4. Square systems of linear equations

We have already seen that equations of the form

$$ax + by + cz = d,$$

represent planes in $\mathbb{R}^3$.

Question 4.1. *What is the plane through the origin normal to the vector $\vec{n} = \langle 1, 4, 8\rangle$?*

If P is a point in $\mathbb{R}^3$ then P lies in the plane if and only if $\vec{P}$ is orthogonal to $\vec{n}$. So the equation of the plane is given by

$$\vec{P} \cdot \vec{n} = 0.$$

If $P = (x, y, z)$, then $\vec{P} = \langle x, y, z, \rangle$ and

$$\vec{P} \cdot \vec{n} = \langle x, y, z\rangle \cdot \vec{n} = \langle x, y, z\rangle \cdot \langle 1, 4, 8\rangle = x + 4y + 8z.$$

So the equation of the plane is

$$x + 4y + 8z = 0.$$

Note that we can recover the vector $\vec{n}$ orthogonal to the plane from the coefficients of x, y and z.

Question 4.2. *What is the plane through the point $P_0 = (-2, 1, 6)$ normal to the vector $\vec{n} = \langle 1, 4, 8\rangle$?*

Now note that $P = (x, y, z)$ is in the plane if and only if the vector $\overrightarrow{P_0P} = \langle x+2, y-1, z-6\rangle$ is in the plane if and only if $\overrightarrow{P_0P}$ is orthogonal to $\vec{n}$ if and only if $\overrightarrow{P_0}P. \cdot \vec{n} = 0$ if and only if

$$(x + 2) + 4(y - 1) + 8(z - 6) = 0 \qquad \text{so that} \qquad x + 4y + 8z = 50.$$

Once again we can recover $\vec{n}$ from the coefficients of x, y and z.

To determine the equation of a plane, the crucial piece of data is therefore the vector $\vec{n}$. For example, if we are given two vectors living in the plane, then the cross product of these vectors gives us $\vec{n}$.

Question 4.3. *What is the relation between the vector $\vec{v} = \langle -1, 1, 1\rangle$ and the plane $2x - y + 3z = 7$?*

Well, the vector $\vec{n} = \langle 2, -1, 3\rangle$ is orthogonal to the plane. Visibly $\vec{v}$ is not a multiple of $\vec{n}$, so $\vec{v}$ is not orthogonal to the plane. It is parallel to the plane, since

$$\vec{n} \cdot \vec{v} = \langle 2, -1, 3\rangle \cdot \langle -1, 1, 1\rangle = -2 - 1 + 3 = 0.$$

Suppose we are given a 3×3 system of equations,

$$\begin{aligned} a_1x + a_2y + a_3z &= a \\ b_1x + b_2y + b_3z &= b \\ c_1x + c_2y + c_3z &= c. \end{aligned}$$

Compactly,

$$A\vec{x} = \vec{b},$$

where

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

is the coefficient matrix, and

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \text{and} \qquad \vec{b} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

are column vectors. Each equation represents a plane. Most of the time two planes intersect in a line and then the third plane intersects the line in a single point. Three equations, three unknowns, with any luck there is a single solution. In fact if A is invertible, the unique solution is

$$\vec{x} = A^{-1}\vec{b}.$$

What can go wrong? Let suppose the three planes are $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$. Here is one possible exception. Let's suppose that the first two planes intersect in a line

$$\mathcal{P}_1 \cap \mathcal{P}_2.$$

But suppose that the third plane is parallel to this line. There are two possibilities; the line is contained in the plane (in which case there are infinitely many solutions) or misses the plane entirely (in which case there are no solutions).

Question 4.4. *How many possible configurations are there?*

Here, I am interested in qualitatively different geometric configurations of all three planes.

Let's list the possible configurations according to the solution set.

If the solution is a single point, there is only one configuration. Similarly if the solution is a whole plane, there is only one configuration, all three planes are the same plane. Suppose the solution is a line. There are two possibilities. All three planes contain this line but are different. Or, two planes are the same and the third plane is different.

Finally how many possibilities when there are no solutions? Well we could have three parallel planes. Or two planes are the same plane and the third plane is parallel to this plane. Or the three planes could intersect in three parallel lines (the tobelerone solution). Or two planes are parallel and the third plane intersects the planes in two parallel lines. Four possibilities for no solutions.

In total this makes $1 + 1 + 2 + 4 = 8$ possible configurations.

All seven of the possible degenerate configurations correspond to a matrix which is not invertible, that is, a matrix whose determinant is zero.

It is interesting to distinguish a special case from the general case.

We say that we have a homogeneous system if $\vec{b} = \vec{0}$, so that we are trying to solve the system

$$A\vec{x} = \vec{0}.$$

In this case $\vec{x} = \vec{0}$ is always a solution. If $\det A \neq 0$, then this solution is unique. Otherwise we get a line or a plane through the origin.

Theorem 4.5. *The homogeneous system of equations*

$$A\vec{x} = \vec{0},$$

always has at least one solution, $\vec{x} = 0$*.*

$\vec{x} = \vec{0}$ *is the unique solution if* $\det A \neq 0$*. Otherwise there are infinitely many solutions.*

Now suppose we consider the general case,

$$A\vec{x} = \vec{b}.$$

If $\vec{b} \neq \vec{0}$, the system is called an inhomogeneous system. Here we shift the three planes from the origin. There might be zero, one or infinitely many solutions. If $\det A \neq 0$, then $\vec{x} = A^{-1}\vec{b}$ is the unique solution. Otherwise there might be no solutions or infinitely many solutions.

Theorem 4.6. *The inhomogeneous system of equations*

$$A\vec{x} = \vec{b},$$

might have zero, one or infinitely many solutions.

If $\det A \neq 0$ *then* $\vec{x} = A^{-1}\vec{b}$ *is the unique solution. If* $\det A = 0$ *then either there are no solutions or infinitely many solutions.*

To see why (4.5) and (4.6) are true, at least in dimension three, we have to go through the eight configurations and see what happens in each case. Note that there are only four cases to consider for (4.5), since the four cases where the planes have no common points doesn't occur.

Now the rows of the matrix A are the normal vectors to the three planes. The determinant is the (signed) volume of the parallelepiped spanned by the three normals.

If there is one solution the three planes all point in different directions, we have a parallelepiped and its volume is non-zero.

If there is a plane of solutions, the three planes are the same and the three normal directions are parallel. The determinant is zero, either algebraically because one row is a multiple, or geometrically, since the parallelepiped is degenerate and the volume is zero.

If there is a line of solutions the normal directions are orthogonal to any vector in the line, that is, the normal directions all live in a plane. But then the volume of the parallelepiped is zero.

This proves (4.5).

Finally, suppose that there are no solutions. If two planes are parallel the corresponding normal directions are parallel, so the parallelepiped is degenerate, it lives in a plane and so the parallelepiped has zero volume. This only leaves one, the case where the three planes meet in three parallel lines. In this case the three normal directions live in the plane orthogonal to the direction of the three lines. So the parallelepiped is again degenerate and the volume is zero.

To summarise. The parallelepiped is non-degenerate in only one case, so that the determinant is non-zero in only one case, when the three planes meet in one point. This is the content of (4.5) and (4.6).