation, let's take $a = 1$. For t close to zero,

$$f(t) = f(0) + f'(0)t + f''(0)t^2/2 + \dots.$$

This gives

$$\sin\theta \approx \theta - \theta^3/6 \qquad \text{and} \qquad \cos\theta \approx 1 - \theta^2/2.$$

So

$$x(\theta) \approx \theta^3/6 \qquad \text{and} \qquad y(\theta) \approx \theta^2/2.$$

So

$$\frac{y(\theta)}{x(\theta)} \approx \frac{3}{\theta},$$

which as $\theta \to 0$ tends to ∞. So we have a vertical tangent.

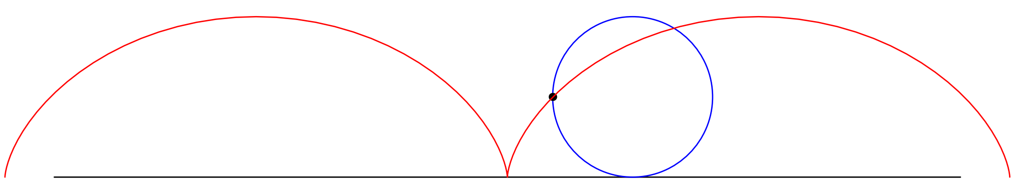

FIGURE 3. Cycloid