

33. REVIEW I

Example 33.1. We have two lines in \mathbb{R}^3 , one given parametrically by

$$\vec{r}_1(t) = \langle -4 + 5t, 1 + t, -2 - t \rangle,$$

and the other given as the intersection of the two planes:

$$2x - y - z = 6 \quad \text{and} \quad x + y - 2z = 3.$$

What is the shortest distance between these two lines?

There are many different ways to solve this problem but all of them start the same way, by first finding the equation of the second line parametrically.

Note that each equation determines a plane and the intersection of two planes is a line. A line is specified by two points. So we want to find two points on the line.

To get a point on a line intersect with a plane. Let's intersect with the plane $z = 0$. The two equations reduce to

$$\begin{aligned} 2x - y &= 6 \\ x + y &= 3. \end{aligned}$$

This is an inhomogeneous system of linear equation. We can rewrite this as a matrix equation:

$$A\vec{x} = \vec{b} \quad \text{where} \quad \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}.$$

This has a unique solution if and only if $\det A \neq 0$.

$$\det A = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 3.$$

As the determinant is not zero, A is invertible:

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}.$$

If we are given A^{-1} , then it is easy to solve $A\vec{x} = \vec{b}$:

$$\vec{x} = A^{-1}\vec{b} \quad \text{that is} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

So one point on the line is $P = (3, 0, 0)$.

If we take the plane $z = -1$ then

$$\begin{aligned} 2x - y &= 5 \\ x + y &= 1. \end{aligned}$$

Arguing as above,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

So another point on the line is $Q = (2, -1, -1)$. The parametric form of the second line is then

$$\vec{r}_2(t) = \vec{P} + t\vec{PQ} = \langle 3, 0, 0 \rangle + t\langle -1, -1, -1 \rangle = \langle 3 - t, -t, -t \rangle.$$

So as to work with pluses rather than minuses, we switch the sign of t to get the same line parametrised with the opposite orientation:

$$\vec{r}_2(t) = \langle 3 + t, t, t \rangle.$$

Now we turn to the problem of finding the two closest points P_1 and P_2 belonging to the two lines. There are many ways to solve this problem. The first three ways use the geometric fact that the vector $\overrightarrow{P_1P_2}$ is orthogonal to the direction of both lines.

Method #1: Find a plane Π containing the second line which is parallel to the first line. Π is orthogonal to the direction of both lines:

$$\vec{u} = \langle 5, 1, -1 \rangle \quad \text{and} \quad \vec{v} = \langle 1, 1, 1 \rangle.$$

The cross product of \vec{u} and \vec{v} is therefore orthogonal to both lines:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \hat{i} \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} - \hat{j} \begin{vmatrix} 5 & -1 \\ 1 & 1 \end{vmatrix} + \hat{k} \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} = 2\hat{i} - 6\hat{j} + 4\hat{k}.$$

So

$$\vec{n} = \langle 1, -3, 2 \rangle$$

is a normal vector to the plane Π . Π contains $(2, -1, -1)$ (set $t = -1$ just to get a more interesting point than $(3, 0, 0)$). So the equation of the plane is

$$\langle x-2, y+1, z+1 \rangle \cdot \langle 1, -3, 2 \rangle = 0 \quad \text{that is} \quad (x-2) - 3(y+1) + 2(z+1) = 0,$$

so that rearranging we have $x - 3y + 2z = 3$. Pick any point of the first line. If we set $t = 0$ we get $R = (-4, 1, -2)$.

The line through R parallel to \vec{n} intersects the plane Π at the closest point R' to R . This line is

$$\vec{r}(t) = \langle -4, 1, -2 \rangle + t\langle 1, -3, 2 \rangle = \langle -4 + t, 1 - 3t, -2 + 2t \rangle.$$

This lies on the plane Π when

$$(t - 4) - 3(1 - 3t) + 2(-2 + 2t) = 3 \quad \text{so that} \quad 14t = 14$$

But then $t = 1$. The closest point R' is $(-1, -2, 0)$.

$$\overrightarrow{RR'} = \langle 1, -3, 2 \rangle,$$

so the shortest distance is

$$\sqrt{(1+9+4)} = \sqrt{14}.$$

Method #2: Find the two closest points P_1 and P_2 directly. Choose different parametrisations for the first and second line:

$$\vec{r}_1(s) = \langle -4 + 5s, 1 + s, -2 - s \rangle \quad \text{and} \quad \vec{r}_2(t) = \langle 3 + t, t, t \rangle.$$

Then

$$\overrightarrow{P_1P_2} = \langle 7 - 5s + t, -1 - s + t, 2 + s + t \rangle.$$

We want this vector to be orthogonal to both \vec{u} and \vec{v} :

$$\langle 7-5s+t, -1-s+t, 2+s+t \rangle \cdot \langle 5, 1, -1 \rangle = 0 \quad \text{and} \quad \langle 7-5s+t, -1-s+t, 2+s+t \rangle \cdot \langle 1, 1, 1 \rangle = 0.$$

$$-27s + 5t = -32$$

$$-5s + 3t = -8.$$

Using guess and check, we see that $s = 1$ and $t = -1$ works. The two closest points are

$$P_1 = (1, 2, -3) \quad \text{and} \quad P_2 = (2, -1, -1).$$

As before

$$\overrightarrow{P_1P_2} = \langle 1, -3, 2 \rangle,$$

and the shortest distance is again $\sqrt{14}$.

Method #3: Pick two random points on both lines.

$$R_1 = \langle -4, 1, -2 \rangle \quad \text{and} \quad R_2 = \langle 3, 0, 0 \rangle.$$

Then the distance we want is given by the projection of

$$\overrightarrow{R_1R_2} = \langle 7, -1, 2 \rangle$$

onto $\vec{n} = \langle 1, -3, 2 \rangle$. The length of the projection is given by

$$\frac{\langle 7, -1, 2 \rangle \cdot \langle 1, -3, 2 \rangle}{|\langle 1, -3, 2 \rangle|} = \frac{7 + 3 + 4}{\sqrt{14}} = \sqrt{14}.$$

Method #4: Like method #2, but now use calculus to minimise the distance between P_1 and P_2 . Note that if we minimise the distance or the distance squared we get the same points. In practice we minimise the distance squared, since this gives much simpler equations:

$$f(s, t) = (7-5s+t)^2 + (-1-s+t)^2 + (2+s+t)^2 = 27s^2 - 10st + 3t^2 - 64s + 16t + 54.$$

Find the critical points. First find partials:

$$f_s = 54s - 10t - 64 \quad \text{and} \quad f_t = -10s + 6t + 16.$$

Set these equal to zero:

$$\begin{aligned}27s - 5t &= 32 \\ -5s + 3t &= -16\end{aligned}$$

We already know $s = 1$ and $t = -1$ will work.

Geometrically it is clear that we must have a minimum and there is no maximum. But, again just to practice, let's use the second derivative test to check we have a minimum:

$$f_{ss} = 54 \quad f_{st} = -10 \quad \text{and} \quad f_{tt} = 6.$$

Hence

$$A = 54 \quad B = -10 \quad \text{and} \quad C = 6.$$

$AC - B^2 > 0$. $A > 0$ so we have a minimum.